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PLURIHARMONIC SYMBOLS OF COMMUTING TOEPLITZ TYPE OPERATORS

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Certain Toeplitz type operators acting on the Bergman space A^1 of the unit ball are considered and pluriharmonic symbols of commuting Toeplitz type operators are characterised by using $\mathcal{M}$-harmonic function theory.

1. INTRODUCTION AND RESULT

Let B be the unit ball of the n-dimensional complex space $\mathbb{C}^n$. The Bergman space A^p $(1 \leqslant p \leqslant \infty)$ is the closed subspace of the Lebesgue space $L^p = L^p(B, V)$ consisting of holomorphic functions where V denotes the Lebesgue volume measure on B normalised to have total mass 1. Let Q be the integral operator on L^1 defined by

$$Q(\psi)(z) = \lambda_n \int_B \frac{\left(1 - |w|^2\right)^{n+1}}{\left(1 - \langle z, w\rangle\right)^{2n+2}} \psi(w)\, dV(w) \quad (z \in B)$$

for functions $\psi \in L^1$ where $\langle\ ,\ \rangle$ is the ordinary Hermitian inner product on $\mathbb{C}^n$ and $1/\lambda_n = \int_B \left(1 - |w|^2\right)^{n+1} dV(w)$. It is known that Q is a bounded linear operator taking L^1 onto A^1. Moreover, Q has the following reproducing properties:

$$Qf = f \qquad \text{and} \qquad Q\overline{f} = \overline{f}(0) \tag{1}$$

for functions $f \in A^1$. See [**6**, Chapter 7] for more informations on the operator Q and related facts. For $u \in L^\infty$, the Toeplitz type operator T_u with symbol u is the linear operator acting on A^1 defined by

$$T_u(f) = Q(uf)$$

for functions $f \in A^1$. Clearly T_u is a bounded operator on A^1. In the Hilbert space context A^2, the *original* Toeplitz operators are defined in terms of the Bergman projection acting on L^2. But since the Bergman projection is unbounded on L^1, we

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natually consider the bounded projection Q on L^1 to define the corresponding Toeplitz type operators on A^1.

A function $u \in C^2(B)$ is said to be pluriharmonic if its restriction to an arbitrary complex line that intersects the ball is harmonic as a function of single complex variable. As is well known [**6**, Theorem 4.4.9], a real-valued function on B is pluriharmonic if and only if it is the real part of a holomorphic function on B. It follows that every pluriharmonic function on B can be expressed, uniquely up to an additive constant, as the sum of a holomorphic function and an antiholomorphic function.

In the present paper, we consider a characterisation problem of two pluriharmonic symbols for which the associate Toeplitz type operators commute on A^1. In the Hilbert space case A^2, the corresponding commuting problem for the *original* Toeplitz operators was studied in [**1**] with harmonic symbols on the unit disk and in [**4, 7**] with pluriharmonic symbols on the ball. The following theorem is the main result of the present paper.

Theorem 1. *Let u and v be bounded pluriharmonic symbols on B. Then $T_uT_v = T_vT_u$ on A^1 if and only if one of the following properties holds:*

(*a*) *u and v are both holomorphic on B;*
(*b*) *u and v are both antiholomorphic on B;*
(*c*) *there exist constants α and β, not both 0, such that $\alpha u+\beta v$ is constant on B.*

In the course of the proof of Theorem 1, we shall use an idea in [**4**] to give a slight variant of the characterisation of $\mathcal{M}$-harmonicity given by the weighted area version of the invariant mean value property (see Section 2 for relevent definitions) and a recent result in [**7**] on $\mathcal{M}$-harmonic products to characterise pluriharmonic symbols of commuting Toeplitz type operators. In Section 2 we collect some facts on $\mathcal{M}$-harmonic functions and then give a characterisation for $\mathcal{M}$-harmonic functions in terms of a weighted area version of the invariant mean value property. The characterisation will be used in Section 3 where we prove Theorem 1 and give a simple application.

2. $\mathcal{M}$-harmonic Functions

For $z, w \in B, z \neq 0$, define

$$\varphi_z(w) = \frac{z - |z|^{-2}\langle w, z\rangle z - \sqrt{1-|z|^2}\left(w - |z|^{-2}\langle w, z\rangle z\right)}{1 - \langle w, z\rangle}$$

and $\varphi_0(w) = -w$. Then $\varphi_z \in \mathcal{M}$, the group of all automorphisms (=biholomorphic self-maps) of B and φ_z is an involution: $\varphi_z \circ \varphi_z$ is the identity on B. Furthermore, each $\varphi \in \mathcal{M}$ has a unique representation $\varphi = U \circ \varphi_z$ for some $z \in B$ and $U \in \mathcal{U}$, the group of all unitary operators on $\mathbb{C}^n$. Then the real Jacobian $J_R\varphi$ of φ is given by

$$(2) \qquad J_R\varphi(w) = \left(\frac{1-|z|^2}{|1-\langle w,z\rangle|^2}\right)^{n+1} \qquad (w \in B)$$

and the useful identity

$$(3) \qquad 1 - \langle \varphi(a), \varphi(b)\rangle = \frac{\left(1-|z|^2\right)(1-\langle a,b\rangle)}{(1-\langle a,z\rangle)(1-\langle z,b\rangle)}$$

holds for every $a, b \in B$. See [**6**, Chapter 2] for details. For $u \in C^2(B)$ and $z \in B$, we define

$$\left(\widetilde{\Delta}u\right)(z) = \Delta(u \circ \varphi_z)(0)$$

where Δ denotes the ordinary Laplacian. The operator $\widetilde{\Delta}$ is called the invariant Laplacian because it commutes with automorphisms of B in the sense that $\widetilde{\Delta}(u \circ \varphi) = \left(\widetilde{\Delta}u\right) \circ \varphi$ for $\varphi \in \mathcal{M}$. We say that a function $u \in C^2(B)$ is $\mathcal{M}$- harmonic on B if it is annihilated on B by $\widetilde{\Delta}$. As is the case for harmonic functions on the disk, $\mathcal{M}$-harmonic functions are characterised by a certain mean value property (see [**6**, Theorem 4.2.4]): A function $u \in C(B)$ is $\mathcal{M}$-harmonic on B if and only if

$$(u \circ \varphi)(0) = \int_S (u \circ \varphi)(r\zeta)\, d\sigma(\zeta) \qquad (0 \leqslant r < 1)$$

for every $\varphi \in \mathcal{M}$. Here σ denotes the rotation invariant probability measure on the unit sphere S, the boundary of B. This is the so-called invariant mean value property. The following weighted area version of this invariant mean value property also gives a characterisation for $\mathcal{M}$-harmonicity of functions continuous up to S. In the case $\alpha = 0$ the following characterisation was obtained in [**6**, Proposition 13.4.4], [**5**, Corollary 3.5] and recently with bounded function in [**3**] on the ball. The case $\alpha > -1$ was proved in [**2**, Proposition 10.2] on the disk. We now have the ball version where the case $\alpha = n+1$ will be used in the course of the proof.

Proposition 2. *Let $u \in C(\overline{B})$ and $\alpha > -1$. Then u is $\mathcal{M}$-harmonic on B if and only if*

$$(4) \qquad (u \circ \varphi)(0) = \lambda_\alpha \int_B (u \circ \varphi)(w)\left(1-|w|^2\right)^\alpha dV(w)$$

for every $\varphi \in \mathcal{M}$. Here and elsewhere $1/\lambda_\alpha = \int_B \left(1-|w|^2\right)^\alpha dV(w)$.

PROOF: First suppose u is $\mathcal{M}$-harmonic and let $\varphi \in \mathcal{M}$. By the invariant mean value property, one obtains

$$(u \circ \varphi)(0) = \int_S (u \circ \varphi)(r\zeta)\, d\sigma(\zeta)$$

for every $0 \leqslant r < 1$. Multiplying both sides by $2nr^{2n-1}(1-r^2)^\alpha$ and then integrating in polar coordinates, we get

$$(u \circ \varphi)(0) \int_B \left(1 - |w|^2\right)^\alpha dV(w) = \int_B (u \circ \varphi)(w) \left(1 - |w|^2\right)^\alpha dV(w),$$

so we have (4). To prove the converse implication, we may assume u is real without loss of generality, and let U be the $\mathcal{M}$-harmonic function which is the invariant Poisson integral of the restriction of u to S. Put $h = U - u$. Then $h \in C(\overline{B})$ and $h = 0$ on S (See [**6**, Chapter 3] for related facts.) Let m be the maximun of h on $\overline{B}$ and suppose $h(z_0) = m$ for some $z_0 \in B$. Note that (4) holds for h. By a change of variables, (2) and (3), one obtains

$$\begin{aligned}
h(z) &= (h \circ \varphi_z)(0) \\
&= \lambda_\alpha \int_B (h \circ \varphi_z)(w) \left(1 - |w|^2\right)^\alpha dV(w) \\
&= \lambda_\alpha \int_B h(w) \left(1 - |\varphi_z(w)|^2\right)^\alpha \left(\frac{1 - |z|^2}{|1 - \langle w, z\rangle|^2}\right)^{n+1} dV(w) \\
&= \lambda_\alpha \int_B h(w) \frac{\left(1 - |w|^2\right)^\alpha \left(1 - |z|^2\right)^{n+1+\alpha}}{|1 - \langle w, z\rangle|^{2n+2+2\alpha}} dV(w)
\end{aligned}$$

for every $z \in B$. On the other hand, by a change of variables and (2), one can easily see that

$$I_z(w) dV(w) = \lambda_\alpha \left(1 - |w|^2\right)^\alpha \left(\frac{1 - |z|^2}{|1 - \langle w, z\rangle|^2}\right)^{n+1+\alpha} dV(w)$$

is a probability measure on B for every $z \in B$. Since u is real, we have by the above observations

$$\begin{aligned}
m - h(z) &= h(z_0) - h(z) \\
&= \int_B h(I_{z_0} - I_z)\, dV \\
&\leqslant m \int_B (I_{z_0} - I_z)\, dV \\
&= 0
\end{aligned}$$

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for every $z \in B$, which implies that $h = m$ on B. It follows that $h = 0$ on B because $h = 0$ on S. Hence $u = U$, so that u is $\mathcal{M}$-harmonic on B. The proof is complete. □

The key step in our proof of Theorem 1 is adapted from that of [4]. That is, to characterise pluriharmonic symbols of commuting Toeplitz type operators, we shall use a slight variant of the characterisation of $\mathcal{M}$-harmonicity given by the weighted area version of invariant mean value property. To state it, let us introduce some notation. We associate with each $v \in C(B)$ its so-called radialisation $\mathcal{A}(v)$ defined by the formula

$$\mathcal{A}(v)(z) = \int_{\mathcal{U}} v(Uz)\,dU \qquad (z \in B)$$

where dU denotes Haar measure on $\mathcal{U}$. Using Proposition 1.4.7 of [**6**], one can easily verify that

$$\mathcal{A}(v)(z) = \int_S v(|z|\,\zeta)\,d\sigma(\zeta) \qquad (z \in B)$$

and hence $\mathcal{A}(v)$ is indeed a radial function on B. We write $\mathcal{A}(v) \in C(\overline{B})$ if $\mathcal{A}(v)$ has a continuous extension up to the boundary S. The following proposition was proved in Proposition 4 of [4] in the case $\alpha = 0$.

Proposition 3. *Let $u \in C(B)$, $\alpha > -1$ and suppose*

$$\int_B |u(z)| \left(1 - |z|^2\right)^\alpha dV(z) < \infty.$$

Then u is $\mathcal{M}$-harmonic on B if and only if

$$(u \circ \varphi)(0) = \lambda_\alpha \int_B (u \circ \varphi)(w)\left(1 - |w|^2\right)^\alpha dV(w) \tag{5}$$

and

$$\mathcal{A}(u \circ \varphi) \in C(\overline{B}) \tag{6}$$

for every $\varphi \in \mathcal{M}$.

Proof: We first prove the easy direction. Suppose that u is $\mathcal{M}$-harmonic on B and let $\varphi \in \mathcal{M}$. By the invariant mean value property again, we have

$$(u \circ \varphi)(0) = \int_S (u \circ \varphi)(r\zeta)\,d\sigma(\zeta) \tag{7}$$

for every $r \in [0, 1)$. Then (5) follows by the same argument as in Proposition 2. Also (7) shows that $\mathcal{A}(u \circ \varphi)$ is constant on B, with value $(u \circ \varphi)(0)$, and therefore (6) holds.

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To prove the other direction (which we need for the proof of Theorem 1 with $\alpha = n+1$), suppose that (5) and (6) hold. Let $\varphi \in \mathcal{M}$ and put $v = \mathcal{A}(u \circ \varphi)$. We first show that v is $\mathcal{M}$-harmonic on B. Since $v \in C(\overline{B})$ by (6), it is sufficient by Proposition 2 to show (4) for v. To do this, fix $\psi \in \mathcal{M}$. Then

$$\lambda_\alpha \int_B (v \circ \psi)(z)\left(1-|z|^2\right)^\alpha dV = \lambda_\alpha \int_B \int_{\mathcal{U}} (u \circ F_U)(z)\left(1-|z|^2\right)^\alpha dU\, dV(z) \tag{8}$$

where $F_U = \varphi \circ U \circ \psi \in \mathcal{M}$.

For a fixed unitary operator $U \in \mathcal{U}$, consider the inverse mapping $G_U \in \mathcal{M}$ of F_U and put $a = F_U(0) = (\varphi \circ U \circ \psi)(0)$. Then, since $\left|\varphi^{-1}(0)\right| = |\varphi(0)|$, we have by (3)

$$1-|a|^2 = \frac{\left(1-|\varphi(0)|^2\right)\left(1-|\psi(0)|^2\right)}{\left|1-\langle\varphi^{-1}(0),(U\circ\psi)(0)\rangle\right|^2} \geqslant \left(1-|\varphi(0)|^2\right)\left(1-|\psi(0)|^2\right). \tag{9}$$

On the other hand, we have by (3) again

$$1-|G_U(w)|^2 = \frac{\left(1-|a|^2\right)\left(1-|w|^2\right)}{|1-\langle w,a\rangle|^2}$$

and by (2)

$$J_{\mathrm{R}}G_U(w) = \left(\frac{1-|a|^2}{|1-\langle w,a\rangle|^2}\right)^{n+1}$$

for every $w \in B$. It follows that

$$\begin{aligned}\left(1-|G_U(w)|^2\right)^\alpha J_{\mathrm{R}}G_U(w) &= \frac{\left(1-|a|^2\right)^{n+1+\alpha}\left(1-|w|^2\right)^\alpha}{|1-\langle w,a\rangle|^{2(n+1+\alpha)}} \\ &\leqslant \left(1-|w|^2\right)^\alpha\left(\frac{4}{1-|a|^2}\right)^{n+1+\alpha} \qquad (w \in B).\end{aligned}$$

Now a change of variables and the above, together with (9) show

$$\begin{aligned}&\int_{\mathcal{U}}\int_B |u \circ F_U(z)|\left(1-|z|^2\right)^\alpha dV(z)\, dU \\ &\qquad = \int_{\mathcal{U}}\int_B |u|\left(1-|G_U|^2\right)^\alpha J_{\mathrm{R}}G_U\, dV\, dU \\ &\qquad < \infty\end{aligned}$$

since $\int_B |u(z)| \left(1-|z|^2\right)^\alpha dV(z) < \infty$ by assumption. Now one can interchange the order of integrations on the right side of (8) to obtain

$$\begin{aligned}
\lambda_\alpha \int_B (v\circ\psi)(z)\left(1-|z|^2\right)^\alpha dV(z) &= \lambda_\alpha \int_{\mathcal{U}}\int_B (u\circ F_U)(z)\left(1-|z|^2\right)^\alpha dV(z)\,dU \\
&= \int_{\mathcal{U}} (u\circ F_U)(0)dU \\
&= \int_{\mathcal{U}} (u\circ\varphi\circ U)(\psi(0))dU \\
&= \mathcal{A}(u\circ\varphi)(\psi(0)) \\
&= (v\circ\psi)(0)
\end{aligned}$$

where the second equality holds by (5). Hence v is $\mathcal{M}$-harmonic on B. Since v is radial, the invariant mean value property shows that v is constant. Consequently,

$$(u\circ\varphi)(0) = v(0) = v(z) = \int_S (u\circ\varphi)(|z|\,\zeta)d\sigma(\zeta) \qquad (z\in B).$$

Since $\varphi\in\mathcal{M}$ is arbitrary, the above shows that u has the invariant mean value property and hence that u is $\mathcal{M}$-harmonic on B as desired. The proof is complete. □

Before turning to the our proof, we need a recent result of Zheng [7] on $\mathcal{M}$-harmonic products to characterise the symbols. (The original statement in [7, Theorem 2] is in a slightly different form.)

Lemma 4. *Let $u = f+\overline{g}$ and $v = h+\overline{k}$ be two bounded pluriharmonic symbols on B. If $f\overline{k}-h\overline{g}$ is $\mathcal{M}$-harmonic on B, then u and v are all holomorphic or antiholomorphic or there exist constants α and β, not both 0, such that $\alpha u+\beta v$ is constant on B.*

3. Proof

First, we recall some well known facts on the Hardy space H^2 consisting of holomorphic functions f on B for which

$$\sup_{0<r<1}\int_S |f(r\zeta)|^2\, d\sigma(\zeta) < \infty.$$

Note that $H^2\subset A^2$ by an integration in polar coordinates. In addition, it is shown in [4] that $\mathcal{A}(f\overline{g})\in C\left(\overline{B}\right)$ for every $f,g\in H^2$.

Next, before turning to the proof of Theorem 1, we prove a couple of lemmas. For $\varphi\in\mathcal{M}$, let U_φ denote the linear operator on L^1 defined by

$$U_\varphi f = (f\circ\varphi)(J\varphi)^2$$

where $J\varphi$ is the complex Jacobian of φ. Since $|J\varphi|^2$ is the real Jacobian of φ, one obtains by a change of variables

$$\int_B |U_\varphi f|\, dV = \int_B |f\circ\varphi|\,|J\varphi|^2\, dV = \int_B |f|\, dV$$

for every $f \in L^1$. Hence U_φ is an isometry of L^1 into L^1 and clearly U_φ takes A^1 onto A^1. Moreover it is easy to see that $U_\varphi U_{\varphi^{-1}} = U_{\varphi^{-1}}U_\varphi$ is the identity operator on L^1. The following lemma is essentially contained in [**8**] (in a slightly different case). But we here give a proof for the sake of completeness.

Lemma 5. *Let $\varphi \in \mathcal{M}$. Then $QU_\varphi = U_\varphi Q$ on L^1.*

Proof: Let $\varphi \in \mathcal{M}$ with the representation $\varphi^{-1} = U\circ\varphi_a$ for some $a \in B$ and $U \in \mathcal{U}$. Note by [**8**, Section 2] that

$$(10)\qquad (J\varphi_a)(z) = (-1)^n\left(\frac{\sqrt{1-|a|^2}}{1-\langle z,a\rangle}\right)^{n+1}\qquad (z\in B).$$

It follows from a straightforward calculation that

$$(11)\qquad (J\varphi)^2\left(\varphi^{-1}(z)\right) = \frac{\left(1-\langle z,a\rangle\right)^{2n+2}}{\left(1-|a|^2\right)^{n+1}}\qquad (z\in B).$$

Let $f \in L^1$ and pick a point $z \in B$. By a change of variables and a simple manipulation using (2) and (3), one can see from (11) that

$$\begin{aligned}
Q(U_\varphi f)(z) &= \lambda_n\int_B \frac{\left(1-|w|^2\right)^{n+1}}{\left(1-\langle z,w\rangle\right)^{2n+2}}(f\circ\varphi)(w)(J\varphi)^2(w)\,dV(w)\\
&= \lambda_n\int_B \frac{\left(1-|\varphi^{-1}(w)|^2\right)^{n+1}}{\left(1-\langle z,\varphi^{-1}(w)\rangle\right)^{2n+2}} f(w)(J\varphi)^2\left(\varphi^{-1}(w)\right)J_R\varphi^{-1}(w)\,dV(w)\\
&= \lambda_n\frac{\left(1-|a|^2\right)^{n+1}}{\left(1-\langle z,Ua\rangle\right)^{2n+2}}\int_B \frac{\left(1-|w|^2\right)^{n+1}}{\left(1-\langle\varphi_a U^{-1}(z),w\rangle\right)^{2n+2}} f(w)\,dV(w).
\end{aligned}$$

On the other hand, (10) shows that the last expression of the above is just the same as $(J\varphi)^2(z)Qf(\varphi(z))$, which is exactly $U_\varphi Qf(z)$. Hence $U_\varphi Q = QU_\varphi$ on L^1, as desired. The proof is complete. ∎

Lemma 6. *Let $\varphi \in \mathcal{M}$ and $u \in L^\infty$. Then*

$$U_\varphi T_u U_{\varphi^{-1}} = T_{u\circ\varphi}.$$

Proof: Let $f \in A^1$. By Lemma 5, one obtains

$$T_{u\circ\varphi}U_\varphi f = T_{u\circ\varphi}[(f\circ\varphi)(J\varphi)^2] = Q[(u\circ\varphi)(f\circ\varphi)(J\varphi)^2]$$
$$= QU_\varphi(uf) = U_\varphi Q(uf) = U_\varphi T_u f.$$

Thus $T_{u\circ\varphi}U_\varphi = U_\varphi T_u$ on L^1. Now use the fact $U_\varphi U_{\varphi^{-1}}$ is the identity operator to get the desired result. This completes the proof. □

We are now ready to prove Theorem 1.

Proof of Theorem 1: We begin with the easy direction. First suppose that (a) holds, so that u and v are holomorphic on B, which means that T_u and T_v are, respectively, the operators on A^1 of multiplication by u and by v by (1). Thus $T_uT_v = T_vT_u$ on L^1. Now assume (b), so that $\overline{u}$ and $\overline{v}$ are holomorphic on B. By the explict formula for the operator Q and an application of Fubini's theorem, one can see that $T_uT_vf = Q(uvf)$ for every bounded function f in A^1. Note that the set of all bounded functions in A^1 forms a dense subset of A^1. It follows from continuity that T_u and T_v commute, as desired. Finally suppose (c) holds and assume $\alpha \neq 0$ (the other case is similar). Then $u = c_1v + c_2$ for some constants c_1 and c_2, which implies $T_u = c_1T_v + c_2$, so that $T_uT_v = c_1T_vT_v + c_2T_v = T_vT_u$ on A^1.

Now we prove the converse implication. Write $u = f + \overline{g}$ and $v = h + \overline{k}$ for some holomorphic f, g, h, and k. It is shown in the proof of Theorem 1 in [4] that functions f, g, h, and k are all in H^2. Since $H^2 \subset A^2 \subset A^1$, in particular, functions f, g, h, and k are all in A^1. Let 1 denote the constant function 1 on B. Then we have by (1)

$$\begin{aligned} T_uT_v1 &= T_u(Qv) = T_u\big(h + \overline{k}(0)\big) \\ &= Q\big(fh + \overline{k}(0)f + h\overline{g} + \overline{g}\overline{k}(0)\big) \\ &= fh + \overline{k}(0)f + Q(h\overline{g}) + \overline{g}(0)\overline{k}(0). \end{aligned}$$

Note that $\int_B F\,dV = F(0)$ for holomorphic functions $F \in L^1$. It follows that

$$\begin{aligned} \int_B (T_uT_v1)\,dV &= (T_uT_v1)(0) \\ &= f(0)h(0) + f(0)\overline{k}(0) + \overline{g}(0)\overline{k}(0) + Q(h\overline{g})(0) \\ &= f(0)h(0) + f(0)\overline{k}(0) + \overline{g}(0)\overline{k}(0) \\ &\quad + \lambda_n \int_B h(w)\overline{g}(w)\left(1 - |w|^2\right)^{n+1} dV(w). \end{aligned} \tag{12}$$

Similarly,

$$\int_B (T_vT_u1)\,dV = f(0)h(0) + h(0)\overline{g}(0) + \overline{g}(0)\overline{k}(0) + \lambda_n \int_B f(w)\overline{k}(w)\left(1 - |w|^2\right)^{n+1} dV(w). \tag{13}$$

Since $T_u T_v = T_v T_u$ by assumption, letting $\delta = f\bar{k} - h\bar{g}$, we have by (12) and (13) that

$$\lambda_n \int_B \delta(w)\left(1 - |w|^2\right)^{n+1} dV(w) = \delta(0). \tag{14}$$

We also have (by a remark mentioned at the beginning of this section) that

$$\mathcal{A}(\delta) \in C(\overline{B}). \tag{15}$$

Let $\varphi \in \mathcal{M}$. Multiplying both sides of the equation $T_u T_v = T_v T_u$ by U_φ on the left and by $U_{\varphi^{-1}}$ on the right, we obtain since $U_{\varphi^{-1}} U_\varphi$ is the identity operator that

$$U_\varphi T_u U_{\varphi^{-1}} U_\varphi T_v U_{\varphi^{-1}} = U_\varphi T_v U_{\varphi^{-1}} U_\varphi T_u U_{\varphi^{-1}}$$

and therefore by Lemma 6

$$T_{u\circ\varphi} T_{v\circ\varphi} = T_{v\circ\varphi} T_{u\circ\varphi}. \tag{16}$$

Equations (14) and (15) were derived under the assumption that $T_u T_v = T_v T_u$. Thus (16) says that (14) and (15) remain valid with $\delta \circ \varphi$ in place of δ. That is,

$$\lambda_n \int_B (\delta \circ \varphi)(w)\left(1 - |w|^2\right)^{n+1} dV(w) = (\delta \circ \varphi)(0)$$

and $\mathcal{A}(\delta \circ \varphi) \in C(\overline{B})$ for any $\varphi \in \mathcal{M}$. It follows from Proposition 3 with $\alpha = n+1$ that $\delta = f\bar{k} - h\bar{g}$ is $\mathcal{M}$-harmonic on B. Now Lemma 4 gives the desired characterisation. This completes the proof. ∎

We conclude this paper with a simple application. We note that pluriharmonic functions are closed under complex conjugation.

Corollary 7. *Let u be a bounded pluriharmonic symbol on B. Then $T_u T_{\bar{u}} = T_{\bar{u}} T_u$ on A^1 if and only if the image of B under u lies on some line in $\mathbb{C}$.*

Proof: If $u(B)$ lies on some line in $\mathbb{C}$, a rotation and a translation show that there exist constants c ($|c| = 1$) and d such that $cu + d$ is real valued on B. Since $T_u = (T_{cu+d} - d)/c$ and $T_{\bar{u}} = \left(T_{cu+d} - \bar{d}\right)/\bar{c}$, one can show that T_u and $T_{\bar{u}}$ commute. Conversely assume $T_u T_{\bar{u}} = T_{\bar{u}} T_u$ on A^1 and then Theorem 1 implies that u and $\bar{u}$ are holomorphic on B or a nontrivial linear combination of u and $\bar{u}$ is constant on B. The first case implies u is constant on B, so we are done. Also, a simple manipulation shows that the latter case implies $u(B)$ lies on some line in $\mathbb{C}$. This completes the proof. ∎

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