# MODELS OF ELLIPTICAL GALAXIES IN 1–1–1 RESONANCE AND THEIR NORMALIZATION: THE 3D HÉNON AND HEILES SYSTEM

S. FERRER, A. VIARTOLA
*Universidad de Zaragoza, Zaragoza, Spain*
J. PALACIÁN, P. YANGUAS
*Universidad Pública de Navarra, Pamplona, Spain*
AND
J. F. SAN JUAN
*Universidad de la Rioja, Logroño, Spain*

**Abstract.** In this Note we study perturbed isotropic harmonic oscillators in 1–1–1 resonance, which is one of the typical cases of galactic potential models. We focus on cubic and quartic axial symmetric potentials giving explicitly their normal form. The 3D normalized Hénon and Heiles case, which requires to reach fourth order, is studied showing its relative equilibria and bifurcations.

## 1. Introduction

We deal with 3D cubic and quartic potentials with axial and discrete symmetries proposed, among others, by de Zeeuw (1985). As far as we know, only first order studies have been done. One of the most typical cases is the Hénon and Heiles dynamical system in two dimensions (Hénon and Heiles, 1964). It has generally been viewed as the paradigm for various problems in mechanics, galactic dynamics (Hénon and Heiles, 1964; Verhulst, 1979) and for more theoretical studies such as integrability conditions (Churchill *et al.*, 1983) and chaos.

We consider the family of Hamiltonians given by

$$\mathcal{H} = \mathcal{H}_0+\varepsilon\omega^2[a_1z^3+a_2(x^2+y^2)z]+\varepsilon^2\omega^2[b_1(x^2+y^2)^2+b_2z^4+b_3(x^2+y^2)z^2], \tag{1}$$

where $\mathcal{H}_0 = \frac{1}{2}[X^2+Y^2+Z^2+\omega^2(x^2+y^2+z^2)]$, the variables $(X, Y, Z)$ refer to the conjugate momenta of $(x, y, z)$, $\omega$ is a frequency and $\varepsilon$ means that

*I. M. Wytrzyszczak, J. H. Lieske and R. A. Feldman (eds.),*
*Dynamics and Astrometry of Natural and Artificial Celestial Bodies*, 425, 1997.

second and third terms of Eq. (1) are small perturbations of $\mathcal{H}_0$. Note that this dynamical system is axially symmetric. Among the cubics ($b_1 = b_2 = b_3 = 0$), the most relevant case is the Hénon and Heiles Hamiltonian, which corresponds to the coefficients $a_1 = -1/3$ and $a_2 = 1$.

In Section 2 we normalize the Hamiltonian $\mathcal{H}$, while the reduced Hénon and Heiles Hamiltonian is presented with more detail in Section 3.

## 2. Normalization

Typically, several studies with polynomial potentials in 1–1–1 resonance have been made. While some have attacked the problem by numerical integration (Robe, 1985), other authors (Verhulst, 1979; de Zeeuw, 1985) have formulated it in terms of action–angle variables and applied the averaging method.

What we do here is to reduce the system by normalization. First, we define a set of complex variables derived from the Cartesians. If $i$ stands for the imaginary unit $\sqrt{-1}$ then

$$u = \tfrac{1}{\sqrt{2}}(x - i\,X/\omega), \quad v = \tfrac{1}{\sqrt{2}}(y - i\,Y/\omega), \quad w = \tfrac{1}{\sqrt{2}}(z - i\,Z/\omega),$$
$$U = \tfrac{1}{\sqrt{2}}(X - i\,\omega x), \quad V = \tfrac{1}{\sqrt{2}}(Y - i\,\omega y), \quad W = \tfrac{1}{\sqrt{2}}(Z - i\,\omega z).$$

The Lie derivative $L_0$ of a function $F$ associated to $\mathcal{H}_0$ is the Poisson bracket $\{F, \mathcal{H}_0\}$, which in the complex becomes the differential operator

$$L_0 = i\,\omega\,\left(u\frac{\partial}{\partial u} - U\frac{\partial}{\partial U} + v\frac{\partial}{\partial v} - V\frac{\partial}{\partial V} + w\frac{\partial}{\partial w} - W\frac{\partial}{\partial W}\right),$$

because the unperturbed Hamiltonian in the complex reads as $\mathcal{H}_0 = i\omega(uU + vV + wW)$.

Yet, as the perturbation remains polynomial in complex variables, for a given monomial $m = u^a v^b w^c U^d V^e W^f$ its Lie derivative yields $L_0(m) = i\,\omega\,(a + b + c - d - e - f)m$. The kernel of $L_0$ is the vector subspace generated by the monomials $m$ such that their corresponding exponents verify the relation

$$a + b + c - d - e - f = 0. \tag{2}$$

Hence, the normalization for a potential in complex variables is as follows. On the one hand, a term $m$ of the perturbing potential which verifies Eq. (2) adds nothing to the generator and remains in the normalized potential. If, on the other hand, it does not verify Eq. (2) its contribution to the normalized potential is zero and the term $i\,m/\omega/(a + b + c - d - e - f)$ must be added to the generator.

From a practical point of view, by means of the Lie–Deprit method one constructs a canonical transformation $(u', v', w', U', V', W'; \varepsilon) \to (u, v, w, U, V, W)$ such that in each step of the process one deals with the partial differential identity $L_0(\mathcal{W}_n) + \mathcal{K}_n = \tilde{\mathcal{H}}_n$. While $\mathcal{W}_n$ stands for the generator,

$\mathcal{K}_n$ refers to the transformed Hamiltonian, both of order $n$ and the terms $\tilde{\mathcal{H}}_n$ are computed by the recursive relations given in the Lie triangle algorithm proposed by Deprit (1969).

The process of the normalization is obviously independent of the set of variables one chooses. Moreover, we have also performed the algorithm in the so called nodal–Lissajous $(\ell, g, \nu, L, G, N)$ (Ferrer and Gárate, 1996), an extension of the planar Lissajous variables (Deprit, 1991), obtaining the same results. The reason to use complex variables in the normalization is that the algorithm is rather suitable to be coded with an algebraic manipulator. Besides, the total number of terms appearing in the intermediate calculations, when one carries out the normalization up to a high order, is much smaller in complex variables than in Lissajous. Let us remark the equivalence between Eq. (2) and the elimination of the variable $\ell$. In other words, normalizing the initial Hamiltonian in complex variables is the same as reducing in one the number of degrees of freedom in Lissajous. Furthermore we have made use of the Lissajous variables to write the reduced Hamiltonian and take advantage of their well defined physical and geometrical meaning, see (Ferrer *et al.*, 1996a): $N$ is the third component of the angular momentum, $G$ its modulus, $L = \mathcal{H}_0/\omega$, $g$ measures the position of the semi–minor axis reckonned from the nodal line, while $\nu$ stands for the argument of the node and $\ell$ gives the position on the ellipse from the semi–minor axis. The case $G = 0$ must be excluded from our study.

The application of the normalization in complex variables to Eq. (1), up to an order $2n$, yields that $\mathcal{K} = \sum_{i=0}^{n} \varepsilon^{2i}\mathcal{K}_{2i}/(2i)! + \mathcal{O}(\varepsilon^{2n+2})$, where $\mathcal{K}_0 = \mathcal{H}_0$ and each Hamiltonian $\mathcal{K}_{2i}$, $i = 1, 2, \cdots, n$, is a polynomial in complex, of degree $2i+2$ and all the terms verify Eq. (2). Notice that odd Hamiltonians are zero because in the original potential, cubic terms appear at first order while quartic terms are placed at second order.

Thus, in Lissajous variables, the normalized Hamiltonian $\mathcal{K} \equiv \mathcal{K}(-, g, -, L, G, N)$ defines a one degree of freedom system (because of the axial symmetry) and, up to second order, reads

$$\mathcal{K} = \omega L + \frac{\varepsilon^2 L^2}{96}(C_0 + C_2 \cos 2g + C_4 \cos 4g) + \mathcal{O}(\varepsilon^4),$$

where

$$\left\{\begin{array}{lcl} C_0 & = & -48a_2^2 + 96b_1 + 8e^2(a_2^2 + 6b_1) \\ & & +16s^2[-9a_1a_2 + a_2^2 - 6b_1 + 3b_3 + 3e^2(3a_1a_2 - 7a_2^2 - 6b_1 + 3b_3)] \\ & & +3s^4[10(-3a_1^2 + 2a_1a_2 + a_2^2) + 12(b_1 + b_2 - b_3) \\ & & \qquad +e^2(5(-3a_1^2 + 2a_1a_2 + a_2^2) + 6(b_1 + b_2 - b_3))], \\ C_2 & = & 4es^2[-30a_1a_2 - 10a_2^2 - 36b_1 + 18b_3 \\ & & \qquad +3s^2(5(-3a_1^2 + 2a_1a_2 + a_2^2) + 6(b_1 + b_2 - b_3))], \\ C_4 & = & 3e^2s^4[5(-3a_1^2 + 2a_1a_2 + a_2^2) + 6(b_1 + b_2 - b_3)], \end{array}\right.$$

$e^2 = 1 - \eta^2$ with $\eta = G/L$ and $s^2 = 1 - c^2$ with $c = N/G$. Primes have been dropped in order to simplify the notation.

In particular, for the Hénon and Heiles the second order is trivial ($C_2 = C_4 = 0$), thus the fourth order has to be calculated. It is

$$\mathcal{K}_{HH} = \omega L + \frac{\varepsilon^2 L^2}{12}(-5 + 7\eta^2 - 8\sigma^2) + \frac{\varepsilon^4 L^3}{432\omega}\sum_{i=0}^{3} C_{4,2i}\cos 2ig + \mathcal{O}(\varepsilon^6),$$

where

$$\begin{cases} C_{4,0} &= -67 - 264\,\sigma^2 - 21\,\eta^2 + 56(9 + 20\sigma^2)c^2 \\ & \quad - 1008(2 + \sigma^2)c^4 + 1680c^6, \\ C_{4,2} &= 56\,c^2 e\,s^2[9(2 + \sigma^2) - 2\eta^2 - 45c^2], \\ C_{4,4} &= 1008\,c^2 e^2 s^4, \\ C_{4,6} &= -168\,e^3 s^6, \end{cases}$$

such that $\sigma = c\eta = N/L$. Indeed, we have reached order six so as to use it in forthcoming work to make estimations of the convergence of the series associated to the reduced problem. Nevertheless, the studies we present in this work, concerning the equilibria and bifurcations of the reduced system, are focused on the Hamiltonian $\mathcal{K}_{HH}$ truncated at order four. The normalization has been implemented with *Mathematica* and PSPC (Abad and San Juan, 1993), an algebraic processor specifically designed to handle problems in the frame of the Theory of Perturbations. We have checked that the results obtained by the two ways are exactly the same.

## 3. Equilibria and Bifurcations

When fixing the integrals $(L, N)$ the reduced space is topologically equivalent to an $\mathrm{S}^2$ sphere. Hence, we choose the coordinates $I_1 = e\eta s\cos 2g$, $I_2 = e\eta s\sin 2g$, $I_3 = \eta^2 - (1 + \sigma^2)/2$, verifying $I_1^2 + I_2^2 + I_3^2 = (1 - \sigma^2)^2/4$. Moreover, with these variables we include the study of circular ($g$ is not defined) and polar ($\nu$ not defined) trajectories.

The equilibria in the reduced space are the roots of the differential system $\dot{I}_i = 0$, $i = 1, 2, 3$. They depend on the parameters $\sigma$ and $\bar{\varepsilon}$, where $\bar{\varepsilon} = \varepsilon^2 L/(36\omega)$. We limit our study to a domain such that $(\sigma, \bar{\varepsilon}) \in (0, 1) \times (0, 0.05)$. Here we only show the results. For details on the process we refer to (Ferrer *et al.*, 1996b; Yanguas, 1996).

We obtain only three valid solutions of the system $\dot{I}_i = 0$:

(i) *The poles* $I_1 = I_2 = 0$. The south pole is always an equilibrium, whereas the north pole is an equilibrium only for $\sigma$ equal to $2/3$.

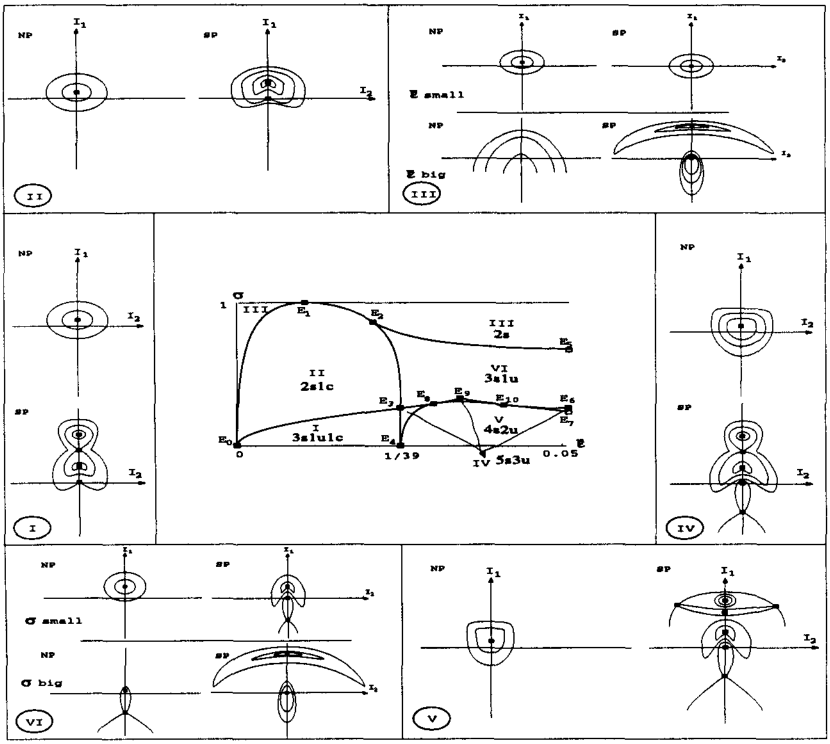

*Figure 1.* Equilibria and their stability. Black squares represent the equilibria. NP means that the picture shows a view of the sphere from its north pole, whereas SP is a view from the south pole. In regions III and VI two different views of the same configuration are shown in order to remark the evolution of the bifurcations and equilibria in the plane of bifurcations. In the figure of the centre the number of equilibria and type of stability (s stable, u unstable and c cusp) are shown in each of the six regions.

(ii) *The meridian* $I_1 \neq 0, I_2 = 0$. We find equilibria in this meridian verifying a system of two polynomial equations of degree six in $e$ whose coefficients are functions of $\sigma$ and $\overline{\varepsilon}$.

(iii) *Points* $I_1 \neq 0, I_2 \neq 0$. We obtain explicit expressions for $I_1$, $I_2$ and $I_3$, corresponding to symmetric points with respect to the axis $I_1$.

The study of the bifurcation lines is accomplished by imposing the following conditions to the equations coming from the three previous cases: the existence of double roots, a root that reaches the north or south pole of the sphere, or two roots on a parallel of the sphere that coalesce in the equidistant meridian. The results are shown in Figure (1), which is a portrait (north and south views) of the flow in the six zones determined by the bifurcation lines. We list now the type of bifurcation corresponding to each line. It is worth to say that we have found analytical expressions for them.

(1) Line $E_0E_1E_2E_3E_4$ is a cusp-centre bifurcation at the south pole.

(2) Lines $E_2E_5$ and $E_0E_9E_6$ are centre-saddle bifurcations coming from the discussion of the existence of double roots of the system of two equations in $e$, mentioned in (ii).

(3) Line $E_4E_8E_{10}E_7$ is a pitchfork bifurcation of a saddle point at $I_1$.

(4) For the points $E_2$ and $E_3$ two bifurcations occur in two different points of $S^2$, while for $E_8$ and $E_{10}$ a double bifurcation takes place at the same point of the sphere.

At present our research is devoted to the study of the stability of the reduced equilibria and to the analysis of the periodic orbits of the system. We refer the reader to (Ferrer *et al.*, 1996b; Yanguas, 1996).

*Acknowledgements.* Research partially supported by CICYT PB 95-0795, PB 95-0807 (Spain) and by Universidad Pública de Navarra (authors J. Palacián and P. Yanguas).

## References

Abad, A. and San Juan, J.F.: 1993, "PSPC: A Poisson Series Processor Coded in C", in: *Dynamics and Astrometry of Natural and Artificial Celestial Bodies*, (K. Kurzyńska, F. Barlier, P.K. Seidelmann, I. Wytrzyszczak, eds), Astronomical Observatory, Poznań, 383–388.

Churchill, R.C., Kummer, M., and Rod, D.L.: 1983, "On averaging, reduction, and symmetry in Hamiltonian systems", *J. Diff. Equat.* **49**, 359–414.

de Zeeuw, T.: 1985, "Motion in the core of a triaxial potential", *Mon. Not. R. Astron. Soc.* **215**, 731–760.

Deprit, A.: 1969, "Canonical transformations depending on a small parameter", *Celest. Mech.* **1**, 12–30.

Deprit, A.: 1991, "The Lissajous transformation. I: Basics", *Celest. Mech. & Dyn. Astron.* **51**, 201–226.

Ferrer, S. and Gárate, J.: 1996, "On perturbed 3D elliptic oscillators: A case of critical inclination in galactic dynamics", in: *New Trends in Hamiltonian Systems and Celestial Mechanics,* (E. Lacomba, J. Llibre, eds), ASND **8** World Scientific, Singapore, 179–197.

Ferrer, S., Lara, M., Palacián, J., and Yanguas, P.: 1996 a, "On perturbed oscillators in 1–1–1 Resonance: Critical inclination in the 3D Hénon and Heiles potential", to appear in: *Hamiltonian Systems with Three or More Degrees of Freedom,* (A. Delshams, ed.), Kluwer, Dordrecht.

Ferrer, S., Lara, M., Palacián, J., San Juan, J.F., Viartola, A., and Yanguas, P.: 1996 b, "The Hénon and Heiles system in three dimensions: The atypical cubic oscillator in 1–1–1 resonance, *Physica D*, submitted.

Hénon, M. and Heiles, C.: 1964, "The applicability of the third integral of motion: Some numerical experiments", *Astron. J.* **69**, 73–79.

Robe, H.: 1985, "Periodic orbits in a triaxial galaxy", *Astron. & Astrophys.* **142**, 351–354 and 1987, *ibid* **182**, 202–206.

Verhulst, F.: 1979, "Discrete symmetric dynamical systems at the main resonances with applications to axi–symmetric galaxies", *Phil. Trans. Roy. Soc. London* **A 290**, 435–465.

Yanguas, P.: 1996, Ph.D. Thesis, in preparation.