INTEGRAL HARNACK INEQUALITY

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Introduction. Let D be a domain in Euclidean space of d dimensions and K a compact subset of D. The well known Harnack inequality assures the existence of a positive constant A depending only on D and K such that $(1/A)u(x) \leq u(y) \leq Au(x)$ for all x and y in K and all positive harmonic functions u on D. In this we obtain a global integral version of this inequality under geometrical conditions on the domain. The result is the following: suppose D is a Lipschitz domain satisfying the uniform exterior sphere condition—stated in Section 2. If u is harmonic in D with continuous boundary data f then

$$\int_D |u|(x)\,dx \leq C \int_{\partial D} |f|\,ds$$

where ds is the $d-1$ dimensional Hausdorff measure on the boundary ∂D. A large class of domains satisfy this condition. Examples are C^2-domains, convex domains, etc.

The lemma on which we base our proof states: For bounded domains satisfying the uniform exterior sphere condition solution of the Poisson equation with Dirichlet boundary conditions and constant forcing term has bounded gradient.

1. Generalities. Let D be a bounded domain in Euclidean space of $d \geq 3$ dimensions. G will denote its Green function: For all x, y

$$G(x, y) = K(x, y) - H(x, y) \tag{1.1}$$

where $K(x, y) = |x-y|^{-d+2}$ and $H(x, y)$ is the solution of the Dirichlet problem for D with boundary data $K(\cdot, y)$. Write

$$\sigma(x) = \int G(x, y)\,dy, \tag{1.2}$$

Then σ satisfies the Poisson equation

$$\begin{aligned} \Delta\sigma &= -A_d \\ \sigma &= 0 \text{ at regular points of } \partial d. \end{aligned} \tag{1.3}$$

where $A_d = (d-2)2\pi^{d/2}/\Gamma(d/2)$.

For any positive Radon measure m on D the function $\int G(x, y)m(dy)$ is locally integrable in D if it is finite at one point. Such a function is called a potential.

With the above notation and terminology we have

PROPOSITION 1.1. *Let z be an arbitrary but fixed point in D. All potentials in D are integrable on D iff there is a constant A depending only on z and D such that*

$$\sigma(y) \leq AG(z, y), \qquad y \in D. \tag{1.4}$$

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Proof. Suppose all potentials in D are integrable. If the assertion were false we could find a sequence y_n such that $\sigma(y_n) \geq n^2 G(z, y_n)$. If m is the measure giving mass $\sigma(y_n)^{-1}$ to y_n we have $\int G(z, y) m(dy) < \infty$. So m determines a potential and this potential is integrable by assumption. However the integral of this potential is $\int \sigma(y) m(dy) = \infty$. A contradiction.

Conversely suppose (1.4) is valid. Let p be the potential of the measure m. We have

$$\int p(x)\, dx = \int \sigma(x)\, dm(x) \leq Ap(z).$$

so that if $p(z) < \infty$ we are done. If $p(z) = \infty$ we proceed as follows: Take a ball contained in D and containing z. The balayage q of p on the complement of B is finite at z. q is thus integrable. Since p is locally integrable and equals q off B we find p is integrable. The proof is complete.

COROLLARY 1.2. *If all potentials on a domain D are integrable so are all positive harmonic functions.*

Proof. Let z be any point in D. From Proposition 1.1 there is a constant A such that $\sigma(y) \leq AG(z, y)$. Let u be positive harmonic. For any compact subdomain E, the reduit of u on E is a potential. The last inequality shows that the integral of this potential is bounded by $Au(z)$. As E expands to D, these reduits increase to u. That completes the proof.

PROPOSITION 1.3. *Let D be a bounded domain. For $x \in D$ let*

$$d(x) = \text{dist}(x, \partial D).$$

Then

$$|\text{grad}\, \sigma(x)| \leq (d/d(x))\sigma(x) \tag{1.5}$$

where d = dimension of space.

Proof. σ satisfies the Poisson equation (see (1.3))

$$\Delta\sigma = -C$$

with Dirichlet boundary conditions. It follows that grad σ is harmonic in D. Let $x \in D$ and B the ball centre x and radius $d(x)$. By the mean value property

$$\text{grad}\, \sigma(x) = \frac{1}{|B|} \int_B \text{grad}\, \sigma(y)\, dy$$
$$= \frac{1}{|B|} \int_{\partial B} \sigma n\, ds$$

where $|B|$ denotes volume of B; the last equality above being a consequence of the divergence theorem. Continuing

$$|\text{grad}\, \sigma(x)| \leq \frac{1}{|B|} \int_{\partial B} \sigma\, ds \leq \frac{d}{d(x)}\, \sigma(x)$$

because σ is superharmonic. The proof is complete.

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COROLLARY 1.4. *Let D be a bounded domain such that all points of ∂D are regular. Then* grad σ *is bounded in D iff*

$$\sigma(x) \leq \text{const } d(x) \tag{1.6}$$

where $d(x)$ as in Proposition 1.3 denotes distance to the boundary.

Proof. Let grad σ be bounded, $x \in D$ and $z \in \partial D$ satisfying $|x-z| = d(x)$. The line joining x and z is in D; z being regular $\sigma(z) = 0$. We have

$$\begin{aligned}\sigma(x) &= \int_0^1 \frac{d}{dt} \sigma(z + t(x-z))\, dt \\ &= \int_0^1 (x-z) \,.\, \text{grad } \sigma\, dt \\ &\leq |x-z|\, \|\text{grad } \sigma\|_\infty.\end{aligned}$$

PROPOSITION 1.5. *Let D be a bounded domain, f measurable with $|f| \leq 1$. Then*

$$|\text{grad } Gf(x)| \leq \frac{d}{d(x)} G\,|f|\,(x) + \text{const} \tag{1.7}$$

where const *is independent of f. In particular if* grad σ *is bounded and all points of ∂D regular then* $\|\text{grad } Gf\|_\infty \leq M$ *where M is independent of f and depends only on the dimension and volume of D.*

Proof. Assume f vanishes outside D and put $\phi = Kf$. ϕ is continuously differentiable [**1**] and

$$Gf = \phi - u \tag{1.8}$$

where u is the Dirichlet solution with boundary data ϕ. Let us estimate the gradients of ϕ and u.

Writing $a = |x-y|$, $b = |z-y|$,

$$\begin{aligned}|K(x,y) - K(z,y)| &= \left|\frac{1}{a^{d-2}} - \frac{1}{b^{d-2}}\right| \\ &= |a-b| \sum_{i+j=d-1} 1/a^i b^j\end{aligned}$$

For $i+j = d-1$, $a^{-i}b^{-j} \leq a^{-d+1} + b^{-d+1}$. We can continue from above

$$|K(x,y) - K(z,y)| \leq |x-z|\, d[1/a^{d-1} + 1/b^{d-1}]. \tag{1.9}$$

$|f| \leq |$ and the integral

$$\int_D \frac{1}{|\xi - y|^{d-1}}\, dy \leq \omega^{1-1/d} \frac{d}{(d-1)^{1-1/d}} |D|^{1/d} \tag{1.10}$$

where $|D|$ is the volume of D, ω the surface area of unit sphere and d is the dimension.

Integrating (1.9) and using (1.10)

$$|\phi(\xi)-\phi(\eta)|\leq A\,|\xi-\eta| \tag{1.11}$$

where A depends only on the volume of D and the dimension. We use (1.11) to estimate the gradient of u.

u being harmonic in D, so is grad u. Let $x\in D$ and B the ball with centre x and radius $d(x)$. By the mean value property

$$\begin{aligned}\text{grad}\,u(x)&=\frac{1}{|B|}\int_B \text{grad}\,u(y)\,dy\\&=\frac{1}{|B|}\int_{\partial B} u\eta\,ds\end{aligned}$$

by the divergence theorem. Let $z\in\partial D$ such that $|x-z|=d(x)$. Continuing from above

$$\begin{aligned}|\text{grad}\,u(x)|&=\left|\frac{1}{|B|}\int_{\partial B} u\eta\,ds\right|\\&=\frac{1}{|B|}\int_{\partial B}(u(y)-\phi(z))\eta\,ds\\&\leq\frac{1}{|B|}\int_{\partial B}|u(y)-\phi(z)|\,ds\end{aligned} \tag{1.12}$$

Let $\tau(y)$ be a point on ∂D satisfying

$$|y-\tau(y)|=\text{dist}(y,\partial D)$$

z being in ∂D,

$$\begin{aligned}&|y-\tau(y)|\leq|y-z|\leq 2\,d(x)\\&|\tau(y)-z|\leq|y-\tau(y)|+|y-z|\leq 4\,d(x)\end{aligned} \tag{1.13}$$

Continuing from (1.12):

$$\begin{aligned}|\text{grad}\,u(x)|\leq&\frac{1}{|B|}\int_{\partial B}|u(y)-\phi(\tau(y))|\,ds\\&+\frac{1}{|B|}\int_{\partial B}|\phi(\tau(y))-\phi(z)|\,ds\end{aligned}$$

(1.8), (1.11) and (1.13) can be used to estimate the integrands above

$$\begin{aligned}|u(y)-\phi(\tau(y))|&\leq G\,|f|\,(y)+|\phi(y)-\phi(\tau(y))|\\&\leq G\,|f|\,(y)+2A\,d(x)\\|\phi(\tau(y))-\phi(z)|&\leq 4A\,d(x)\end{aligned}$$

Using these and continuing from (1.14) and remembering that $G\,|f|\,(y)$ is superharmonic

$$|\text{grad } u(x)| \leq \frac{d}{d(x)} G\,|f|\,(x) + 6DA \tag{1.15}$$

Finally using (1.8), (1.11) and (1.15) we get (1.7). Since $G\,|f|\,(x) \leq \sigma(x)$, the second statement of the proposition follows from Corollary 1.4.

2. Domain condition. In this section we assume that the domain D is nice enough to satisfy the uniform R-sphere condition:

There exists $R>0$ such that for each $z \in \partial D$ corresponds a point ζ such that $|\zeta - z| = R$ and the open ball with centre ζ and radius R is completely contained in the complement of D.

This is a well known condition. See for example Courant–Hilbert [**1**]. Examples of such domains are domains with c^2-boundaries convex domains etc.

PROPOSITION 2.1. *Let D be a domain satisfying the uniform R-sphere condition. Let σ be as in* (1.2). *Then for $x \in D$*

$$|\text{grad } \sigma(x)| \leq M \tag{2.1}$$

where M depends only the diameter of D, the dimension of space and R.

Proof. Let $x \in D$ and $z \in \partial D$ such that $|z - x| = d(x)$. By assumption there is a ball $B(\zeta, R)$ in the complement of D and $|\zeta - z| = R$. The function

$$\phi(y) = \frac{1}{R^{d-1}} - \frac{1}{|\zeta - y|^{d-1}}$$

is positive and superharmonic in the complement of $B(\zeta, R)$. And for all $y \in D$

$$\Delta\phi(y) = -(d-1)\,|\zeta - y|^{-d-1} \leq -(d-1)(A+R)^{-d-1}$$

where A = diameter of D. Since by (1.3) $\Delta\sigma = -A_d$ in D, $N\phi$ with $N = A_d((A+R)^{d+1}/d-1)$ satisfies $\Delta(N\phi - \sigma) \leq 0$ in D. This means that $N\phi - \sigma$ is superharmonic in D and since $\sigma = 0$ on ∂D, $N\phi - \sigma \geq 0$ on ∂D. By the boundary minimum principle $N\phi \geq \sigma$ in D. Because $\phi(x) \leq R^{-d}\,|z - x|$ we obtain

$$\sigma(x) \leq R^{-d} N\,|z - x||$$

Proposition (1.3) then gives (2.1).

THEOREM 2.2 (Harnack inequality). *Let D be a bounded Lipschitz domain satisfying the uniform exterior R-sphere condition. If u is harmonic in D with boundary data $f \geq 0$*

$$\int u\,dx \leq \frac{M}{A} d \int f\,ds$$

where ds is the $(d-1)$ dimensional Hausdorff measure on ∂D, M and A_d are given in (2.1) *and* (1.3).

Proof. Let A be a smooth subdomain of D and $F \geq 0$ smooth on R^d. Then

$$F + \frac{1}{A_d} \int_D G(x, y)\, \Delta F(y)\, dy = u \tag{2.2}$$

where u is the harmonic function in D with boundary data F. Using Green's identity for A and from (1.3)

$$\int_A \sigma\, \Delta F + A_d \int_A F = \int_{\partial A} \sigma \frac{\partial F}{\partial \eta} - \int_{\partial A} F \frac{\partial \sigma}{\partial \eta}$$

In this last equality if we let A increase to D, and note that $\sigma = 0$ on ∂D:

$$\int_D \sigma\, \Delta F + A_d \int_D F = -\lim \int_{\partial A} F \frac{\partial \sigma}{\partial \eta} \tag{2.3}$$

Integrate both sides of (2.2) on D, compare with (2.3) and use (2.1) to get

$$\int_D u \leq \frac{M}{A_d} \int_{\partial D} F\, ds \tag{2.4}$$

where ds is the Hausdorff dimensional measure on ∂D.

REMARK. An easy conclusion from (2.4) is that for each $x \in D$ the harmonic measure at x is absolutely continuous relative to ds and has bounded density. Indeed let $m(x, dz)$ denote the harmonic measure at x and put $m(dz) = \int_D m(x, dz)\, dx$. If u and F are as above

$$\int_D u = \int_{\partial D} F\, dm$$

and (2.4) immediately tells us that m is absolutely continuous relative to ds and has density bounded by M/A_d. On the other hand if $f \in L^1(m)$, then necessarily $f \in L^1(m(x, \cdot))$ for each $x \in D$ i.e. for each $x \in D$, $m(x, \cdot)$ has bounded density relative to m. In particular $m(x, \cdot)$ has bounded density relative to ds as claimed.

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