INTEGRAL MEANS OF FUNCTIONS WITH POSITIVE REAL PART

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1. We denote by $\mathcal{P}$ the class of functions of the form

$$h(z) = 1 + \sum_{n=1}^{\infty} c_n z^n$$

that are regular in $\Delta = \{ z : |z| < 1 \}$ and satisfy Re $h(z) > 0$ there. For $0 \leq r < 1$, we write

$$I_p(r) = I_p(r, h) = \frac{1}{2\pi} \int_0^{2\pi} |h(re^{i\theta})|^p \, d\theta, \quad (p > 0),$$

$$I(r) = I_1(r),$$

$$A(r) = A(r, h) = \sup \{ \text{Re} \, h(z) : |z| = r \},$$

$$M(r) = M(r, h) = \sup \{ |h(z)| : |z| = r \}.$$ 

We note that, for $\theta \in \mathbb{R}$, the inequality

$$1 + r M(r) \leq 1 - r$$

is classical.

Let now $h \in \mathcal{P}$ and write $h(z) = u(r, \theta) + iv(r, \theta)$ for $z = re^{i\theta} \in \Delta$. Then

$$I(r) \leq \frac{1}{2\pi} \int_0^{2\pi} u(r, \theta) \, d\theta + \frac{1}{2\pi} \int_0^{2\pi} |v(r, \theta)| \, d\theta$$

$$= 1 + \frac{1}{2\pi} \int_0^{2\pi} |v(r, \theta)| \, d\theta$$

by the normalization $h(0) = 1$. Furthermore, by Zygmund’s theorem [1, p. 58],

$$\int_0^{2\pi} |v(r, \theta)| \, d\theta \leq \int_0^{2\pi} u(r, \theta) \log^+ u(r, \theta) \, d\theta + 6\pi e$$

$$\leq 2\pi \log A(r) + 6\pi e,$$

since $\log^+ u(r, \theta) = \max\{ \log u(r, \theta), 0 \} \leq \log A(r)$, as $A(r) \geq 1$. We have thus proved the first part of our opening theorem.

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THEOREM 1. Let \( h \in \mathcal{P} \) and \( 0 \leq r < 1 \). Then
\[
I(r) \leq \log A(r) + B,
\]
and, for \( p > 1 \),
\[
I_p(r) \leq B_p A(r)^{p-1}. \tag{1.1}
\]
(Throughout this paper \( B, C, K \) denote positive absolute constants and \( B_p, C_p \) denote positive constants which depend only on \( p \), but the constant denoted by each symbol may differ at different occurrences.)

To prove the second part of Theorem 1 we need only note that, by M. Riesz's theorem [1, p. 54], for \( p > 1 \),
\[
\int_0^{2\pi} |h(r e^{i\theta})|^p d\theta \leq B_p \int_0^{2\pi} u(r, \theta)^p d\theta \leq 2\pi B_p A(r)^{p-1}.
\]

For \( h \in \mathcal{P} \), therefore,
\[
I(r) = O(\log A(r)),
I_p(r) = O(A(r)^{p-1}), \quad p > 1,
\]
as \( r \to 1 \). For \( 0 < p < 1 \), of course, it is well known that
\[
I_p(r) = O(1), \quad r \to 1,
\]
for such \( h \). The question now arises whether, in some sense, the relations in (1.2) are best possible. One might ask, for instance, whether there is a positive function \( \phi \) on \( (0, 1) \) such that if \( h \in \mathcal{P} \) and \( A(r) = O(\phi(r)) \), then
\[
I(r) = o(\log A(r))
\]
as \( r \to 1 \). Using examples constructed by Salem, we prove a general theorem which implies that the answer to this question is in the negative.

THEOREM 2. Let \( \phi \) be any positive function continuous and increasing to infinity on \( [0, 1) \) such that \( (1 - r) \phi(r) \) decreases on \( [0, 1) \). Then there is a function \( f \in \mathcal{P} \) with \( A(r) = O(\phi(r)) \), \( r \to 1 \), for which
\[
\liminf_{r \to 1} \frac{I(r)}{\log \phi(r)} > 0, \tag{1.3}
\]
and
\[
\liminf_{r \to 1} \frac{I_p(r)}{\phi(r)^{p-1}} > 0 \tag{1.4}
\]
for each \( p > 1 \).

Remark. The hypothesis that \( (1 - r) \phi(r) \) is decreasing is not an unnatural one here since it can be shown (cf. Lemma 5 below) that, for \( h \in \mathcal{P} \), \( (1 - r)(1 + r)^{-1} A(r) \) is a decreasing function of \( r \) on \( [0, 1) \).
The proof of Theorem 2 is given in Sections 2 and 3 but we conclude this section by noting that the first part of the theorem extends a recent result due to Lewis [6]. This author has shown that, given any number \( \epsilon \) in \( (0, 1) \), there exists a function \( h \in \mathcal{P} \) satisfying \( M(r) = O((1 - r)^{-\epsilon}) \) for which

\[
(1.5) \quad \liminf_{r \to 1} \frac{I(r)}{\log \frac{1}{1 - r}} > 0.
\]

Hayman [4] had earlier established a similar result but with, in (1.5), \( \lim sup \) in place of \( \lim inf \). If we take \( \phi(r) = (1 - r)^{-\epsilon} \) above and note that \( A(r) = O((1 - r)^{-\epsilon}) \) implies \( M(r) = O((1 - r)^{-\epsilon}) \) for \( h \in \mathcal{P} \) (this follows easily from (5.7) below, for example), then we see that (1.5) is a special case of (1.3).

2. Proof of theorem 2. In this section we state and prove two lemmas that we need.

**Lemma 1.** Let \( h \in \mathcal{P} \) and write \( u(r, \theta) = \text{Re} \, h(re^{i\theta}) \). Then, for \( p > 1 \),

\[
(2.1) \quad \int_0^{2\pi} u(r, \theta)^p d\theta \log \left\{ \frac{1}{2\pi} \int_0^{2\pi} u(r, \theta)^p d\theta \right\} \leq (p - 1) \int_0^{2\pi} u(r, \theta)^p \log u(r, \theta) d\theta
\]

for \( 0 \leq r < 1 \).

**Proof.** Fix \( r \in [0, 1) \) and, for \( h \in \mathcal{P} \), set

\[
\mu_r(\theta) = \frac{1}{2\pi} \int_0^\theta u(r, t) dt
\]

for \( \theta \in [0, 2\pi] \). Then \( \mu_r \) is an increasing function of \( \theta \) and \( \mu_r(2\pi) - \mu_r(0) = 1 \). Hence, if

\[
J_p(r) = \frac{1}{2\pi} \int_0^{2\pi} u(r, \theta)^p d\theta = \int_0^{2\pi} u(r, \theta)^{p-1} d\mu_r(\theta),
\]

then [9, p. 73] \([J_p(r)]^{1/(p-1)}\) is an increasing function of \( p \) in \((1, \infty)\). Consequently,

\[
\frac{d}{dp} \left\{ \frac{1}{p - 1} \log J_p(r) \right\} \geq 0
\]

for \( p > 1 \), from which it easily follows that

\[
\frac{1}{p - 1} J_p(r) \log J_p(r) \leq \frac{d}{dp} J_p(r) = \frac{1}{2\pi} \int_0^{2\pi} u(r, \theta)^p \log u(r, \theta) d\theta,
\]

and we have proved Lemma 1.
Some preliminaries are necessary before we can state our second lemma. If \( h \in \mathcal{P} \), then, by the Herglotz representation theorem, there is a function \( \mu \) increasing on \(( -\infty, \infty )\) satisfying

\[
\mu(t + 2\pi) - \mu(t) = 1
\]

for \( t \in ( -\infty, \infty ) \), such that

\[
(2.2) \quad h(z) = \int_{\mathbb{R}} e^{it} \frac{e^{it} + z}{e^{it} - z} d\mu(t)
\]

for \( z \in \Delta \). Then, for \( 0 \leq r < 1 \),

\[
(2.3) \quad \text{Re} h(re^{i\theta}) = \int_0^{2\pi} P(r, \theta - t) d\mu(t)
\]

where \( P(r, \psi) = \frac{1 - r^2}{1 - 2r \cos \psi + r^2} = \frac{1 - r^2}{(1 - r)^2 + 4r \sin^2 \frac{1}{2} \psi} \) is the Poisson kernel. We note that, for \( |\theta - t| < \pi \) and \( \frac{1}{2} \leq r < 1 \),

\[
(2.4) \quad P(r, \theta - t) \leq \frac{1 - r^2}{(1 - r)^2 + 4r \pi^2 (\theta - t)^2} \leq \frac{\pi^2 (1 - r)}{(1 - r)^2 + (\theta - t)^2}.
\]

Finally, for \( \delta > 0 \), we write

\[
\omega(\delta, \mu) = \sup \{ \mu(\theta + \delta) - \mu(\theta) : 0 \leq \theta < 2\pi \}
\]

so that \( \omega(\delta, \mu) \) is the "modulus of continuity" of \( \mu \).

**Lemma 2.** Let \( h \in \mathcal{P} \) and \( \mu \) be related as in (2.2). Then

\[
A(r, h) \leq K \omega(1 - r, \mu)(1 - r)
\]

for \( \frac{1}{2} \leq r < 1 \).

**Proof.** For \( m = 0, 1, 2, \ldots \), write

\[
F_m = F_m(r, \theta) = \{ t : m(1 - r) \leq |\theta - t| < (m + 1)(1 - r) \}.
\]

Then, for \( r \in \left[ \frac{1}{2}, 1 \right) \) and \( \theta \in [0, 2\pi] \),

\[
\text{Re} h(re^{i\theta}) \leq \int_{\frac{\theta}{\pi}}^{\theta + \frac{\pi}{\theta}} \frac{\pi^2 (1 - r)}{(1 - r)^2 + (\theta - t)^2} d\mu(t)
\]

\[
\leq \pi^2 (1 - r) \sum_{m=0}^{\infty} \int_{F_m} \frac{d\mu(t)}{(1 - r)^2 + (\theta - t)^2}
\]

\[
\leq \frac{2 \pi^2 \omega(1 - r, \mu)}{1 - r} \sum_{m=0}^{\infty} \frac{1}{1 + m^2} = K \frac{\omega(1 - r, \mu)}{1 - r}
\]

by (2.3) and (2.4). This proves Lemma 2.
3. As we have already said in Section 1, our proof of Theorem 2 is based on a result of Salem [10] which we now state as

**Lemma 3.** Let the function $\psi$ be defined and increasing on $(0, \infty)$ and satisfy $\psi(\delta)/\delta \to \infty$ as $\delta \to 0$. Suppose also that, for every integer $n > 1$ and $\delta \in (0, \infty)$, $\psi(n\delta) \leq n\psi(\delta)$. Then there exists a function $F$ defined and increasing in $(-\infty, \infty)$ with $F(t + 2\pi) - F(t) = 1$ for $t \in (-\infty, \infty)$ whose modulus of continuity $\omega(\delta, F)$ satisfies

\begin{equation}
\omega(\delta, F) \leq \psi(\delta), \quad 0 < \delta \leq 2\pi,
\end{equation}

and such that if

\begin{equation}
c_n = \int_0^{2\pi} e^{-inz}dF(t), \quad n \geq 1,
\end{equation}

then

\begin{equation}
\sum_{k=1}^n |c_k|^2 \geq Cn\psi\left(\frac{1}{n}\right), \quad n \geq 1.
\end{equation}

Suppose now that $\psi$ is the function defined in Theorem 2 and set

\begin{equation}
\psi(\delta) = \begin{cases} 
\delta \phi(1 - \delta), & 0 < \delta < 1, \\
\delta \phi(0), & \delta \geq 1.
\end{cases}
\end{equation}

Then it is easily verified that $\psi$ satisfies the conditions of Lemma 3. Let $F$ be the corresponding function obtained in the lemma and write

\begin{equation}
f(z) = \int_0^{2\pi} \frac{e^{itz}}{e^{it} - z} dF(t) = 1 + 2 \sum_{n=1}^\infty c_n z^n, \quad z \in \Delta.
\end{equation}

Then $f \in \mathcal{P}$ and, by Lemma 2 and (3.1),

\begin{equation}
A(r, f) \leq K \frac{\omega(1 - r, F)}{1 - r} \leq K \frac{\psi(1 - r)}{1 - r} = K\phi(r)
\end{equation}

for $\frac{1}{2} \leq r < 1$. We show next that

\begin{equation}
I_2(r, f) \geq B\phi(r), \quad \frac{1}{2} \leq r < 1.
\end{equation}

Let $r \in \left[\frac{1}{2}, 1\right)$ and let $n$ be the integer that $1 - 1/n \leq r < 1 - 1/(n + 1)$. Then, using Parseval's theorem and (3.2),

\begin{equation}
I_2(r, f) = 1 + 4 \sum_{n=1}^\infty |c_m|^2 r^{2m} \geq 4 \sum_{n=1}^{n+1} |c_m|^2 \left(1 - \frac{1}{n}\right)^{2m} \geq K \sum_{n=1}^{n+1} |c_m|^2
\end{equation}

\begin{equation}
\geq B(n + 1)\psi\left(\frac{1}{n + 1}\right) = B\phi\left(1 - \frac{1}{n + 1}\right) \geq B\phi(r),
\end{equation}

since $\phi$ is increasing. This proves (3.4), and (1.4) follows for the case $p = 2$. 


We next use (3.4) to establish (1.4) for arbitrary \( p > 1 \). By Holder's inequality, with \( 1/q = 1 - 1/p \) and \( p > 1 \), we have

\[
I_2(r) \leq I_p(r)^{1/p} I_0(r)^{1/q}
\]

and so, by (3.4), (1.1) and (3.3),

\[
B \Phi(r) \leq I_p(r)^{1/p} C_p \Phi(r)^{1/p}
\]

i.e.,

\[
I_p(r) \geq B_p \Phi(r)^{p-1}
\]

for \( \frac{1}{2} \leq r < 1 \). This clearly gives (1.4).

It remains only to show that (1.3) holds. Write \( u(r, \theta) = \text{Re} f(z) \) for \( z = re^{i\theta} \in \Delta \). Then by (2.1), with \( p = 2 \), (3.4) and (3.3),

\[
B \Phi(r) \log B \Phi(r) \leq \int_0^{2\pi} u(r, \theta)^2 \log u(r, \theta) d\theta
\]

\[
\leq \int_0^{2\pi} u(r, \theta)^2 \log^+ u(r, \theta) d\theta \leq K \Phi(r) \int_0^{2\pi} u(r, \theta) \log^+ u(r, \theta) d\theta,
\]

that is,

\[
(3.5) \quad \int_0^{2\pi} u(r, \theta) \log^+ u(r, \theta) d\theta \geq C \log B \Phi(r),
\]

for \( \frac{1}{2} \leq r < 1 \). But, by a converse [1, p. 60] to the theorem of Zygmund used in Section 1, it follows that, since \( f \in \mathcal{P} \),

\[
\int_0^{2\pi} u(r, \theta) \log^+ u(r, \theta) d\theta \leq \frac{\pi}{2} \int_0^{2\pi} |\text{Im} f(re^{i\theta})| d\theta + K
\]

\[
\leq \frac{\pi}{2} \int_0^{2\pi} |f(re^{i\theta})| d\theta + K
\]

for \( 0 \leq r < 1 \), and this, together with (3.5), clearly implies (1.3). The proof of Theorem 2 is now complete.

4. Some refinements of theorems 1 and 2. The function \( F \) constructed by Salem in [10] to establish Lemma 3 is a singular function, so it is natural to ask whether Theorem 2 can be proved with a function \( f \in \mathcal{P} \) which is 'generated', according to (2.2), by an increasing function \( \mu \) which is absolutely continuous. The subclass of \( \mathcal{P} \) of such functions will be denoted by \( \mathcal{P}_{ac} \). That the answer to the question is in the negative, at least in a special case, has already been proved by Keogh [5] who has shown, essentially, that if \( h \in \mathcal{P}_{ac} \), then

\[
I(r) = o \left( \log \frac{1}{1 - r} \right)
\]
as \( r \to 1 \). We prove here the following more complete result.

**Theorem 3.** Let \( h \in \mathcal{P}_{ac} \) and suppose that \( A(r, h) \to \infty \) as \( r \to 1 \). Then, as \( r \to 1 \),

\[
I(r, h) = o(\log A(r, h))
\]

and, for each \( p > 1 \),

\[
I_p(r, h) = o(A(r, h)^{p-1}).
\]

**Proof.** To prove (4.1) it is enough, by Zygmund’s theorem again, to show that

\[
\int_0^{2\pi} u(r, \theta) \log^+ u(r, \theta) d\theta = o(\log A(r, h)),
\]

where \( u(r, \theta) = \text{Re} \, h(re^{i\theta}) \). Let \( \mu \) be the absolutely continuous increasing function related to \( h \) by (2.2). Then (as in Section 2)

\[
u(r, \theta) = \int_0^{2\pi} P(r, \theta - t) d\mu(t) = \int_0^{2\pi} P(r, \theta - t) g(t) dt,
\]

for some \( g \in L[0, 2\pi] \), and it is familiar from harmonic function theory that we then have

\[
\lim_{r \to 1} u(r, \theta) = g(\theta)
\]
a.e. in \([0, 2\pi]\). We write next

\[
\int_0^{2\pi} u(r, \theta) \log^+ u(r, \theta) d\theta = \int_0^{2\pi} [\log^+ u(r, \theta)] g(\theta) d\theta
\]

\[
+ \int_0^{2\pi} [\log^+ u(r, \theta)] [u(r, \theta) - g(\theta)] d\theta
\]

\[
= J_1 + J_2, \text{ say.}
\]

Now \( \log^+ u(r, \theta) \{\log A(r)\}^{-1} \) is uniformly bounded in \( \Delta \) and, because of (4.3), tends to 0 a.e. in \([0, 2\pi]\) as \( r \to 1 \). Hence, by Lebesgue’s dominated convergence theorem,

\[
J_1 \{\log A(r)\}^{-1} \to 0 \text{ as } r \to 1.
\]

Also

\[
J_2 \{\log A(r)\}^{-1} \leq B \int_0^{2\pi} |u(r, \theta) - g(\theta)| d\theta
\]

and, since, trivially,

\[
\int_0^{2\pi} u(r, \theta) d\theta \to \int_0^{2\pi} g(\theta) d\theta \text{ as } r \to 1,
\]

it follows from (4.3) (see, for example, [1, p. 21]) that the integral on the right of (4.4) tends to 0 as \( r \to 1 \). This completes the proof of (4.1).
The relation (4.2) can be obtained by a similar argument or it can be deduced from (4.1) by means of Lemma 1; in either case the details are easy and are left to the reader.

Both (4.1) and (4.2) are best possible but before proving this we mention another result of Keogh [loc. cit.]. The result is stated by the author in terms of starlike functions but it can be given an equivalent formulation for $\mathcal{P}_{ac}$ as follows: given any positive function $\eta$, defined in $[0, 1)$ with $\eta(r) \to 0$ ($r \to 1$), there exists $h \in \mathcal{P}_{ac}$ such that

$$\sup \left\{ \left| \int_0^r \frac{\Re h(re^{i\theta}) - 1}{\rho} \, d\rho \right| : z = re^{i\theta} \in \Delta \right\} < \infty$$

and

$$\limsup_{r \to 1} \frac{I(r)}{\eta(r) \log 1/(1 - r)} > 0.$$

We strengthen and extend this result by proving

**Theorem 4.** Let $\phi$ be as in Theorem 2 and let $\eta$ be any positive function defined in $[0, 1)$ with $\eta(r) \to 0$ ($r \to 1$). Then for $p \geq 1$, there are functions $g_p \in \mathcal{P}_{ac}$ satisfying $A(r, g_p) = O(\phi(r))$, $r \to 1$, such that

$$\liminf_{r \to 1} \frac{I(r, g_p)}{\eta(r) \log \phi(r)} > 0$$

and

$$\liminf_{r \to 1} \frac{I_p(r, g_p)}{\eta(r) \phi(r)^{p-1}} > 0$$

for each $p > 1$.

This theorem will be shown to be a consequence of Lemma 3 and the following lemma.

**Lemma 4.** Let $f(z) = 1 + 2 \sum_{n=1}^\infty c_n z^n \in P$ and let $(\lambda_n)_0^\infty$, where $\lambda_0 = 1$, be a convex sequence of positive numbers which converges to $0$. Let

$$g(z) = 1 + 2 \sum_{n=1}^\infty \lambda_n c_n z^n, \quad z \in \Delta.$$

Then $g \in \mathcal{P}_{ac}$ and $A(r, g) \leq A(r, f)$ for $0 \leq r < 1$.

**Proof.** Since, for $0 \leq r < 1$, $(\lambda_n r^n)_0^\infty$ is a convex sequence which converges to $0$, it follows [12, p. 183] that

$$\Re \left\{ \frac{1}{2} + \sum_{n=1}^\infty \lambda_n z^n \right\} \geq 0$$

for $z = re^{i\theta} \in \Delta$. Hence, by (2.2),

$$\lambda_n = \int_0^{2\pi} e^{-i\alpha t} d\mu(t)$$
for some function $\mu$ increasing on $[0, 2\pi]$ with $\mu(2\pi) - \mu(0) = 1$. Thus, for $z \in \Delta$,

$$g(z) = \int_0^{2\pi} \left(1 + 2 \sum_{n=1}^{\infty} c_n e^{in(\theta-1)}\right) d\mu(t) = \int_0^{2\pi} f(re^{it}(\theta-1))d\mu(t)$$

and so $g \in \mathcal{P}$ and $A(r, g) \leq A(r, f)$. We next use the fact [12, p. 179] that if $\frac{1}{2}a_0 + \sum_1^{\infty} (a_n \cos nx + b_n \sin nx)$ is a Fourier-Stieltjes series and $(\lambda_n)_0^{\infty}$ is a convex sequence tending to 0, then $\frac{1}{2}a_0\lambda_0 + \sum_1^{\infty} (a_n \cos nx + b_n \sin nx)\lambda_n$ is a Fourier series. This result implies here that there is a function $G \in L[0, 2\pi]$ such that

$$\lambda_n c_n = \frac{1}{2\pi} \int_0^{2\pi} e^{-in\theta} G(t) dt \quad (n \geq 0)$$

(where $c_0 = 1$), from which we obtain

$$g(z) = \frac{1}{2\pi} \int_0^{2\pi} \frac{e^{it} + z}{e^{it} - z} G(t) dt.$$

Since

$$G(t) = \lim_{r \to 1} \text{Re } g(re^{it}) \geq 0$$

a.e. on $[0, 2\pi]$, it follows that $g \in \mathcal{P}_{ac}$. This completes the proof of the lemma.

**Proof of Theorem 4.** For $n \geq 1$, let

$$\epsilon_n = \sup\{\eta(r)^{1/2}: 1 - 1/n \leq r \leq 1 - 1/(n + 1)\}.$$

Then $\epsilon_n \to 0$ as $n \to \infty$. Let $(\lambda_n)_0^{\infty}$ be a convex decreasing sequence such that $\lambda_{n+1} \geq \epsilon_n$ ($n \geq 1$) and $\lambda_n \to 0$. (Such a sequence is easily constructed.) Let $f(z) = 1 + 2\sum c_n z^n$ be the function defined in the proof of Theorem 2 and let

$$g(z) = 1 + 2 \sum_{n=1}^{\infty} \lambda_n c_n z^n, \quad z \in \Delta.$$

Then, by Lemma 4, $A(r, g) \leq A(r, f) = O(\phi(r))$ and $g \in \mathcal{P}_{ac}$. Now fix $r \in (0, 1)$ and choose $n$ such that $1 - 1/n < r \leq 1 - 1/(n+1)$. By the argument used to prove (3.4),

$$I_2(r, g) \geq B\lambda_{n+1}^2 \phi(r) \geq B\eta(r) \phi(r)$$

and it is clear that (4.6) follows in the case $p = 2$ on taking $g_2 = g$.

The method used in Section 3 to deduce (1.4) from (3.4) now gives

$$I_p(r, g_2) \geq B_p\eta(r)^p \phi(r)^{p-1} \left(\frac{1}{2} \leq r < 1\right)$$

for every $p > 1$, and it is clear from this that a $g_\rho \in \mathcal{P}_{ac}$ exists for which (4.6) holds for all such $p$. 

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Finally, by the argument used to prove (1.3),
\[ I(r, g_2) \geq C\eta(r) \log [B\eta(r)\phi(r)] \]
and assuming, as we may, that \( \eta(r) \geq K\phi(r)^{-1/2} \), say, for all \( r \) sufficiently near 1, we immediately deduce (4.5) with \( g_1 = g_2 \). This completes the proof of Theorem 4.

5. The maximum modulus. As a consequence of Theorem 1 we have, for \( h \in \mathcal{P} \),
\[ I(r) \leq \log M(r) + A \]
and
\[ I_p(r) \leq B_p M(r)^{p-1} \quad (p > 1) \]
for \( 0 \leq r < 1 \). We now turn our attention, in this final section, to the problem of obtaining lower estimates for the integral means in terms of the maximum modulus. We begin by deriving a simple inequality of this type for functions that are merely regular in \( \Delta \).

Suppose, initially, that \( f(z) = \sum_0^\infty a_n z^n \) is regular in \( \Delta \) and continuous in the closure \( \bar{\Delta} \). For \( 0 \leq r < 1 \), we have
\[ M(r, f) \leq \sum_0^\infty |a_n|r^n \leq \left( \sum_0^\infty |a_n|^2 \right)^{1/2} \left( \sum_0^\infty r^{2n} \right)^{1/2} \]
and so, by Parseval’s theorem,
\[ (1 - r^2)M(r, f)^2 \leq \frac{1}{2\pi} \int_0^{2\pi} |f(e^{i\theta})|^2 d\theta. \]
Fix now \( r \in (0, 1) \) and let \( g \) be a function regular in \( \Delta \). Let the zeros of \( g \) in \( \{z : |z| \leq r\} \) be, with due account of multiplicity, \( z_1, z_2, \ldots, z_n \) and write
\[ B_n(z) = \prod_{k=1}^n \frac{z - z_k}{1 - \overline{z_k}z}, \quad z \in \bar{\Delta}, \]
Then \( |B_n(z)| < 1 \) for \( z \in \Delta \) and \( |B_n(z)| = 1 \) when \( |z| = 1 \). Now write
\[ F(z) = g(rz)B_n(z)^{-1}, \quad z \in \Delta. \]
Then \( F \) is regular and non-zero in \( \Delta \) and continuous in \( \Delta \) and so, given any \( p > 0 \), we can define a regular branch of \( F^{p/2} \) in \( \Delta \) which is also continuous on \( \bar{\Delta} \). Hence, using (5.1) with \( f = F^{p/2} \),
\[ (1 - r^2)M(r', g)^p \leq (1 - r^2)M(r, F^{p/2})^2 \leq \frac{1}{2\pi} \int_0^{2\pi} |F(e^{i\theta})|^p d\theta \]
\[ = \frac{1}{2\pi} \int_0^{2\pi} |g(re^{i\theta})|^p d\theta, \]
that is,
\[(5.2) \quad (1 - r^2) M(r^2, g)^p \leq I_p(r, g)\]
for \(p > 0\).

This inequality is sharp for each \(p > 0\), as the example \(g(z) = (1 - z)^{-2/p}\) shows, and is essentially a known result. The case \(p = 1\), for instance, is proved in [7] and the general result, in however a less precise form, is obtained in [2]. Although (5.2) is sharp, it can be improved in one direction to yield the following more delicate result: if \(g\) is regular in \(\Delta\), then
\[(5.3) \quad \int_0^r M(t, g)^p dt \leq r \pi I_p(r, g)\]
for \(p > 0\). A proof of this result can be found in [8]; but see also [3]. We note also that if, for some \(p > 0\), \(g\) belongs to the Hardy class \(H^p\), i.e., if
\[
\sup_{0 \leq r < 1} I_p(r, g) < \infty,
\]
then it is known [3], and is, in fact, an easy consequence of (5.3), that in this case (5.2) can be improved to
\[(5.4) \quad M(r, g) = o((1 - r)^{-1/p}), \quad r \to 1.\]

For the class \(\mathcal{P}\), of course, inequalities (5.2) and (5.4) are of interest only when \(p \geq 1\), since (as already noted in Section 1) \(h \in \mathcal{P}\) implies \(M(r, h) = O((1 - r)^{-1})\), \(r \to 1\). In the case \(p = 1\) both (5.2) and (5.4) can be improved for the class \(\mathcal{P}\), as we now show.

**Theorem 5.** Let \(h \in \mathcal{P}\). Then, for \(0 < r < 1\),
\[(5.5) \quad M(r, h) \leq \frac{2r \pi I(r, h)}{(1 - r) \log 1/(1 - r)}.\]

If, further, \(h \in H^1\), then
\[(5.6) \quad M(r, h) = o\left(\left(1 - r \log \frac{1}{1 - r}\right)^{-1}\right), \quad r \to 1.\]

We show that this theorem is a consequence of inequality (5.3) and the following lemma.

**Lemma 5.** Let \(h \in \mathcal{P}\). Then
\[
\frac{1 - r}{1 + r} M(r, h)
\]
is a decreasing function of \(r\) on \([0, 1)\).

**Proof.** For \(h \in \mathcal{P}\),
\[(5.7) \quad |h'(z)| \leq \frac{2}{1 - r} \Re h(z), \quad z \in \Delta.\]
(This classical inequality follows easily from (2.2)). Fix $\theta \in [0, 2\pi]$. Then, for $0 < r < 1$,
\[
\frac{\partial}{\partial r} \left( \frac{1 - r}{1 + r} |h(re^{i\theta})| \right) = \frac{1 - r}{r(1 + r)} |h(z)| \left\{ \frac{\text{Re} \left( \frac{zh'(z)}{h(z)} \right) - \frac{2r}{1 - r^2} \} (z = re^{i\theta}) \right. 
\]
\[
\leq 0 \text{ by (5.7). Hence } |(1 - r)/(1 + r)|h(re^{i\theta})| \text{ decreases on } [0, 1) \text{ for each fixed } \theta \in [0, 2\pi]. \text{ Let now } 0 < r_1 < r_2 < 1 \text{ and choose } \theta_0 \text{ such that } |h(r_2 e^{i\theta_0})| = M(r, h). \]

Then
\[
\frac{1 - r_2}{1 + r_2} M(r_2) = \frac{1 - r_2}{1 + r_2} |h(r e^{i\theta_0})| \leq \frac{1 - r_1}{1 + r_1} |h(r e^{i\theta_0})| \]
\[
\leq \frac{1 - r_1}{1 + r_1} M(r_1),
\]
and we have established Lemma 5.

We now prove Theorem 5. By (5.3), with $p = 1$,
\[
r \pi I(r, h) \geq \int_0^r M(t, h) dt \geq \frac{1 - r}{1 + r} M(r, h) \int_0^r \frac{1 + t}{1 - t} dt
\]
\[
\geq \frac{1}{2} (1 - r) M(r, h) \log \frac{1}{1 - r},
\]
where we have used Lemma 5. This proves (5.5).

If $h \in H^1$ then, by (5.3) again,
\[
\int_0^1 M(r, h) dr < \infty
\]
and an obvious refinement of the above argument gives (5.6). We omit the details but remark that they can be found in [11] where similar arguments have been used.

Our last theorem shows that (5.6) is, in a certain sense, best possible.

**Theorem 6.** Let $\varepsilon(r)$ be any positive function defined on $[0, 1)$ such that $\varepsilon(r) \to 0$ ($r \to 1$). Then there exists a function $h \in \mathcal{P}$ such that $h \in H^1$ and
\[
\limsup_{r \to 1} \frac{(1 - r) M(r, h) \log 1/(1 - r)}{\varepsilon(r)} > 0.
\]

(A lim inf result is clearly not possible in general here because of (5.3).)

**Proof.** Let $(r_n)$ be a sequence of positive real numbers increasing to 1 such that
\[
\sum_{n=1}^{\infty} \varepsilon(r_n) < \infty.
\]
Let
\[ A^{-1} = \sum_{n=1}^{\infty} \epsilon(r_n) \left( \log \frac{1}{1 - r_n} \right)^{-1} \]
and set
\[ \lambda_n = A \epsilon(r_n) \left( \log \frac{1}{1 - r_n} \right)^{-1}, \quad n \geq 1, \]
so that \( \sum \lambda_n = 1 \). For \( z \in \Delta \), we now define
\[ h(z) = \sum_{n=1}^{\infty} \lambda_n \frac{1 + r_n z}{1 - r_n z}. \]
Then \( h \in \mathcal{P} \) and, for \( 0 \leq r < 1 \),
\[ I(r, h) \leq \sum_{n=1}^{\infty} \lambda_n \int_{0}^{2\pi} \left| \frac{1 + r_n e^{i\theta}}{1 - r_n e^{i\theta}} \right| d\theta \leq B \sum_{n=1}^{\infty} \epsilon(r_n), \]
so that \( h \in H^1 \). Finally, for \( n \geq 1 \),
\[ M(r_n, h) \geq \Re h(r_n) \geq \lambda_n \frac{1 + r_n^2}{1 - r_n^2} \geq \frac{A \epsilon(r_n)}{2(1 - r_n) \log 1/(1 - r_n)}, \]
and (5.8) follows. This proves Theorem 6.

Inequality (5.4) is also best possible in the same sense for the class \( \mathcal{P} \) for each \( \rho > 1 \). This can be proved using examples similar to those constructed in the proof of Theorem 6 above. The details are left to the reader.

References

2. G. H. Hardy and J. E. Littlewood, Theorems concerning mean values of analytic or harmonic functions, Quart. J. Math. 12 (1941), 221–236.

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