# 7. COMMISSION DE LA MECANIQUE CELESTE

PRÉSIDENT: Professor Y. Hagihara, Department of Astronomy, Faculty of Science, University of Tokyo, Bunkyo-ku, Tokyo, Japan.

VICE-PRÉSIDENT: Dr W. J. Eckert, Watson Scientific Computing Bureau, 612 West 115th Street, New York 25, N.Y., U.S.A.

COMITÉ D'ORGANISATION: G. A. Chebotarev, J. Kovalevsky, G. Lemaître†, P. J. Message, K. Stumpff, F. Zagar.

MEMBRES: Aksenov, Barlier, Belorizky, Böhme, Brouwer†, Brumberg, Cesco, Clemence, Contopoulos, Cook (A. H.), Cox (J. F.), Cunningham, Danby, Duboshin, Duncombe, Fabre, Ferraz Mello, Garfinkel, Hamid, Heinrich†, Herget, Herrick, Hertz, Hori, Izsak†, Jeffreys (H.), Katsis, Kozai, Kustaanheimo, Martin†, Merman (G. A.), Michkovitch, Musen, Popovič, Rabe, Rjabov, Roy, Sadler (D. H.), Schubart, Sconzo, Sehnal, Siegel, Slavenas, Subbotin, Szebehely, Thiry, Wilkens, Wilkins.

MEMBRES CONSULTANTS: Arnold, Diliberto, Kaula, King-Hele, Moser, Vinti.

## INTRODUCTION

The number of works on celestial mechanics has been growing rapidly in these three years owing to the development of powerful computers and the modern progress in mathematics, being stimulated by the launching of space vehicles.

The revision of Brown's lunar theory by Eckert and his colleagues is unprecedentedly important because of the increase of the accuracy of observations in recent years, notably in radio technique, not only for the theory itself but also for unveiling the past history of the Moon.

The planetary theory on the motion of major planets is being continued at the Naval Observatory by Duncombe and his colleagues by its tradition. It is noted that Musen's modification of Hansen's method of general perturbation allows to compute the perturbation by iteration to any desired high orders. Several papers were published on the motion of the satellites of Mars. Maxwell's theory of Saturn's rings has been reviewed. The discovery of Mercury's rotation period by means of radio-echo observations invoked a new dynamical problem of resonance. It is a matter of joy that the theory of Jupiter's Galilean satellites is being attacked for completing the unfinished work of de Sitter.

Since the radio-echo observations of artificial Earth satellites are now available, the determination of the geopotential with higher tesseral harmonics has become possible by means of satellite observations, both photographic and radio-echo. Izsak, Kozai, King-Hele and Cook, Kaula, and Guier and Newton determined the tesseral harmonic coefficients by discussing satellite observations, as well as the corrections to the station coordinates. The theory of motion of satellites under the influence of such complicated geopotential is very interesting. At the same time the clarification of the nature of orbits with high eccentricities and inclinations is requested. The study of motion in interplanetary probes is now opening a new era in the history of celestial mechanics. New problems of optimization have been raised for a space vehicle with the minimum time of transit and with the minimum fuel or thrust for transfer or rendez-vous between two types of space trajectories.

By the use of electronic computers the complete survey of periodic and non-periodic orbits in the restricted three-body problem is in its culmination, not only for clearing up Brown's conjecture on the horse-shoe orbits by Rabe, Deprit and Goodrich, and for testing the hypothesis of capture or escape of an asteroid from Jupiter, but also for the whole domains



https://doi.org/10.1017/S0251107X00030388 Published online by Cambridge University Press

of the plane of motion by Bartlett and Hénon. Hénon applied Birkhoff's idea of surface transformations for studying the whole evolution of periodic, asymptotic and ergodic orbits and greatly enlarged the scope of study by Strömgren on the restricted three-body problem.

Marvellously enough algebraic calculations, addition, multiplication, differentiation, and integration, in a completely literal fashion can now be programmed and the manipulation of multiple Fourier series was made possible. The operations of Delaunay can be carried out on computers.

Also modern electronic computers facilitate the numerical solution of the $n$-body problem even for $n$ sufficiently large and a new tendency is opened of studying the global characteristics of stellar assemblies by tracing each individual trajectories and of comparing with the theory of statistical stellar dynamics. The nature of the third integral of motion, originally found in some specialized potential fields for galactic study, is now being cleared up by numerical works.

The regularizing transformations of Thiele, Levi-Civita, Sundman and Birkhoff are generalized by Lemaître, Arenstorf and are applied for computing close encounters of space vehicles.

The recent progress in the theories of non-linear differential equations, non-linear integral equations, almost-periodic functions in analysis, as well as in the theories of invariant points for surface transformations, measurability, ergodicity in topology is enabling us to attack some simplified models of the three-body problem. Krylov-Bogoliubov's averaging method in non-linear analysis and Diliberto's periodic surface theory are now being applied to the motion of celestial bodies. Wintner's and Lichtenstein-Hölder's theories are based on non-linear integral equations. Siegel-Moser's theory on the behaviour of motion around equilibrium points are founded on Birkhoff's surface transformations and Poincaré's invariant point theorem. Merman's revision of Chazy's work, Kolmogorov-Arnold's theorems on Hamiltonian systems and the generalization of Liapounov's idea on stability are worth mentioning.

*Publications in book-form appeared during this tri-annual period*

J. Kovalevsky, *Introduction à la Mécanique Céleste*, éd. Armand Colin, Paris, 1963.
D. G. King-Hele, *Theory of Satellite Orbits in an Atmosphere*, Butterworths, London, 1964.
R. Deutsch, *Orbital Dynamics of Space Vehicles*, Prentice-Hall, Engelwood Cliffs, N.J., 1963.
K. Stumpff, *Himmelsmechanik II*, Deutscher Verlag d. Wiss., Berlin, 1964.
P. R. Escobal, *Methods of Orbit Determination*, John Wiley, New York, 1965.
I. I. Muller, *Introduction to Satellite Geodesy*, F. Ungar, 1964.
G. V. Groves, ed., *Dynamics of Rockets and Satellites*, North-Holland Pub. Co., Amsterdam, 1965.
W. M. Kaula, *Theory of Satellite Geodesy*, Blaisdell, 1966.
V. G. Szebehely, *Theory of Orbits in the Restricted Problem of Three Bodies*, Acad. Press, New York, 1967.
V. G. Szebehely, ed., *Celestial Mechanics and Astrodynamics*, AIAA Astrodynamics Specialist Conference held at Yale University in 1963, Acad. Press, New York, 1964.
R. L. Duncombe and V. G. Szebehely, ed., *Methods in Astrodynamics and Celestial Mechanics*, AIAA Astrodynamics Specialist Conference held at Montery, Calif. in 1965, Acad. Press, New York, 1966.
E. Stiefel, ed., *Mathematische Methoden der Himmelsmechanik und Astronautik*, Bericht einer Tagung des Math. Forschungsinstitut Oberwolfach, 1964, Bibliographisches Inst. AG, Mannheim, 1966.
J. Kovalevsky, ed., IUTAM-IAU-COSPAR Symposium on the *Trajectories of Artificial Celestial Bodies as determined by Observations*, held in Paris, 1965, Springer, Berlin, 1966.

Z. Kopal, ed., Proc. *International Conference on Selenodesy*, Reidel, Utrecht, 1966.

W. N. Hess, ed., *Introduction to Space Science*, Gordon and Breach, New York, 1965.

G. Veis, ed., Proc. IAG Symposium on the *Use of Artificial Satellites for Geodesy*, Athens, 1966.

M. Roy, ed., Proc. IUTAM Symposium on *Dynamics of Satellites* at Paris, Springer, Berlin, 1963.

G. Contopoulos, ed., *The Theory of Orbits in the Solar System and in Stellar Systems*, Proc. IAU Symposium No. 25, Thessaloniki, 1964, Acad. Press, 1966.

Y. Hagihara, *Celestial Mechanics*, I. Transformation Theory, MIT Press, 1967; II. Perturbation Theory; III. Form of Integrals and Convergency; IV. Periodic Solutions; V. Topological Study.

*Lectures in Applied Mathematics*, V, VI, VII, *Space Mathematics*, I, II, III, Amer. Math. Soc., 1966, Rhode Is.

The following monographs have been published in Soviet Union, as reported by Chebotarev:

M. B. Bank, *Elements of Dynamics of a Space Flight*, Moscow, 1965.

V. V. Beletski, *Motion of an Artificial Satellite with respect to the Centre of Mass*, Moscow, 1965.

E. A. Grebenikov and V. G. Demin, *Interplanetary Flights*, Moscow, 1965.

G. N. Duboshin, *Celestial Mechanics, Analytical and Qualitative Methods*, Moscow, 1964.

V. A. Egorov, *Non-Plane Problem of reaching the Moon*, Moscow, 1965.

G. A. Chebotarev, *Analytical and Numerical Methods of Celestial Mechanics*, Moscow-Leningrad, 1965.

P. E. Eliasberg, *Introduction to the Theory of the Flight of Artificial Earth Satellites*, Moscow, 1965.

The National Commission on Celestial Mechanics, U.S.S.R., organized a conference on the theory of the motion of artificial Earth satellites in Riga on May 19–23, 1964. About 20 papers were read. The Commission sponsored a conference on the general problems in celestial mechanics held in Kiev on January 25–27, and 16 papers were read and discussed. A Plenum of the Commission on Celestial Mechanics was held in Tbilisi in October, 1965. The program included the scientific topics and the organizational part. The following were elected at the Plenum as new staffs of the Bureau of the Commission on Celestial Mechanics: G. W. Duboshin (chairman), V. A. Brumberg (vice-chairman), E. P. Aksenov (scientific secretary), D. E. Okhozimski, G. A. Chebotarev, I. D. Zhongolovich, E. A. Grebenikov.

## TWO-BODY PROBLEM

New methods of orbit determination have been proposed by Sconzo (1) and Kranjc (2). Sconzo referred to the expressions for $p$ and $q$ of Charlier and applied Lambert's theorem to elliptic motions. Fredrich (3) published a new approximation process for solving Kepler's equation. Dommanget (4) studied the evolution of a binary system by considering the two-body problem of variable mass. Hadjidemetriou (5) obtained analytic solutions of the two-body problem with variable mass and Huang (30) those with mass ejection, and both applied the idea to binary systems.

Goodyear (6) obtained the solution in a completely general closed form for coordinates and their derivatives of the two-body problem. The solutions are valid for circular, elliptic, parabolic, hyperbolic and rectilinear orbits of the attracting force, and for the hyperbolic and rectilinear orbits of the repulsive force. A digital computer programme capable of generating and manipulating symbolic mathematical expressions has been used by Sconzo and his colleagues (7) to derive explicit expressions for the coefficients of the $f$- and $g$- series of Keplerian motions. Terms up to the twenty-seventh order have been obtained as polynomials in the local invariants of Stumpff (8). Recurrent formulae for computing the coefficients of $f$- and $g$- series were given by Bond (26). Herrick (9) reviewed and generalized the universal variables to be used with elliptic, hyperbolic and parabolic orbits, including circular and

https://doi.org/10.1017/S0251107X00030388 Published online by Cambridge University Press

rectilinear orbits, and showed their relationship to Stumpff's formulae. Herget (10) has devised a programme for an IBM 1620 electronic computer for computing preliminary orbits by the variation of geocentric distances on the basis of as many observations as may be available. Stumpff (11) obtained unified expressions for elliptic, hyperbolic and parabolic motions by taking a new variable in place of the time and by developing the rectangular coordinates in Taylor series.

Jarnagin (12) published tables of the expansion of functions for the elliptic motion by revising the tables by Cayley.

Barlier (13) published a method for determining the osculating elements of an artificial satellite by using short geocentric arcs of the orbit of about ten degrees with the assumed value of the semi-major axis. The same problem by using the whole revolution is solved for determining the semi-major axis. He showed that a refined analysis of observations, passage by passage, permits the determination of zonal harmonics with a small number of observations and the detection of short-period perturbations. Muller, Barlier, Chassaing (14) studied theoretically the positions of the instantaneous pole and the values of the velocity on an arc along the apparent orbit of an artificial satellite at transit. Mrs Morael-Courtois (15) has given an automatic process and the computing programme for reconstructing a trajectory from visual, photographic and radio observations, and applied to the first French satellite D-1. She studied the use of interferometric observations and gave an useful procedure for treating numerically the atmospheric drag, while Barlier (16) gave a synthetic discussion on satellite orbits by means of a small number of observations. On the other hand, Zhongolovich (17) published a method for determining the position of an artificial satellite from simultaneous observations of its topocentric directions from known stations on the Earth's surface. Sejnalov (18) gave a schema for circular orbit computation of an artificial satellite by means of an electronic computer and Mamedov (19) that for a nearly parabolic orbit from three observations of a comet.

Popovič published papers on orbit determination method in a Yugoslavian journal.

Sconzo is working on the Fourier series expansion in terms of the mean anomaly of a product of the form $r^m \cdot \dot{r}$ by his symbols, in order to use in a guidance problem for low thrust powered trajectories.

The Smithsonian Astrophysical Observatory, the Royal Aircraft Establishment, the Leningrad Institute of Theoretical Astronomy and the Centre National d'Etude Spatiale have determined the orbits of a number of artificial satellites by means of a new and improved programme (20) and carried out the orbit corrections. Kranjc (21) proposed three new methods for orbit determination. Hertz (22) used short arcs in orbit determination.

Sochilina and Makarova (23) discussed the problem of accuracy of the determination of the elements of an artificial Earth satellite from optical observations. Miss Sochilina (24) obtained analytical formulae for errors of the determined elements, which depend on the distribution of observations along an arc of the orbit. On the basis of these formulae a criterion for the exclusion of some elements from the improvement was derived for an ill-conditioned normal system. She also considered the problem of choosing a time interval in which the systematic errors due to the imperfection of the theory are small.

Batrakov (25) derived a unique system of formulae for the improvement of an orbit by using optical observations as well as range and range velocity data, which are suitable both in the case of small and large eccentricities. Herrick (27) substituted, in place of the difference of two single integrals for a double integral in the equations for the variation of parameters, other combinations of integrals which are more effective with the parameters associated with the universal variables. Böhme (28) published tables for computing the initial values in the special perturbation computation by Kulikov's method (29) which is based on Cowell's.

BIBLIOGRAPHY

1. Sconzo, P. *Astr. Nachr.*, **287**, 149, 1963.
2. Kranjc, A. *Pubbl. Oss. astr. Univ. Bologna*, **8**, no. 18, 1963.
3. Fredrich, J. *Astr. Nachr.*, **288**, 141, 1964.
4. Dommanget, J. *Ann. Obs. R. Belgique*, (3) **9**, no. 5, 1963.
5. Hadjidemetriou, J. *Icarus*, **5**, 34, 1966; *Z. Astrophys.*, **63**, 116, 1966; Proc. *IAU Symposium No.* 25, 129, Thessaloniki, 1964.
6. Goodyear, W. H. *Astr. J.*, **70**, 189, 1965.
7. Sconzo, P., LeShack, A. R., Tobey, R. *Astr. J.*, **70**, 269, 1965.
8. Stumpff, K. *Himmelsmechanik*, Bd. I. Deutscher Verlag d. Wiss., Berlin, 1959; *NASA Techn. Note*, D–1415, 1962.
9. Herrick, S. *Astr. J.*, **70**, 309, 1965.
10. Herget, P. *Astr. J.*, **70**, 1, 1965.
11. Stumpff, K. *Mathematische Methoden der Himmelsmechanik und Astronautik*, 1, 1966.
12. Jarnagin, M. P. *Astr. Pap. Amer. Eph.*, **18**, 1965.
13. Barlier, F. *Bull. astr.*, **25**, 325, 1965.
14. Muller, P., Barlier, F., Chassaing, J. P. *Bull. astr.*, (3) **1**, 37, 1966.
15. Morael-Courtois, N. Centre Nat. Etude Spatiale, Note Techn. 5, 1966.
16. Barlier, F. *Trajectories of Artificial Celestial Bodies as determined by Observations*, 36, 1966.
17. Zhongolovich, I. D. *Bjull. Inst. Teor. Astr.*, **10**, no. 8 (121), 509, 1966.
18. Sejnalov, R. A. *Bjull. Inst. Teor. Astr.*, **10**, no. 8 (121), 537, 1966.
19. Mamedov, M. A. *Bjull. Inst. Teor. Astr.*, **10**, no. 8 (121), 549, 1966.
20. For example:
    Gooding, R. H. *Planet. Space Sci.*, **12**, 37, 1964; **14**, 1173, 1966.
    Gooding, R. H., Taylor, R. J. *R. Aircraft Estab. Techn. Rep.*, 66190, 1966.
    Hiller, H. *R. Aircraft Estab. Techn. Rep.*, 66020, 1966.
    Merson, B. H. *Dynamics of Satellites*, 83, Springer, Berlin, 1963.
    Merson, B. H. *R. Aircraft Estab. Techn. Rep.*, 66255, 1966; Proc. IUTAM–IAU–COSPAR Symposium in Paris, on *Trajectories of Artificial Celestial Bodies as determined by Observations*, 1966.
    Gaposchkin, E., in the *Smithsonian Astrophys. Obs., Special Reports*.
21. Kranjc, A. *Pubbl. Oss. astr. Univ. Bologna*, **8**, no. 18, 1963.
22. Hertz, H. G. *NASA Techn. Note*, D–3172, 1966.
23. Sochilina, A. S., Makarova, E. N. Beobachtungen künstlicher Erdsatelliten, no. 3, Berlin, 1964.
24. Sochilina, A. S. *Bjull. Inst. Teor. Astr.*, 1966 (in press).
25. Batrakov, Yu. V. Proc. XVI. Intern. Astronaut. Congress, 1965.
26. Bond, V. R. *Astr. J.*, **71**, 8, 1966.
27. Herrick, S. Proc. *IAU Symposium No.* 25, 274, Thessaloniki, 1964.
28. Böhme, S. *Astr. Nachr.*, **287**, 241, 1963.
29. Kulikov, D. K. *Bjull. Inst. Teor. Astr.*, **7**, no. 10 (93), 770, 1960.
30. Huang, S.-S. *Astrophys. J.*, **138**, 471, 1963.

## PERTURBATION THEORY

Bohan (1) applied Wilkens' numerical-analytic method (2) of expanding the disturbing function. Martynenko (3) published tables for Newcomb's operators in the expansion of the disturbing function in powers of eccentricities in the elliptic problem of three bodies. Liakh (4) derived the general expressions for Newcomb's operators.

Goodyear (5) has applied the method of variation of parameters for computing the perturbations. The initial coordinates of an osculating two-body trajectory were taken as parameters. Perlin (6) published tables for computing the perturbation of elements with eccentric anomaly as the independent variable.

https://doi.org/10.1017/S0251107X00030388 Published online by Cambridge University Press

Danby (7) used his matrix method to derive formulae for the calculation of planetary theory in rectangular coordinates. Now he (8) developed a theory in a set of polar coordinates which, he says, is simpler and more convenient in practice, after practical evaluation of the constants.

Musen (9) expanded the perturbation of position vectors of planets in purely periodic series in powers of planetary masses. The effect of lower order perturbations to higher order perturbations is expressed as the corresponding terms in the multiple potential in Maxwell's expansion. He reduced the differential equations in the form which can be integrated by mere quadratures. Further Musen (10) has taken Cartan's integral invariants as foundation of the theory of variation of elements and obtained the differential equations for the general perturbation as the first system of Pfaffian equations associated with the linear differential form appearing in the integral invariant. The equations for general perturbations of the Gibbsian unit vector, of the Gibbsian rotation vector and of Euler's parameters are defined. The utilization of the Gibbsian position vectors represents an extension of B. Strömgren's and Musen's ideas on special perturbations to the problems of general perturbations. Euler's parameters find their application in Hansen's lunar theory. Bailie and Fisher (11) obtained an analytical representation of Musen's theory of artificial satellites in terms of the orbital true longitude, including terms with small divisors derived from the third and fourth harmonics of the geopotential. Musen (12) included the long-period terms caused by near-commensurability of mean motions in his theory of the long-range effect based on Halphen's method, at first by removing all short-period terms by numerical process, because no convenient expansion of the disturbing function is available for large values of eccentricity and inclination and the ratio of semi-major axes.

The theory of general perturbations in rectangular coordinates is the most direct of all methods of expansion of the perturbations into series, because it is intimately associated with the computation of ephemerides and has not the singularity of the zero eccentricity. Brouwer's theory of general perturbations in rectangular coordinates makes use of the variation of elements in the canonical form. But this is not of any advantage if the perturbations are expanded in trigonometric series with purely numerical coefficients. Davis (13) rewrote Brouwer's formulae (14) in terms of the standard elliptic elements but the formulae contain two terms of degree $-1$ in the eccentricity. Musen (15) suggests to use Eckert-Brouwer's formulae (16) for the orbit correction as a foundation of the planetary theory. Musen and Carpenter (17) referred to a vectorial expression for perturbation, which is free from those disadvantages and is convenient for numerical computation, and suggested to use the method of iteration to compute the effects of higher orders. Such effects are important not only in the planetary case but also in the case of artificial celestial bodies moving in orbits in cis-lunar space far away from the Earth.

Musen (18) developed a numerical lunar theory which can be used to obtain the rectangular coordinates of a satellite moving in a highly inclined orbital plane. The arguments of the theory are of the Laplacian type, that is, the linear functions of the true orbital longitudes of the satellite and of the Sun. He made use of Hansen's device to perform the integration and introduced a fictitious satellite whose true orbital longitude is considered as a constant until the integration is completed. The perturbations in the orbital plane are obtained by means of a $W$-function analogous to the one of the classical Hansen theory. Musen has shown that the combination of the ideas of Laplace, Hansen and Hill represents a convenient way to obtain a numerical lunar theory, at the same time as this work represents a further development and a simplification of the results given by Musen in his previous article (19). Musen's modification (20) of Hansen's solution of the lunar problem has been programmed and verified by duplicating some of Hansen's series. Charnow (21) developed methods for manipulating trigonometric series with numerical coefficients and literal arguments. The programme described will be used to calculate perturbations and ephemerides of both natural and artificial satellites.

Hori (22) applied a theorem by Lie in canonical transformations to the transformation theory of perturbed dynamical systems and worked out a theory of general perturbations with unspecified canonical variables. All formulae are in the form of canonical invariance, the osculating variables are given explicitly in terms of the mean variables and the theory is applicable to the case in which the unperturbed portion of the Hamiltonian depends on angular variables as well as the momentum variables.

Grebenikov (23) discussed the averaging method in celestial mechanics originally devised by Krylov-Bogoliubov, in particular by considering the case of near-commensurability of mean motions. He has given a mathematical proof for the justification of Delaunay-Hill's method of averaging. They proved the stability of the Lagrangian triangular solution in the elliptic restricted three-body problem for small eccentricities of the disturbing body for an infinite time interval, provided that the triangles formed by the undisturbed three bodies and by the disturbed three bodies differ by an infinitesimal amount and that the masses are subject to certain restrictions. Musen (24) extended the method of Krylov-Bogoliubov to higher orders and compared with Poincaré's method. The determination of the elements affected by the long-period and secular perturbations, as well as the elimination of the short-period effects, is reduced to the solution of a set of partial differential equations step by step.

Morrison (25) applied the generalized method of averaging to a perturbed vector system of differential equations by assuming no resonance. Von Zeipel's procedure is a particular case of the generalized method of averaging corresponding to an appropriate choice of the arbitrary functions arising in the averaged equations.

Gormally (26) comments van der Corput's theory of asymptotic series as a consistent basis for the perturbation method.

Ritz's variational method has been applied by Galerkin (27) for solving non-linear periodic systems of differential equations. Cesari (28) and Urabe (29) worked out the method for obtaining the solution in trigonometric polynominals with the estimation of errors.

Brumberg (30) discussed the theory of planetary motion represented in purely trigonometric form. Two methods by using the rectangular coordinates are suggested for the practical elaboration of such theory with the aid of digital computers. The first method is based on the use of an intermediary orbit according to Hill's method in his lunar theory. The second method consists in determining the formal trigonometric series for the coordinates and substituting in the differential equations of the planetary motion, thus it is reduced to solving an infinite non-linear algebraic system. The disturbing function is expanded by Tisserand-Lebeuf polynomials, which are simpler and more suitable for close lunar satellites than by an ordinary method (31).

Kovalevsky (32) has worked on the problem of the motion of a satellite with large inclination and eccentricity. He showed (33) how one can eliminate the terms of the first and second orders in the ratio of the mean motions in a closed form in inclination and eccentricity by von Zeipel's method. He also discussed the character of the motion of the pericentre whether there is revolution or libration as a function of the eccentricity and inclination. The theory is now being applied to Nereid.

Meffroy (34, 35) eliminated short-period terms of a planetary theory of the first order by means of von Zeipel's method. Newcomb's operators are used in the expansion of the disturbing function with the Delaunay variables. He carried out the computation by keeping the third degree terms in eccentricity and inclination in the disturbing function.

Ferraz Mello (36, 37) analysed von Zeipel's method and showed that it is equivalent to the method of Lindstedt-Poincaré in which only the reduction of the degree of freedom is intended. The Hamiltonian systems are classified according to the possibility of total reduction or a partial reduction. In the first case of lunar type the disturbing function does not contain

https://doi.org/10.1017/S0251107X00030388 Published online by Cambridge University Press

long-period terms of the first order. This is the case of non-canonical equations including the effect of solar radiation pressure on the motion of an Earth satellite. In the second case of planetary type the disturbing function contains long-period terms of the first order with moderate eccentricity and inclination.

Miatchine (38) estimated the error of numerical integration of a differential equation by Störmer's method. He (39) also discussed a criterion for changing the integration steps by Runge-Kutta's method. Subbotin (40) studied the estimate of accuracy in the methods of computing the ephemeris of the inner planets. Foursenko (41) discussed the computation of the Moon's ephemeris on Brown's expansions and estimated the accuracy of the Moon's coordinates.

Gröbner (42) proposed a method for numerical treatment of ordinary differential equations by means of Lie's series. The method has been applied by Knapp (43) to the *n*-body problem. Filippi (44) discussed the accuracy of the method of Runge-Kutta-Fehlberg in the solution of the *n*-body problem.

Seidelmann of Naval Observatory is developing an iterative solution of Hansen's method.

## BIBLIOGRAPHY


1. Bohan, N. A. *Bjull. Inst. Teor. Astr.*, **9**, no. 6 (109), 448, 1964.
2. Wilkens, A. *Astr. Nachr.*, **190**, 289, 1911.
3. Martynenko, B. K. *Bjull. Inst. Teor. Astr.*, **10**, no. 7 (120), 457, 1965.
4. Liakh, R. A. *Publ. Astr. Obs. Univ. Leningrad*, **20**, 1964.
5. Goodyear, W. H. *Astr. J.*, **70**, 524, 1965.
6. Perlin, F. H. *Bjull. Inst. Teor. Astr.*, **9**, no. 9 (112), 607, 1964.
7. Danby, J. M. A. *Astr. J.*, **67**, 287, 1962.
8. Danby, J. M. A. *Astr. J.*, **70**, 155, 1965.
9. Musen, P. *J. Observateurs*, **48**, 11, 1965.
10. Musen, P. *NASA Techn. Note*, D–2301, 1964.
11. Bailie, A. E., Fisher, D. *NASA Techn. Note*, D–1468, 1963.
12. Musen, P. *NASA Techn. Note*, D–2152, 1964.
13. Davis, M. *Astr. J.*, **56**, 188, 1952.
14. Brouwer, D. *Astr. J.*, **51**, 37, 1944.
15. Musen, P. *J. geophys. Res.*, **71**, 1966 = NASA X–641–66–119, 1966.
16. Eckert, W. J., Brouwer, D. *Astr. J.*, **46**, 125, 1937.
17. Musen, P., Carpenter, L. *J. geophys. Res.*, **68**, 2727, 1963.
18. Musen, P. *J. Observateurs*, **49**, 54, 1966.
19. Musen, P. *J. geophys. Res.*, **66**, 403, 1961.
20. Musen, P. *J. geophys. Res.*, **68**, 1439, 1963.
21. Charnow, M. L. *NASA Techn. Note*, D–3378, 1966.
22. Hori, G. *Publ. astr. Soc. Japan*, 18, no. 4, 1966.
23. Grebenikov, E. A. *Astr. Zu.*, **41**, no. 3, 567, 1964; **42**, no. 1, 190, 1965; *Publ. Sternberg Inst.*, **35**, 1966.
24. Musen, P. *J. Astronaut. Sci.*, **12**, 129, 1965.
25. Morrison, J. A. *Methods in Celestial Mechanics and Astrodynamics*, 117, 1966; *SIAM J.*, **13**, 96, 1965. cf., Kyner, W. T. *SIAM J.*, **13**, 136, 1965.
26. Gormally, J. M. *Methods in Celestial Mechanics and Astrodynamics*, 199, 1966. van der Corput, J. G. *J. Analyse Math.*, **4**, 341, 1956; **5**, 315, 1957.
27. Kantorowitsch, L. W., Krylov, W. I. *Näherungsmethoden der höheren Analysis*, Deutsche Verlag d. Wiss., Berlin, 1956.
28. Cesari, L. *Contr. to Diff. Eqs.*, **1**, 149, 1963.
29. Urabe, M. *Archive Rat. Mech. Analysis*, **20**, 120, 1965.
30. Brumberg, V. A. *Trudy Inst. Teor. Astr.*, **9**, 3, 1966.
31. Brumberg, V. A. *Bjull. Inst. Teor. Astr.*, **11**, no. 2 (125), 1966.
32. Kovalevsky, J. *C. R. Acad. Sci.*, Paris, **258**, 4435, 1964.


https://doi.org/10.1017/S0251107X00030388 Published online by Cambridge University Press

33. Kovalevsky, J. Proc. *IAU Symposium No.* 25, 326, Thessaloniki, 1964.
34. Meffroy, J. *C. R. Acad. Sci.*, Paris, **260**, 1571, 1965.
35. Meffroy, J. *C. R. Acad. Sci.*, Paris, **260**, 1863, 1965.
36. Ferraz Mello, S. *Mem. Soc. astr. Italiana*, **37**, no. 2, 1966.
37. Ferraz Mello, S. *C. R. Acad. Sci.*, Paris, **258**, 463, 1964.
38. Miatchine, V. F. *Bjull. Inst. Teor. Astr.*, **9**, no. 10 (113), 668, 1964; **10**, no. 3 (116), 1965.
39. Miatchine, V. F. *Bjull. Inst. Teor. Astr.*, **7**, no. 4 (87), 257; **8**, no. 8 (101), 134, 1961.
40. Subbotin, N. S. *Bjull. Inst. Teor. Astr.*, **10**, no. 2 (115), 143, 1965.
41. Foursenko, M. A. *Bjull. Inst. Teor. Astr.*, **10**, no. 4 (117), 272, 1965.
42. Gröbner, W. *Die Lie-Reihen und ihre Anwendungen*, Berlin, 1960.
43. Knapp, H. *Monatshefte f. Math.*, **68**, 33, 1964.
44. Filippi, S. *Mathematische Methoden der Himmelsmechanik und Astronautik*, 307, Math. Forschungsinstitut, Oberwolfach, 1966.

## PLANETARY AND SATELLITE THEORY

Cohen and Hubbard (**1**) computed the orbits of five outer planets by means of the method of special perturbation over 120 000 years. A remarkable libration of the close approaches of Pluto to Neptune was noticed such that the distance between these planets is never less than 18 A.U. It is concluded that the orbit of Pluto is safe from any very close approach to Neptune and no particular instability resulting from the fact that the perihelion of Pluto is inside the orbit of Neptune. The period of libration is about 19 670 years and the amplitude is about 76°. In addition there is a 2° or 3° modulation with a 4300 year period which corresponds to the period of the great inequality of Neptune and Uranus. A detailed analytical theory of the motion of Pluto has been published by Sharaf and Budnikova (**2**). The improved values of Pluto's elements based on the observations for the period 1914–18 are found to be in good accord with observations. Hori and Giacaglia (**3**) studied the secular perturbation of Pluto. After short-period terms are eliminated the equations of motion for the secular perturbation are of two degrees of freedom. They eliminated the critical argument of period 20 000 years by a method analogous to that of Hori (**3a**) applied to the motion of an artificial satellite with critical inclination. The first order theory agrees with the results of numerical integration by Cohen and Hubbard (**1**). In particular the theory shows the revolution of the argument of perihelion in a period of 15 000 000 years, the change of the eccentricity $0{\cdot}243 < e < 0{\cdot}266$ and that of the inclination $14\overset{\circ}{.}1 < I < 15\overset{\circ}{.}4$.

Choudhry (**4**) studied the motion of asteroids of the Hecuba type by means of von Zeipel's method. Kovalevsky (**5**) investigated the behaviour of the long-period terms in the motion of a satellite as disturbed by the Sun, and found two types of motion: the one type is the class of revolutionary motions with rotating pericentres and the orbit is oscillatory in which the pericentre oscillates around the maximum latitude. Orlov (**6**) applied Delaunay-von Zeipel's method to the differential equations in Hill's lunar theory and obtained an approximate analytic representation of the coordinates. These two authors did not refer to the expansion in powers of the eccentricities and inclinations, similarly to Hori's method (**7**) on the motion of the Moon.

Herget's programme (**8**) for the method of variation of arbitrary vectorial constants for the IBM 1410 includes the perturbations by any or all of the planets from Mercury to Pluto or Venus to Pluto and has been applied with complete success to more than a dozen asteroids of small eccentricity and to about 40 other asteroids in all. He also formed a differential correction programme associated with this programme.

Morando (**9**) developed a numerical method for a general theory of an asteroid disturbed by Jupiter, in which secular terms appear only in the mean motions and the time is not present in the coefficients of periodic terms. The preliminary orbit is not Keplerian, but includes the motions of the node and perihelion. The first order long-period terms are then obtained. Finally the short-period terms as well as higher order long-period terms are computed. The method has been successfully applied to Vesta, for which the maximum amplitude of long-period terms in eccentricity and inclination does not exceed their mean values and for which there is no resonance. The results are comparable with those by Perrotin in 1880 computed by means of Le Verrier's method as far as short-period terms are concerned. The method can be easily generalized to more complete planetary problems.

Wilkens is continuing his work on the commensurabilities in the system of asteroids through an investigation of the spatial motion of the Hecuba type asteroids, after having discussed in former papers (**10**) the plane motion of asteroids near the Hestia and Hecuba gaps. The study is extended to the spatial motion of the Hestia type asteroids. Schubart has generalized his theoretical work on the characteristic asteroids by including the influence of the orbital eccentricity of Jupiter, mainly for the Hilda group by neglecting the effect of short-period terms. He has seen no indication of instability in the motion of asteroids of this group.

Kotsakis (**11**) compared the computed orbit in the restricted three-body problem and the corresponding Kepler orbit and found that the orbits will fill rings round the Sun. In the case of an exploding planet at a certain distance from the centre of mass of the Sun and Jupiter the difference is small, but at certain distances the area filled by the fragments is larger in the restricted three-body problem.

R. B. Hunter, according to a letter from Roy, has carried out a preliminary computation dealing with the application of the three-dimensional elliptical restricted three-body problem to the motions of asteroids and Jovian satellites. From the numerical integration of a large number of orbits an attempt has been made to map out the regions around Jupiter of stable and unstable orbits, also the region of the asteroid distribution highly unstable due to the Jovian perturbation. The Glasgow group, R. B. Hunter, M. W. Ovendon, A. E. Roy, has obtained two results of interest: (*i*) orbits have been found which take asteroids from the asteroid region to become temporary satellites of Jupiter before being sent into a narrow belt of quasi-stable orbits about 7 A.U. from the Sun, (*ii*) the group of real asteroids at the mean motion commensurability 3/2 separates an inner group of fictitious asteroids which become direct satellites of Jupiter from an outer group of fictitious asteroids which become retrograde satellites of Jupiter.

Galibina (**12**) studied the original and future orbits of comets with eccentricities near unity. The result of computation of 26 original and 13 future orbits of long-period comets with the true anomaly as the independent variable shows that 21 of the original orbits have been elliptic and 5 hyperbolic, and that 8 of the future orbits are elliptic and 5 hyperbolic. Kastel (**13**) computed the close approach of Comet Brooks II with Jupiter in 1886 by taking into account the perturbation due to five planets from Venus to Saturn and the oblateness of Jupiter, and found that the capture to a major planet of a short-period comet is a more favourable hypothesis. Brady (**14**) studied the motion of 30 nearly parabolic cometary orbits and showed that 75% of the original orbits were elliptic before the comet entered the region of the planet and picked up energy by increasing their eccentricities and might attain hyperbolic velocity and be ejected. He suggested to study the planetary perturbations after the perihelion passage.

Makover (**15**) points out that there exists a stellar encounter causing considerable changes in cometary orbits owing to the presence of the comet cloud of Oort (**16**) and thus shows that there is no difficulty against the capture hypothesis of comets in the solar system, the ejection hypothesis as the origin of comets being inconsistent. Chebotarev (**17**) studied the motion of an infinitesimal mass in the outer region of the solar system under the action of the galactic centre and saw considerable perturbation caused on the orbital elements. The same problem has been attacked by Chebotarev (**18**) by assuming the disturbing body to be the galactic nucleus. It is shown that the stable motion of a comet with $e_0 = 0{\cdot}6$ is possible at a distance

https://doi.org/10.1017/S0251107X00030388 Published online by Cambridge University Press

of 80 000 A.U. from the Sun and that the boundaries of the cloud of comets are approximately 60 000 to 100 000 A.U.

Ferraz Mello (19) has studied the planar motion of the four Galilean satellites of Jupiter in rectangular coordinates. He has shown that the use of Hill's method leads to difficulties and that higher order harmonics of Jupiter's gravitational potential as well as the relativity corrections are negligible. He thus proposed a new method based on a normalized form of the equations by using a functional relation which leads to integro-differential equations as suggested by the Laplace integral for the motion of the perijoves. The method takes into account from the very first approximation the effects of the resonances between the satellites I and II as well as between II and III. His numerical application gave perturbations of I by II in good agreement with the classical results. The second order theory is now being worked out by Ferraz Mello, while Sagnier is engaged in the generalization of the method for orbits with non-vanishing inclinations.

Lanzano (20) computed the third order theory for the equilibrium configuration of a rotating planet and its effect on the secular variation of the orbital elements.

Marsden (21) studied the motion of the Galilean satellites of Jupiter. Von Zeipel's method was applied for the elimination of the short-period terms and the number of degrees of freedom was reduced from fourteen to twelve. The long-period and critical terms were handled essentially by the *ad hoc* procedure first utilized by Laplace. He is now engaged in the programming, the addition, multiplication, differentiation and integration of Fourier series for a computer and planning to extend to a higher degree of approximation in order to determine new constants by comparing the theory with observations.

Miss A. Bec is working on the determination of the mass of Jupiter by means of the motion of the satellites VI to XII, while Kovalevsky computed the planetary perturbation of Jupiter VIII. An important program of algebraic operations on literal trigonometric series of Delaunay type with four angles and five small parameters in the coefficients is now being carried out in the Bureau des Longitudes by means of an electronic computer for future theoretical works on the lunar and satellite theories.

Wilkins (58) is continuing his analysis of the observations of the satellites of Mars.

Fish (22), Redmond and Fish (23) discussed whether the observed secular acceleration of Phobos is caused by the bodily tidal friction in Mars. They think the hypothesis not inconsistent that Phobos has been captured or formed in an orbit just inside a synchronous orbit $4.5 \times 10^9$ years ago. Kotsakis (24) discussed the dispersion of the fragments of an exploding planet. Alfvén (25) by basing on the Yerkes' photometric observations of asteroids pointed out that the rotational period was not systematically related with the size of asteroids and hence that the current hypothesis on the fragmentation of a planet into asteroids could not be reconciled. He thinks that there must have been some process by which all planets including asteroids obtain this period of rotation when they are formed by condensation of interplanetary material. Alfvén suggested a stepwise condensation process. Jaschek and Jaschek (26) found relations among the absolute magnitudes, frequency and the age of asteroidal families. Anders (27) found the absolute magnitude distribution of asteroids to be similar to the Gaussian for brighter members and grading into a logarithmic curve for the fainter. If the asteroids belonging to Hirayama families were re-assembled into their parent asteroids, then the Gaussian portion of the frequency curve would be enhanced at the expense of the logarithmic portion. Anders concluded that the present state of the asteroidal belt is not in a highly fragmented one and estimated the life of Hirayama families.

Sconzo (28) integrated the equations of motion of a secondary body in the equatorial plane of a rotational symmetric central body rigorously by using the Weierstrassian elliptic functions. The results obtained are used to evaluate the shift of the apsidal line of the secondary body due to the oblateness of the central body. Applications are made to the two Mars satellites as well as to the fifth satellite of Jupiter. The numerical results are in good agreement with observation. The computed perihelion shift of Mercury due to the Sun's oblateness with the zonal harmonics coefficient $J_2 \sim 0{\cdot}001$ amounts to 34% of the observed value.

Vinogradova and Radzievskii (**29**) tried to explain the discrepancy of the motion of Phobos and Deimos by the solar light pressure due to the non-homogeneity of the figure of Mars. The light pressure accelerates or decelerates the motion according as the orientation of Mars's figure.

An interesting paper was published by Goldreich (**30**) on the secular change of the eccentricity of satellite orbits in the solar system. One cause will be the tidal effect on a planet by its satellite which has been the subject of past frequent discussions, and the other will be the tidal effect on the satellite by the planet. Urey, Elsässer and Rochester (**31**) proposed this latter mechanism for facing the satellite's same phase towards the planet and decreasing the eccentricity of the satellite's orbit. Goldreich has computed the dissipation due to the tidal effect on the satellite and seen the importance of the second effect, except for Phobos, Deimos, the Moon and Jupiter V.

McCord (**53**) studied the Neptunian system. By assuming the tidal friction as the dominant mechanism he has seen that Triton's orbit was nearly parabolic in the past.

Antonakopoulos (**54**) studied the restricted three-body problem in a resisting medium and showed that all three-dimensional orbits developed into planar circular forms, by thinking it to be the cause of the present configuration of the solar system.

In the absence of satellites an oblate spinning planet will precess about the normal to its orbital plane. Goldreich (**32**) considered how a planet might keep satellite orbit in its equatorial plane as it precesses. He discarded the tidal effect for the cause. The major effect produced by the oblateness of a planet on its satellite orbits is a secular motion of their pericentres and nodes. If the satellite's nodal period is much shorter than the planet's precessional period, then the inclination of the satellite orbit relative to the planet's equator will not vary as the planet precesses. A satellite formed in an equatorial orbit or brought into the equatorial plane by the tide will continue to move in the equatorial plane as its planet precesses. Goldreich showed why satellites are formed in equatorial orbits, thus explained the reason why Phobos and Deimos are always on the equatorial plane of Mars.

A. F. Cook and Franklin (**33**) discussed in detail Maxwell's theory on the stability of Saturn's rings. Maxwell's two uniformly dense models with collisions neglected on one hand and incompressibility assumed on the other hand are supplemented by three additional models, based on a Maxwellian velocity distribution of the constituent particles. The first model neglects collisions, the second assumes adiabatic compression, and the third isothermal compression. They criticized Maxwell's assumptions that the tangential force resulting from tangential displacements was more important in determining the stability than the radial force resulting from radial displacements, and that the variation of the angular velocity across the ring did not significantly alter any stability criterion. They showed that the shearing effect of differential rotation must not be so underestimated and that radial oscillations which are unaffected by shear are the ones that govern the ring stability. They made new photometry of Saturn's rings and interpreted the result in the most general way. They preferred their first model for which the radius of frozen transparent droplets on particles is 7 $\mu$.

The stability of infinitely thin self-gravitating galaxies, in analogy to Saturn's ring, has been considered by Toomre (**34**), Hunter (**35**) and Yabushita (**36**). Toomre applied Fourier-Bessel analysis to obtain equilibrium figures for galaxies with infinite extension and replaced the galaxies by a finite number of concentric rings for stability analysis. Hunter referred to a spheroidal coordinate system and employed Legendre polynomials, and obtained the

https://doi.org/10.1017/S0251107X00030388 Published online by Cambridge University Press

frequencies of free oscillations as functions of an infinite matrix. Yabushita considered the stability of a ring with differential rotation and that of disk galaxies by making use of a cylindrical coordinate system and appropriate Bessel functions. Yabushita showed that the frequencies of free oscillations are given by the eigen-values of a certain infinite matrix and solved the eigen-value problem numerically. The upper limit of the mass of the ring which is stable against axisymmetric perturbations was given for several model rings. The numerical values differed greatly from Maxwell's due to the consideration of the effect of both edges of the ring, which was entirely neglected by Maxwell.

Chebotarev and Bozhkova (**37**) have obtained trigonometric formulae giving precessional data for long time intervals. The perturbed values of the eccentricity of the Earth are given up to $3 \times 10^7$ years backward. Volkov (**38**) obtained the equilibrium surface of a liquid mass slightly different from an ellipsoid in the gravitational field of a remote material point (Roche satellite problem) by means of Lichtenstein-Liapounov's method.

Carpenter (**39**) programmed the computation of first order planetary perturbations using Musen's method (loc. cit. PERTURBATION THEORY), and is now working on higher order effects. He obtained accurate integrations of the equations of motions of the five outer planets from 1800 through 2000 using the rapidly convergent Chebyshev series. These series were also used in the method of variation of elements for studies of asteroidal motions. Kolenkiewics applied the method to periodic orbits obtaining a harmonic solution of the very restricted four-body problem.

O'Keefe and Liu (**41**), Colombo (**42**), Colombo and Shapiro (**43**) suggested that Mercury's rotation rate, which had been considered to be 3/2 times faster than its orbital mean motion, required for stabilizing this resonant spin a sufficient deviation from axial symmetry. Pettengill and Dyce (**45**), Peale and Gold (**46**) determined the sidereal rotation period of Mercury to be 59 days in contrast to its revolution period of 88 days by means of radar-echo observation. Liu (**40**) studied theoretically the libration of Mercury. Goldreich and Peale (**44**) showed that tidal friction would bring an axially symmetric planet to an asymptotic rotation rate which is somewhat faster than its orbital mean motion. In an asymptotic spin state tidal torque averaged over an orbital period vanishes, and the maximum torque occurs at the perihelion. The precise value of the final spin is determined by the amplitude and frequency dependence of the planet's dissipation function. The existence of resonant spin states at rotation rates of any half-integer, negative or positive, multiple of the orbital mean motion has been shown by Goldreich and Peale (**44**). They derived $(B - A)/C \approx 10^{-8}$ for stabilizing the resonance against the disruptive influence of the solar tidal torque, and thought that there may exist stable spin states which are both faster and slower than the observed, since it is known that for the Moon there exist stable resonant spin states 0·5, 1·5, 2, 2·5 times the orbital mean motion, and hence that Mercury and the Moon have bypassed some of these stable resonances before attaining the present spin. It is remarked that Iapetus rotates synchronously because of the high orbital eccentricity.

Goldreich (**48**) discussed the spin of a planet or a satellite, which is losing its angular momentum through tidal friction and may approach one of its possible final states, and derived a criterion whether the final state attained is one of synchronous rotation.

Goldreich and Peale (**47**) studied in more detail the spin-orbit coupling and the capture probability by assuming the type of tidal torques with which MacDonald (**49**) re-discussed the tidal effect on the orbital elements of the Moon studied formerly by Darwin (**51**) and Jeffreys (**52**). They noticed two types of resonant spin rate for planets and satellites: the first occurs in eccentric orbits at rotation rate of each half-integer multiple of the orbital mean motion, the simplest being the synchronous rotation; and the second involves the presence of another planet or satellite with a resonant spin, in which case the planet or satellite always aligns the same axis toward the second planet or satellite at each conjunction. They derived



https://doi.org/10.1017/S0251107X00030388 Published online by Cambridge University Press

the averaged equations of motion for the complete revolution period and formulated the stability criterion for both types of resonance. Probabilities of capturing a planet or satellite into one of the commensurable rotation states as it is being despun by tidal friction are calculated. Application to Mercury reveals the very small value of $(B - A)/C \approx 10^{-8}$ to be sufficient to stabilize Mercury's rotation period at two-thirds of its orbital period. The probability that Mercury would be captured at this resonance is calculated for several assumed forms of tidal torques. They say that Venus may be in a resonant spin state of the second type, and that the sidereal rotation period of 243·16 days retrograde, determined by Carpenter (**50**) and others, would be commensurable with its synodic motion, requiring $(B - A)/C \geqslant 10^{-4}$ for stabilizing this rotation, so that the capture probability at this resonance appears to be very small.

According to a letter from Miss Roman of NASA, D. O. Muhleman of the Jet Propulsion Laboratory has constructed a least-square computer programme that allows for the computation of the astronomical unit and corrections to eight orbital elements of the Sun and Venus relative to a provisional theory. JPL radar observations of Venus taken during the period from 1961 to 1964 yield almost 55 000 normal equations resulting in an estimate for the A.U. of 149 598 388 ± 50 km. The most significant orbital element correction is to the longitude of Venus relative to that of the Earth, which amounts to 0″96 ± 0″02 nearly twice the value derived by Duncombe. A second solution included the optical observations of Venus from 1943 to 1949. The resulting A.U. estimate was 149 598 439 ± 50 km. These estimates are based on the value 299 792·5 km s$^{-1}$ for the speed of light. Future progress on this problem awaits an improvement in the provisional theory and the inclusion of modern optical observations.

Hertz wrote me that he was now engaged in an attempt to determine the mass of Vesta from perturbations on the asteroid (197) Arete which has approached Vesta to within a few hundredths of A.U. five times since its discovery in 1879. A very preliminary result for the mass of Vesta is $(1{\cdot}17 \pm 0{\cdot}10) \times 10^{-10}$ solar mass.

According to a letter from P. O. Lindblad, Thomas Giuli at the Royal Institute of Technology in Stockholm is investigating by means of numerical computations the effect of particle capture on the rotation of a planet. The rotational angular momentum acquired by a planet is calculated as it gravitationally attracts particles which are initially on heliocentric orbits in the ecliptic. The calculations have been completed for the case where the particles are initially on circular heliocentric orbits. The case for particles initially on eccentric heliocentric orbits is now under investigation.

Duncombe wrote me about the activity of the Naval Observatory. Duncombe himself has completed nearly the fitting of Clemence's new theory of the motion of Mars to observations extending from 1750 to 1960. Jackson has derived improved elements of Ceres, Pallas, Juno and Vesta from a discussion of over three thousand meridian transit observations extending from 1920 to 1958. Generation of precise ephemerides and comparison with observations since discovery, for six asteroids with nearly commensurable mean motions in the ratio 2:1 with Jupiter, has been continued by Klepczynski, O'Handley, Fiala and Duncombe. It is hoped to derive corrections to the mass of Jupiter from these analysis. Pascu is determining new positions of the satellites of Mars from photographs with the 61-inch astrometric reflector.

Message (**55**) and Schubart (**56**) discussed analytically the motion in the restricted three-body problem in which the mean motions of the infinitesimal mass and the smaller mass around the larger mass of the two finite masses are nearly in the ratio $(p + 1)/p$. There exist periodic solutions of the second sort of Poincaré when the commensurability is very close and of the first sort when it is less close. Message studied numerically the equations for the long-period variations of elements for the Hecuba type asteroids, while Schubart isolated the secular and critical terms by a numerical averaging process. Message has seen that the effect of the large amplitude perturbations near commensurability on a distribution of asteroids, which was originally uniform over mean motions, shows a draining off from the vicinity of exact commensurability of a magnitude large enough to account for the observed gap in the mean motion distribution. On the other hand, Goldreich (**57**) showed that special cases of near-commensurate mean motions are stable under tidal forces.

Lyttleton and Yabushita (**59**) studied the effect of stellar encounters on planetary motions by the method of variation of elements. In order to estimate the cumulative effects for a long series of encounters they assumed a Gaussian distribution of star velocities and derived the standard deviations of the changes of the orbital elements in terms of the elements, the average stellar velocity, the mean stellar density, the age of the solar system and the lower bound of the encounter parameter. Yabushita (**60**) examined the changes in orbital elements of a binary star by a distant stellar encounter and their cumulative effects.

It is added that an interesting paper was published by Urey (**61**) on the chemical evidence relative to the origin of the solar system.

## BIBLIOGRAPHY

**1.** Cohen, C. J., Hubbard, E. C. *Astr. J.*, **70**, 10, 1965.
**2.** Sharaf, S. G., Budnikova, N. A. *Trudy Inst. Teor. Astr.*, **10**, 1964; **11**, 89, 1966.
**3.** Hori, G., Giacaglia, G. E. O. *Astr. J.*, (recent issue).
**3a.** Hori, G. *Astr. J.*, **65**, 291, 1960.
**4.** Choudhry, R. K. *J. Indian Math. Soc.*, new series, **28**, 99, 1964.
**5.** Kovalevsky, J. *C. R. Acad. Sci.*, Paris, **258**, 4435, 1964.
**6.** Orlov, A. A. *Bjull. Inst. Teor. Astr.*, **10**, no. 5 (118), 360, 1965.
**7.** Hori, G. *Astr. J.*, **68**, 125, 1963.
**8.** Herget, P. *Astr. J.*, **67**, 16, 1962.
**9.** Morando, B. *Bull. astr.*, (3) **1**, fasc. 4, 1966.
**10.** Wilkens, A. *Sitzungsber. Bay. Akad. Wiss., Math-Phys. Kl.*, 1963.
**11.** Kotsakis, D. *Z. Astrophys.*, **60**, 273, 1965; *Bull. Soc. Math. Grèce*, **5**, fasc. 1, 1964.
**12.** Galibina, I. V. *Bjull. Inst. Teor. Astr.*, **9**, no. 1 (104), 46, 1963; **9**, no. 7 (110), 465, 1964.
**13.** Kastel, G. R. *Bjull. Inst. Teor. Astr.*, **10**, no. 2 (115), 118, 1965.
**14.** Brady, J. L. *Astr. J.*, **70**, 279, 1965.
**15.** Makover, S. G. *Bjull. Inst. Teor. Astr.*, **9**, no. 8 (111), 525, 1964.
**16.** Oort, J. H. *Bull. Astr. Inst. Netherlands*, **11**, 91 and 259, 1950.
**17.** Chebotarev, G. A. *Astr. Zu.*, **40**, no. 5, 1963; **41**, no. 5, 983, 1964.
**18.** Chebotarev, G. A. *Astr. Zu.*, **43**, 435, 1966.
**19.** Ferraz Mello, S. *Bull. astr.*, (3) **1**, fasc. 4, 1966.
**20.** Lanzano, P. *Icarus*, **1**, 121, 1962.
**21.** Marsden, B. G. The Motion of the Galilean satellites of Jupiter, Dissertation, Yale Univ., 1966.
**22.** Fish, F. F. *Icarus*, **3**, 87, 1964.
**23.** Redmond, J. C., Fish, F. F. *Icarus*, **4**, 442, 1965.
**24.** Kotsakis, D. D. *Bull. Soc. Math. Grèce*, **5**, 3, 1964; *Z. Astrophys.*, **60**, 273, 1965.
**25.** Alfvén, H. *Icarus*, **3**, 52 and 57, 1964.
**26.** Jaschek, C., Jaschek, M. *Astr. J.*, **68**, 108, 1963.
**27.** Anders, E. *Icarus*, **4**, 399, 1965.
**28.** Sconzo, P. IBM Pub. Independent Res. and Develop. Program, Washington System Center, Task No. 0232–G, 1965.
**29.** Vinogradova, V. P., Radzievskii, V. V. *Astr. Zu.*, **42**, no. 2, 424, 1965.
**30.** Goldreich, P. *Mon. Not. R. astr. Soc.*, **126**, 257, 1963.
**31.** Urey, H. C., Elsässer, W. M., Rochester, M. G. *Astrophys. J.*, **129**, 842, 1959.
**32.** Goldreich, P. *Astr. J.*, **70**, 5, 1965.
**33.** Cook, A. F., Franklin, F. A. *Astr. J.*, **69**, 173, 1964; **70**, 704, 1965.
**34.** Toomre, A. *Astrophys. J.*, **138**, 385, 1963; **139**, 1217, 1964.
cf., Milder, D. M. *Astrophys. J.*, **145**, 109, 1966.

35. Hunter, C. *Mon. Not. R. astr. Soc.*, **126**, 299, 1963; **129**, 321, 1965.
36. Yabushita, S. *Mon. Not. R. astr. Soc.*, **133**, 247, 1966.
37. Chebotarev, G. A., Bozhkova, A. I. *Bjull. Inst. Teor. Astr.*, **9**, no. 3 (106), 169, 1963.
38. Volkov, M. S. *Bjull. Inst. Teor. Astr.*, **11**, no. 3 (126), 1967.
39. Carpenter, L. *NASA Techn. Note*, D–3168, 1964.
40. Liu, H. S. *J. geophys. Res.*, **71**, 3099, 1966.
41. O'Keefe, J. A., Liu, L. S. *Science*, **150**, 1717, 1965.
42. Colombo, G. *Nature*, **208**, 575, 1965.
43. Colombo, G., Shapiro, I. I. *Smithson. astrophys. Obs., Spec. Rep.*, no. 1881, 1965; *Astrophys. J.*, **145**, 296, 1966.
44. Goldreich, P., Peale, S. J. *Nature*, **209**, 1078 and 1117, 1966.
45. Pettengill, G. H., Dyce, R. B. *Nature*, **206**, 1240, 1965.
46. Peale, S. J., Gold, T. *Nature*, **206**, 1241, 1965.
47. Goldreich, P., Peale, S. J. *Astr. J.*, **71**, 425, 1966.
48. Goldreich, P. *Astr. J.*, **71**, 1, 1966.
49. MacDonald, G. J. F. *Revs. of Geophys.*, **2**, 467, 1964.
50. Carpenter, R. L. *Astr. J.*, **69**, 2, 1964; **71**, 142, 1966.
51. Darwin, G. H. *Phil. Trans.*, **170**, 447, 1879; **171**, 713, 1880 = Sci. Pap., II, 36 and 208, Cambridge Univ. Press, 1908.
52. Jeffreys, H. *Mon. Not. R. astr. Soc.*, **122**, 339, 1961.
Ruskol, E. L. *Astr. Zu.*, **43**, 829, 1966.
53. McCord, T. B. *Astr. J.*, **71**, 585, 1966.
54. Antonakopoulos, G. Thesis, Univ. of Athens, 1966.
55. Message, P. J. Proc. *IAU Symposium No.* 25, 197, Thessaloniki, 1964.
56. Schubart, J. *ibid.*, 187.
57. Goldreich, P. *ibid.*, 268; *Mon. Not. R. astr. Soc.*, **130**, 159, 1965.
58. Wilkins, G. A. See Report of Commission 20, p. 399.
59. Lyttleton, R. A., Yabushita, S. *Mon. Not. R. astr. Soc.*, **129**, 105, 1965.
60. Yabushita, S. *Mon. Not. R. astr. Soc.*, **133**, 133, 1966.
61. Urey, H. C. *Mon. Not. R. astr. Soc.*, **131**, 199, 1966.

## LUNAR THEORY

A series of important works in the lunar theory is being continued by Eckert and his colleagues (**1**, **2**, **3**, **4**).

Because of the high quality and long acceptance as the standard of comparison Brown's lunar theory will, for years to come, play a key role in the discussion of the observed lunar motion and in the critical examination of new and more precise theoretical developments. In the modification of Brown's basic solution to facilitate the comparison with observation the full precision of the solution was not preserved since this was not at first considered necessary. Some of this loss of precision was regained by the 'Improved Lunar Ephemeris' in 1952. Eckert, Walker, Eckert (**1**) have made the full accuracy of Brown's solution available for the comparison with observation and to increase the precision of the relations between the computed coordinates and the parameters on which they are based. The precision of the solar terms in sine of the parallax is improved by more than one order of magnitude.

Eckert (**2**) discovered a large concentration of mass immediately inside the lunar surface by comparing his new determination of the motion of the Moon's perigee and node with observations. Eckert based his important discovery on the new solution of the lunar theory by Eckert and Smith (**3**) and on the dynamical ellipticity of the Earth determined by artificial satellites. The parameter giving the radial distribution of the lunar mass is known to be $g'$, which is 0·6 for a homogeneous body and 1·00 for a hollow spherical shell. Brown has taken $g' = 0{\cdot}50$, the same value as for the Earth, but the value of the ellipticity of the Earth came out a little larger than the one usually adopted. Spencer Jones determined $g' = 0{\cdot}87$ from the

https://doi.org/10.1017/S0251107X00030388 Published online by Cambridge University Press

adopted value of the ellipticity. From Eckert's values $d\pi = -3''.1$, $d\Omega = -27''.9$, he obtained $g' = 0{\cdot}965$.

Eckhardt (5) solved the rotationary motion of the Moon by numerical integration and studied the forced physical libration.

Klock and Scott (6) determined the coefficients of periodic term 182 in Brown's lunar theory from the 6-inch transit observations during 1952–64. The result is in good accord with the theory of Eckert and Smith (3).

Curott (7) analysed 32 ancient solar eclipse reports to determine the secular decrease in angular velocity of the Earth. The computation is based on Ephemeris Time and recently adopted astronomical constants. Secular trends in the Earth's acceleration are noted although ambiguous. He obtained $\dot{\Omega}/\Omega = -(1{\cdot}13 \pm 0{\cdot}20) \times 10^{-10}$.

Musen's attempt in the lunar theory has been already described in the section of PERTURBATION THEORY, page 16.

Petrovskaya (8) continued her study on the convergence of Hill's series and estimated the magnitude of neglected terms in the power series in powers of $\mathfrak{m}$ when the expansion is truncated at the term of $\mathfrak{m}^6$. Schubart (9) solved Hill's problem and obtained plane and space periodic solutions. He referred to the method of variation of elements in terms of the components of the disturbing force. He also discussed the convergence of the solution. Hochstadt (10) discussed the discriminant $\Delta(\lambda)$ of Hill's equation as regards the stability of the solution. The $\lambda$-axis is separated to an infinite sequence of the stability and instability intervals. Hochstadt obtained the representation of $\Delta(\lambda)$ in terms of $\lambda$, based on the fact that $\Delta(\lambda)$ is an entire function of order 1/2. Blumen (11) studied the eigen-values and Levy and Keller (12) the instability intervals.

It is remarked that Brillouin (13) proposed new methods for solving Hill's equation, one is based on recurrence formulae and the other on the Brillouin-Wentzel-Kramers' method often employed in quantum mechanics. De Vogelaere (14) proposed again a new method in connection with the Störmer problem of the motion of charged particles in a magnetic field.

Szebehely (15) cleared up with negation the question whether zero-velocity curves are orbits, and gave the restriction to the initial condition and the field in order to obtain the affirmative answer, that is, the restriction is that the absolute value of the gradient of the potential should be proportional to the curvature of the equi-potential line.

It is remarkable that Barton (16) of Cambridge has written a scheme of programmes for performing the manipulation of multiple Fourier series in an entirely literal fashion on a computer. The programmes have been used to derive the lunar disturbing function in terms of the elliptic elements and the mean anomalies to the tenth order of small quantities. He has duplicated the first Delaunay operations completely and reproduced a large part of the second operation, and thus reproduced Delaunay's lunar theory by means of these programmes. He was successful in producing Hill's variational curve on the computer. Stumpff (17) discussed again his third order differential equation for the radius vector in Hill's lunar problem.

Polanuer (18) worked out a new theory on the physical libration of the Moon, which is free from resonance terms, on the basis on Lindstedt's method.

Lyttleton (19) supplemented the effect of meteoritic accretion on the lunar orbit, causing the contraction of the orbital size, in addition to the tidal dissipative effect. Lyttleton (20) further discussed the reduction of the lunar distance through meteroritic impart. He estimated that in order to reduce the lunar distance by 1/3 an amount of matter of about 1/6 of the lunar mass would be necessary to interact with the Moon. Smalley (23) concluded the maximum capture distance of the Moon was 1·3 times the present distance. Kopal and Lyttleton (21) argued against the permanent binary hypothesis as the remote history of the Earth-Moon system that the closed nature of the zero-velocity curve did not guarantee the permanent binary hypothesis.

Belorizky (22) compared the acceleration of the Moon in its non-perturbed motion, and the gravity acceleration at the Earth's equator by basing on the new values of the astronomical constants.

Van Flandern is making a combined solution of grazing occultations and selected central occultations by the Moon for the period 1956–66.

## BIBLIOGRAPHY


1. Eckert, W. J., Walker, M. J., Eckert, D. *Astr. J.*, **71**, 314, 1966.
2. Eckert, W. J. *Astr. J.*, **70**, 787, 1965.
3. Eckert, W. J., Smith, H. F. *Astr. Pap. Amer. Ephem.*, **19**, Part 2, 1966; Proc. *IAU Symposium No.* 25, 242, Thessaloniki, 1964.
4. Eckert, W. J. *Trans. IAU*, **12 B**, 113, 1964.
5. Eckhardt, D. H. *Astr. J.*, **70**, 466, 1965.
6. Klock, B. L., Scott, D. K. *Astr. J.*, **70**, 335, 1965.
7. Curott, D. R. *Astr. J.*, **71**, 264, 1966.
8. Petrovskaya, M. S. *Bjull. Inst. Teor. Astr.*, **7**, no. 6 (89), 441, 1959; **9**, no. 4 (107), 257, 1963.
9. Schubart, J. *Math. Annalen*, **149**, 131, 1963.
10. Hochstadt, H. *Math. Zeitschr.*, **82**, 237, 1963; *Archive Rat. Mech. Analysis*, **19**, 353, 1965; *Trans. N. Y. Acad. Sci.*, (2) **26**, 887, 1964; *Comm. Pure Appl. Math.*, **17**, 25, 1964.
11. Blumen, L. E. *Comm. Pure Appl. Math.*, **16**, 261, 1963.
12. Levy, D. M., Keller, J. B. *Comm. Pure Appl. Math.*, **16**, 469, 1963.
13. Brillouin, L. *Quart. Appl. Math.*, **6**, 167, 1948.
14. de Vogelaere, R. *Can. J. Math.*, **2**, 440, 1950.
15. Szebehely, V. G. *Amer. Rocket Soc. J.*, **32**, 1938, 1962.
16. Barton, D. *Astr. J.*, **71**, 438, 1966.
17. Stumpff, K. *Mathematische Methoden der Himmelsmechanik und Astronautik*, 51, 1960; *NASA Tech. Note* D–1540, 1541, 1963.
18. Polanuer, D. *Bjull. Inst. Teor. Astr.*, **9**, no. 8 (111), 550, 1964; **10**, no. 3 (116), 230, 1965.
19. Lyttleton, R. A. *Icarus*, **1**, 137, 1962.
20. Lyttleton, R. A. *Icarus*, **5**, 162, 1966.
21. Kopal, Z., Lyttleton, R. A. *Icarus*, **1**, 455, 1963.
22. Belorizky, D. *J. Observateurs*, **49**, 46, 1966.
23. Smalley, V. G. *Icarus*, **5**, 491, 1966.


## RELATIVITY EFFECT

Aoki (1) derived the difference of the coordinate time and the proper time for the planetary motion on the basis of Schwarzschild's line element of the general relativity theory, and identified the coordinate time as the Ephemeris time. Geisler (2) obtained the orbit of a particle with the coordinates expressed parametrically in terms of the coordinate time, and Geisler and McVittie (3) derived the orbital period in Schwarzschild's field of gravitation, in order to meet the observational test of the relativity theory proposed by Kustaanheimo and Lehti (10), that is, the difference between the sidereal period and the anomalistic period for an orbit of high eccentricity.

Podurets (4) derived Einstein's equations for a spherically symmetrical motion of a continuous medium, and Zeldovich (5) considered the motion of matter under the mutual gravitational force in uniform density and velocity distribution. The Newtonian and Einsteinian solutions of the problem are compared. Atkinson (6) derived two general integrals for Einstein's field equations in static spherically symmetric case, and computed light tracks near a very massive star.

https://doi.org/10.1017/S0251107X00030388 Published online by Cambridge University Press

Dicke (7) proposed the regression of the node of Icarus as sensitive measure of the oblateness of the Sun. Francis (8) derived the observable difference of 50 km between the Newtonian and Einsteinian theories at a close approach of Icarus to the Earth in 1968. Kustaanheimo (9), Kustaanheimo and Lehti (10) pointed out the route dependence of the gravitational red shift.

Florides and Synge (11) computed by successive approximation, in a modified procedure from that of Das, Florides and Synge (12) the stationary gravitational field due to a fluid mass, not necessarily of constant density, rotating steadily and slowly about an axis of symmetry. It was found (12) unexpectedly that the internal structure affects the metric outside the mass.

Geisler and McVittie (13) have found a coordinate transformation which transforms the Schwarzschild metric for the field of a spherical body into a special case of the general axi-symmetric metric originally derived by Levi-Civita (14) and by Weyl (15). The transformed Schwarzschild metric is perturbed in such a way as to obtain this special form and the field equations are solved approximately for the perturbing terms. Krause (16) obtained the relativistic potential of an oblate spheroid by referring to the energy tensor in the interior. Geisler and McVittie (13) transformed so as not to include the energy tensor in the interior. With values of the arbitrary constants appearing in the solution, which is formed by a reduction to the Newtonian theory, the metric obtained is taken to be that of the oblate Earth's field. The equatorial orbits of a free particle are discussed and the advance of perigee is calculated.

Roxburgh (17) raised a hypothesis that the inner layers of the Sun rotate more rapidly than the outer layers which, he says, is not inconsistent with the current cosmogonical hypothesis and tried to explain the perihelion advance of Mercury as gravitational effect. Similarly Winer (18) computed the advance of Mercury's perihelion due to the polytropic internal structure of the Sun.

## BIBLIOGRAPHY


1. Aoki, S. *Astr. J.*, **69**, 221, 1964.
2. Geisler, P. A. *Astr. J.*, **68**, 715, 1963.
3. Geisler, P. A., McVittie, G. C. *Astr. J.*, **70**, 14, 1965.
4. Podurets, M. A. *Astr. Zu.*, **41**, no. 1, 28, 1964.
5. Zeldovich, Y. B. *Astr. Zu.*, **41**, no. 5, 873, 1964.
6. Atkinson, R. d'E. *Astr. J.*, **70**, 513 and 517, 1965.
7. Dicke, R. H. *Astr. J.*, **70**, 395, 1965.
8. Francis, M. P. *Astr. J.*, **70**, 449, 1965.
9. Kustaanheimo, P. Information, Dept. Appl. Math., Univ. Helsinki, No. 1, 1965.
10. Kustaanheimo, P., Lehti, R. *Astr. J.*, **68**, 392, 1963.
11. Florides, P. S., Synge, J. L. *Proc. R. Soc.*, **A 270**, 467, 1962.
12. Das, A., Florides, P. S., Synge, J. L. *Proc. R. Soc.*, **A 263**, 451, 1961.
13. Geisler, P. A., McVittie, G. C. *Mon. Not. R. astr. Soc.*, **131**, 483, 1966.
14. Levi-Civita, T. *Atti Accad. Lincei, Rendiconti*, **26**, 307, 1917; **27**, 3, 183, 220, 240, 283, 343, 1918; **28**, 3, 101, 1919.
15. Weyl, H. *Annalen d. Physik*, **54**, 117, 1917; **59**, 101, 1919.
16. Krause, H. G. L. *NASA Techn. Note*, R-188, 1964.
17. Roxburgh, I. W. *Icarus*, **3**, 92, 1964; **4**, 220, 1965.
18. Winer, I. M. *Mon. Not. R. astr. Soc.*, **132**, 401, 1966.


## ARTIFICIAL SATELLITES

Vinti (1) computed his intermediary orbit (2) under the special geopotential, for which the separation of variables in Hamilton-Jacobi's method is capable, to the third order harmonic terms (3). By the inclusion of the third zonal harmonics the potential corresponds to a potential fitted exactly through the third zonal harmonics and to about two-thirds of the fourth, and

https://doi.org/10.1017/S0251107X00030388 Published online by Cambridge University Press

gains in accuracy due to the absence of small denominators in the eccentricity and inclination. Vinti (4) studied the invariant properties of the spheroidal potential of an oblate planet by means of a metric-preserving transformation of the associated Cartesian system. This presumes separability when the coefficients $J_2$ and $J_3$ are taken into account. The inclusion of $J_3$ depends on translating the origin of the spheroidal coordinates by a distance $\frac{1}{2} J_2^{-1} | J_3 |$ equatorial radii, which amounts to 7 km.

Vinti (5) discussed by von Zeipel's method the effects of a constant force on a Keplerian orbit, similarly to the case of Keplerian motion of an electron around a charged nucleus in a constant electrostatic field. He referred to the parabolic coordinates in analogy to Epstein's theory of Stark effect in quantum-mechanics.

Differential corrections to Vinti's and Izsak's (6) solution were derived by Allen and Knolle (7), and the Izsak intermediary orbit specified by only six parameters was shown to be accurate enough compared with the rotating and precessing ellipse which requires eight parameters. Borchers (8) has, by extending the result of Izsak on the second order solution of Vinti's problem, presented a computational procedure for obtaining the coordinates and velocity of a satellite from the initial values. Borchers also derived the exact expression for the velocity in Vinti's intermediary orbit by using Izsak's orbital elements, for comparing with other methods of perturbation by means of the so-called Newton-Raphson iteration method. One of the drawbacks of the ordinary perturbation method is that they necessitate frequent multiplications of Fourier series which are inevitably accompanied with the question of convergence or divergence of the expansions.

Aksenov, Grebenikov, Demin (9) discussed the two fixed centre problem in connection with Vinti's and Kislik's (10) potentials. Aksenov (11) developed an analytical theory of the motion of an artificial satellite in the Earth's gravitational field, with the disturbing action of the Moon and the Sun taken into account. New formulae are derived for providing with the perturbations of the elements of the intermediary motion, which is described by the equations of the two fixed centre problem. The properties of the disturbed motion are studied. They (12) and Yarov-Yarovoy (13) solved the problem and, as well as Vinti (14), showed that the physical significance of Vinti's potential is the translation of the origin of the spheroidal coordinates by a distance $-\frac{1}{2} J_3/J_2$ times the equatorial radii, which is larger than any change produced by $J_3$. According to Vinti (14) it is related to the long-period terms of perturbation theory and turns out to be equal to the displacement by $J_3$ of the plane of symmetry of those exactly elliptical polar orbits which are possible solutions with the spheroidal model. Recently, Alekseev (15) and Deprit (16) discussed in detail the motion of a particle attracted by two fixed mass centres and classified various kinds of possible real motions in three dimensions. Degtjarov and Ewdokimowa (17) analysed the stability of circular orbits in the two-fixed centre problem and plotted the regions of stability and instability. Marchel (18, 19) studied the effect of perturbations due to other bodies in this problem. If the masses of those centres are different, then the odd harmonics $J_3$ can be accounted for. Perturbations of this solution due to other causes are developed.

Garfinkel (20) published an improved theory of his former method of 1959. He revised the potential for his intermediary orbit so as to incorporate the known secular variations up to the second order furnished by the perturbation theory. The assumed form of the geopotential preserved the gross features of the actual geopotential such as axial and equatorial symmetry, singularity at the origin and the vanishing at infinity like the reciprocal of the distance, at the same time as it satisfies the Stäckel conditions for Hamilton-Jacobi separability by leading to a solution in terms of elliptic integrals. The revised algorithm provides a more accurate final orbits, incorporating a substantial fraction of the Keplerian secular variation of the third order. An internal check of the theory is carried out by comparison of the natural frequencies of the orbit with those of a perturbed ellipse. Garfinkel and McAllister (21) extended the

https://doi.org/10.1017/S0251107X00030388 Published online by Cambridge University Press

known solution of the main problem of the artificial satellite theory to include the effects of all higher order zonal harmonics of the geopotential. By supposing that $J_m$, $m > 2$, are at least of order $J_2^2$ Garfinkel and McAllister used the von Zeipel method to derive general expressions for all the secular and long-period variations of orders $J_m$ and $J_m/J_2$ respectively for any $m$. With the aid of the addition theorem due to Groves (**22**), Garfinkel (**23**) and Izsak (**24**) and of Legendre's integral form for the Legendre polynomials the result is exhibited in terms of the associated Legendre polynomials in $(1 - e^2)^{-1/2}$ and in cos $I$.

Then Garfinkel (**25**) extended the research to tesseral harmonic perturbation of an artificial satellite. For a close satellite the method of von Zeipel is used for calculating the long-period variations of order $n J_{m,\lambda}/\omega$, where $n$ is the mean motion and $\omega$ the angular velocity of the Earth's rotation. With the aid of the addition theorem (**23**) for spherical harmonics the results are expressed in a most compact form. Further Garfinkel (**26**) expanded the spherical harmonics of the type $r^{-m-1}\ Y_{m,\lambda}\ (\Theta, \varphi)$ in Fourier polynomials of argument $\Theta = \alpha v + \mu g + \lambda h$, where $v$ is the true anomaly, $g$ and $h$ are the Delaunay variables, and then of argument $\Theta = \alpha l + \mu g + \lambda h$ with Newcomb's polynomials in $e$ as coefficients. He also formulated the long-period part for an Earth satellite in resonance with the Earth's rotation.

Aksnes (**27**) is attempting to include some of the main secular variations in an intermediary orbit and to solve in finite form with the separation of variables.

Kovalevsky (**28**) studied various applications of separability of the equations of motion in the solution of the artificial satellite problem. A. H. Cook (**29**) discussed the condition under which exact solutions may be obtained to the problem of orbits in free space. He considered the problem of finding potentials, which satisfy Laplace's equation and enable the Hamilton-Jacobi equation to be separated.

Grebenikov (**30**) applied Hill's method of rectangular coordinates to the motion of artificial satellites. Giacaglia (**31**) extended Brouwer's theory (**32**) to higher order terms. Von Zeipel's method is shown to be very powerful still. The secular and long-periodic variations in Delaunay's elements of an artificial satellite are obtained for zonal harmonics of any order. Fisher (**33**) compared the von Zeipel and the modified methods as applied to artificial satellites. Formulae were derived for osculating elements when the modified Hansen theory of Musen (**34**) is expressed in terms of orbital true longitude, instead of eccentric anomaly as in Bailie and Bryant's (**35**) work, to any prescribed order of $J_2$, in which high precision is aimed at.

Orlov (**36**) computed the secular and long-period perturbations which agreed with Brouwer's (**32**) and Kozai's (**37**), and developed an analytical theory of the influence of the Sun's attraction on the motion of satellites. On the basis of the circular and elliptical restricted three-body problem the formulae are derived for defining the perturbations up to the third order. Kholshevnikov (**38**, **39**), by studying the secular perturbations in the disturbed motion of celestial bodies, discussed the stability of the orbital motion in the field of an oblate planet. Allan and Cook (**40**), Cook and Scott (**41**) and Gooding (**42**) discussed the long-periodic motion of the plane of a distant circular orbit by taking the oblateness and the luni-solar gravitational force into account, where the two perturbing effects become comparable, in analogy to the motion of Iapetus. A general solution is obtained for simultaneous precession of any number of fixed axes around which the precessions occur. Cook (**43**) has shown that for nearly circular orbits with $e < 0{\cdot}005$ the rate of rotation of the major axes can be markedly non-linear or even oscillatory while the variation in eccentricity is no longer sinusoidal, in contrast to the case of moderately eccentric orbits. Especially Cook discussed the effect of general odd harmonics on the motion in orbits of small eccentricity.

Kaula (**44**) discussed the gravitational and other perturbations, including radiation pressure but excluding drags either mechanical or electromagnetic, of a satellite orbit and formulated analytically the partial derivatives of the orbital rectangular components of position and velocity.

The usual expressions, which are suitable for not too small values of eccentricity, for perturbation of the Keplerian elements are extended by Batrakov (45) for the computation of coordinates without loss of accuracy to the case of small eccentricity.

Miss Shute (52), for studying Moon-to-Earth trajectories, employed a reduced form, obtained by considering the selenocentric velocity asymptotes, of the patch conic method for determining the initial elements of a particle launched or ejected from the Moon's surface with any arbitrary starting conditions, and derived explicit analytic functions for geocentric energy, the Jacobi constant, the angular momentum, standard orbital elements and the conditions for Moon-to-Earth trajectories. Percents of randomly ejected material which initially strike the Earth are seen to be in retrograde motion or to go into heliocentric orbits.

Miss Shute and Chiville (46) estimated the lunar-solar effect on the orbital life-times of artificial Earth satellites with highly eccentric orbits.

Gaska (47) determined periodic orbits of artificial satellites by considering the harmonics up to the sixth order in the geopotential and obtained the radius of convergence of the periodic series.

Sconzo and Benedetto (48) obtained for the motion of an equatorial satellite a simplification of expressing the radius vector in a form similar to that for a revolving conic, and determined the apsidal motion which agreed not only with the orbits of artificial Earth satellites but also with those of Martian satellites. They say that the apsidal motion of Mercury is partly due to the oblateness of the Sun.

Finally it is remarked that King-Hele and Quinn (49) published tables of the launched artificial Earth satellites. Herzberg (50) found unpredicted period, and Hiller (51) discrepancies in the eccentricity and argument of perigee, in the orbital motion of the Alouette I artificial Earth satellite.

Recently Kyner and Bennett (53) modified the classical Encke method for integrating the equations of motion of a near-Earth satellite so that the intermediary, which they call nominal, and the instantaneous orbits are forced to remain close for many revolutions by rotating the nominal ellipse slowly in a plane which is rotating slowly in space.

## BIBLIOGRAPHY

1. Vinti, J. P. *J. Res. nat. Bur. Stand.*, **67 B**, 191, 1963; *Mathematische Methoden der Himmelsmechanik und Astronautik*, 97, 1966.
2. Vinti, J. P. *Trans. IAU*, **12 A**, 22, 1965.
3. Vinti, J. P. *J. Res. nat. Bur. Stand.*, **70 B**, 17, 1966.
4. Vinti, J. P. *J. Res. nat. Bur. Stand.*, **70 B**, 1, 1966.
5. Vinti. J. P. Proc. *IAU Symposium No.* 25, 355, Thessaloniki, 1964.
6. Izsak, I. G. *Smithson. Astrophys. Obs., Spec. Rep.*, no. 52, 1960.
7. Allen, W. A., Knolle, W. E. *Astr. J.*, **69**, 393, 1964; **70**, 152, 1965.
8. Borchers, R. V. *NASA Techn. Note*, D–1539, 1963.
9. Aksenov, E. P., Grebenikov, E. A., Demin, V. G. *Astr. Zu.*, **40**, no. 2, 363, 1963.
10. Kislik, M. D. *Iskusstvennye Sputniki Zemli*, **4**, 3, 1960; **13**, 23, 1961.
11. Aksenov, E. P. *Space Research USSR* (Kosm. Issledovanie), **1**, 1963; *Vestnik Sternberg Inst.*, no. 137, 1965; *Publ. Sternberg Inst.*, **35**, 1966.
12. Aksenov, E. P., Grebenikov, E. A., Demin, V. G. *Iskusstvennye Sputniki Zemli*, **16**, 163 and 173, 1963.
13. Yarov-Yarovoy, M. S. *Bjull. Inst. Teor. Astr.*, **8**, no. 9 (102), 647, 1962.
14. Vinti, J. P. *J. Res. nat. Bur. Stand.*, **70 B**, 1, 1966.
15. Alekseev, V. M. *Bjull. Inst. Teor. Astr.*, **10**, no. 4 (117), 241, 1965.
16. Deprit, A. *Bull. Soc. Math. Belg.*, **14**, 12, 1962.
17. Degtjarov, W. G., Ewdokimowa, L. S. *Bjull. Inst. Teor. Astr.*, **10**, no. 8 (121), 516, 1966.

**18.** Marchel, Ch. Actes du 15ᵉ Congrès Intern. d'Astronautique, **1**, 159, 1964.
**19.** Marchel, Ch. *Bull. astr.*, (3) **1**, fasc. 3, 1966.
**20.** Garfinkel, B. *Astr. J.*, **69**, 223, 1964.
**21.** Garfinkel, B., McAllister, G. T. *Astr. J.*, **69**, 453, 1964.
**22.** Groves, G. V. *Proc. R. Soc.*, **A 254**, 48, 1960.
**23.** Garfinkel, B. *Astr. J.*, **69**, 567, 1964.
**24.** Izsak, J. G. *J. geophys. Res.*, **69**, 2623, 1964.
**25.** Garfinkel, B. *Astr. J.*, **70**, 784, 1965.
**26.** Garfinkel, B. *Astr. J.*, **70**, 699, 1965.
**27.** Aksnes, K. *Astrophys. Norv.*, **10**, no. 4, 1965.
**28.** Kovalevsky, J. *Rendiconti Sem. Mat. e Fisico di Milano*, **35**, 1965.
**29.** Cook, A. H. *Mon. Not. R. astr. Soc.*, **134**, 253, 1966.
**30.** Grebenikov, E. A. *Bjull. Inst. Teor. Astr.*, **7**, 811, 1960.
**31.** Giacaglia, G. E. O. *Astr. J.*, **69**, 303, 1964; Correction, *Astr. J.*, **71**, 228, 1966.
**32.** Brouwer, D. *Astr. J.*, **64**, 378, 1959.
**33.** Fisher, D. *Astr. J.*, **68**, 718, 1963.
**34.** Musen, P. *J. geophys. Res.*, **64**, 2271, 1959; **66**, 403, 1961.
**35.** Bailie, A., Bryant, R. *Astr. J.*, **65**, 451, 1960.
**36.** Orlov, A. A. *Bjull. Inst. Teor. Astr.*, **10**, no. 1 (114), 6, 1965; **10**, no. 5 (118), 360, 1965; Proc. XV. Intern. Astronaut. Congr., **1**, 1965.
**37.** Kozai, Y. *Astr. J.*, **67**, 440, 1962.
**38.** Kholshevnikov, K. V. *Bjull. Inst. Teor. Astr.*, **10**, no. 5 (118), 332, 1965.
**39.** Kholshevnikov, K. V. *Publ. Astr. Obs. Univ. Leningrad*, no. 20, 1964; no. 21, 1964; *Vestnik Univ. Leningrad*, no. 13, 1965.
**40.** Allan, R. R., Cook, G. E. *Proc. R. Soc.*, **A 280**, 97, 1964.
**41.** Cook, G. E., Scott, D. W. *R. Aircraft Estab. Techn. Rep.*, 65292, 1965.
**42.** Gooding, R. H. *R. Aircraft Estab. Techn. Rep.*, 66018, 1966.
**43.** Cook, G. E. *Planet. Space Sci.*, **14**, 433, 1966.
**44.** Kaula, W. M. *Dynamics of Rockets and Satellites*, ed. by Groves, G. V., 178, 1965; *Aerospace Corp. Spec. Proj. Techn. Rep.*, SP–TR–65–4, 1965.
**45.** Batrakov, Yu. V. Symposium on the *Trajectories of Artificial Celestial Bodies as determined by Observations*, Paris, 1965 (in press).
**46.** Shute, B. E., Chiville, J. *Planet. Space Sci.*, **14**, 361, 1966.
**47.** Gaska, S. *Bull. Polish Observ. Artif. Earth Satell.*, Warsaw, no. 10, 26, 1964; *Vestnik Sternberg Inst.*, Moskow, no. 138, 321, 1966.
**48.** Sconzo, P., Benedetto, J. Federal System Div., Intern. Business Machine Corp., Task No. 0232–G, 1965.
**49.** King-Hele, D. G., Quinn, E. *Planet. Space Sci.*, **12**, 681, 1964; **13**, 707, 1965; **14**, 817, 1966.
**50.** Herzberg, L. *Planet. Space Sci.*, **14**, 451, 1966.
**51.** Hiller, H. *R. Aircraft Estab. Techn. Rep.*, 66020, 1966.
**52.** Shute, B. E. *Astr. J.*, **71**, 602, 1966.
**53.** Kyner, W. T., Bennett, M. M. *Astr. J.*, **71**, 579, 1966.
Kyner, W. T. *Astr. J.*, **71**, 583, 1966.

## *Critical inclination*

Chapront (1) has shown that the libratory motion in the orbits of the critical inclination is produced around the value of the argument of latitude of the perigee equal to $\pm 90°$ if the coefficient $J_2$ is only considered, but it occurs around different points if higher harmonics are taken into account, especially the harmonics $J_6$ and $J_8$ play a fundamental role, the character of the libration of the perigee around the node can be reversed, and the odd harmonics shift the libration centre by a large amount. Chapront (2) then used the energy integral and saw how great is the influence either of the variation of the value of the even harmonics or of the introduction of odd harmonics.

BIBLIOGRAPHY

1. Chapront, J. *C. R. Acad. Sci.*, Paris, **260**, 2131, 1965.
2. Chapront, J. *Bull. astr.*, (3) **1**, 23, 1966.

### *Effect of tesseral harmonics*

The asymmetric character of the Earth's equator has been discovered by Kozai, Izsak, Kaula, King-Hele and others (**1**) by the observable effects on the orbits of Earth satellites. In the first approximation the motion is studied by taking the tesseral harmonic expansion of the geopotential. If $J_{22}$ is only considered, then there are two equilibrium points in the orbit of a satellite along the major and the minor axes of the equatorial section of the geopotential level surface, the former being unstable and the latter stable. The motion of a 24-hour satellite is of libration around the stable equilibrium point and is of revolution around the unstable one. Allan (**2**) discussed the libration of period about two years by a differential equation in the form of the one for a spherical pendulum by taking the regression of the orbital plane due to the luni-solar gravitational force into account. Stumpff (**3**) discussed the libration at the equilibrium points of a 24-hour satellite in analogy to the libration around the equilibrium points of the restricted three-body problem.

Cook (**4**) computed the long-period effect of all the coefficients of the tesseral harmonic expansion. Blitzer (**5**, **6**), on the other hand, computed the perturbation of the polar coordinates. He takes the first approximate positions of the equilibrium points as those corresponding to $J_{22}$. The latitudinal deviation of the first approximate equilibrium points occurs from odd zonal harmonics and tesseral harmonics with $n - m$ odd in $J_{nm}$, and the longitudinal deviation from tesseral harmonics with $n - m$ even. There are $2m$ equilibrium longitudes corresponding to $J_{nm}$ in the form

$$\lambda_0 = \lambda_{22} + \frac{s\pi}{2} \qquad (s = 0,\ 1,\ 2,\ 3)$$

for $m = n = 2$, for example. The equations of motion for the deviation from the equilibrium positions are solved.

Recently Doppler shifts in radio observation of satellites revealed the coefficients of harmonic terms of the geopotential. The experiment and method of observation have been described by Newton (**7**) and Guier (**8**). Guier (**9**) determined the non-zonal tesseral harmonics from Doppler data. Guier and Newton (**10**) gave the coefficients of the tesseral harmonic terms. They noticed the resonance effects of satellite orbits with the geopotential. Anderle (**11**) analysed the Doppler data and gave values for the gravity coefficients for $(n, m) = (15, 13)$, $(13, 13)$ and $(15, 14)$. Yionoulis (**12**) studied more in detail and gave the equations which enable to determine which harmonics will contain near-resonant contributions for a given satellite orbit. The coefficients associated with the harmonics of degree and order $(n, m) = (13, 13)$, $(14, 14)$ and $(15, 14)$ are found. The resonance effects provide with a means of obtaining additional harmonic coefficients whose contributions might otherwise have been too small to be detected.

In order to explain the resonance effects of satellite orbits with the geopotential Blitzer (**13**) discussed the motion of a satellite under the influence of the longitude-dependent terms of the geopotential in a frame of reference rotating with the mean rotation of the satellite by using the differential equations of Kaula (**14**) and the coordinate transformation formulae of B. Jeffreys (**15**). The orbits are assumed to be nearly circular but unrestricted with respect to inclination. In general, the effect of the tesseral harmonics is to induce short-period perturbations of small amplitude. However, when the satellite's mean motion is commensurable with the Earth's rotation, Blitzer has seen two distinct types of resonance to set up. Dynamical resonances occur when the impressed frequency due to some $J_{nm}$ term is equal to the natural

https://doi.org/10.1017/S0251107X00030388 Published online by Cambridge University Press

frequency of the satellite's orbital motion, and the amplitude of the induced oscillation builds up with time, the eccentricity increases and the orbit is unstable in the sense that it is driven out of the resonance state. For the other commensurabilities librational resonances occur when the satellite is in a localized potential well in the moving frame and the motion is analogous to a pendulum motion, and is a long-period libration in longitude of the ground track of the satellite. For nearly equatorial motion only the 24-hour orbit is librational and orbits in dynamical resonance are confined to the range from 12 to 48 hours, clustering around but not including the 24-hour orbit.

Bennett (**16**), Steg and De Vries (**17**) studied the effect of the Sun on the Earth-Moon libration point.

Garfinkel (**18**) treated the theory of resonance in dynamics in a formal solution in the problem of small divisors. There the Bohlin-von Zeipel technique was used to solve the ideal resonance problem to $O(k^{1/2})$, where $k$ is the small parameter, and to express the results in terms of the elliptic functions sn, cn, dn, and the Jacobi Zeta function. The general results have been applied to the problem of a 24 hour satellite and the critical inclination. The results confirm the work of Kevorkian *et al.* (**19**) who need the methods of non-linear mechanics to match the resonance solution in the vicinity of the critical inclination with the classical solution with a singularity.

BIBLIOGRAPHY


1. *Trans. IAU*, **12 A**, 23, 1965.
2. Allan, R. R. *Planet. Space Sci.*, **11**, 1325, 1963; **12**, 283, 1964; *R. Aircraft Estab. Techn. Rep.*, 64078, 1965; *Proc. R. Soc.*, **A 288**, 60, 1965; Proc. 16th IAF Conference, Athens, 1965, **6**, 119, 1966; *Planet. Space Sci.*, 1966 (to appear).
3. Stumpff, K. *Mathematische Methoden der Himmelsmechanik und Astronautik*, 113, 1966.
4. Cook, G. E. *Planet. Space Sci.*, **11**, 797, 1963.
5. Blitzer, L. *J. geophys. Res.*, **70**, 3987, 1965.
6. Blitzer, L. *Amer. Rocket Soc. J.*, **32**, 714, 1962.
7. Newton, R. R. *Appl. Phys. Lab. Rep.*, TG–571, 1963; TG–627, 1964; *J. geophys. Res.*, **69**, 3619, 1964.
8. Guier, W. H. *Appl. Phys. Lab. Rep.*, TG–503, 1963.
9. Guier, W. H. *Nature*, **200**, 124, 1963.
10. Guier, W. H., Newton, R. R. *J. geophys. Res.*, **70**, 4613, 1965.
11. Anderle, R. J. *J. geophys. Res.*, **70**, 2453, 1965.
12. Yionoulis, S. M. *J. geophys. Res.*, **70**, 5991, 1965; **71**, 1289, 1966; *Appl. Phys. Lab. Rep.*, TG–633, 1964.
13. Blitzer, L. *J. geophys. Res.*, **71**, 3557, 1966.
14. Kaula, W. M. *Geophys. J.*, **5**, 104, 1961.
15. Jeffreys, B. *Geophys. J.*, **10**, 141, 1965.
16. Bennett, A. G. Boeing Sci. Res. Lab., D2–23307–1964.
17. Steg, L., De Vries, J. P. *Space Sci. Rev.*, **5**, 210, 1966.
18. Garfinkel, B. *Astr. J.*, **71**, 657, 1966.
19. Eckstein, M., Shi, Y., Kevorkian, J. Proc. *IAU Symp. No.* **25**, 291, Thessaloniki, 1964.


### *Light-pressure effect*

The study of the effect of the solar-light pressure on the motion of an Earth satellite has been continued by Poljakchova (**1**) by the method of variation of elements with special reference to the resonance case in which the motion is affected by long-periodic variations with large amplitude. Sehnal (**2**) discussed the Poynting-Robertson effect (**3**). He (**4**) also computed the perturbations of the orbital elements of artificial satellites caused by the radiation pressure reflected from the Earth. Cunningham (**5**) defined and computed the eclipse factor for the solar radiation effect. His eclipse factor is the time interval between the immersion in and

https://doi.org/10.1017/S0251107X00030388 Published online by Cambridge University Press

the emersion from the shadow of the Earth divided by the orbital period. Escobal (6) computed the orbital entrance and emergence from the shadow.

Eremenko (7) described a method of solving the equations for obtaining the point of intersection of an orbit of a satellite with the Earth's shadow. These points of intersection are necessary for the calculation of the perturbation due to the solar radiation pressure. The same problem was discussed by Batrakov (8) for the case of small eccentricities.

BIBLIOGRAPHY

1. Poljakchova, E. N. *Bjull. Inst. Teor. Astr.*, **9**, 15, 1963.
2. Sehnal, L. *Bull. astr. Inst. Csl.*, **14**, 133, 1963.
3. Robertson, H. P. *Mon. Not. R. astr. Soc.*, **97**, 423, 1937.
4. Sehnal, L. Proc. COSPAR-IUTAM-IAU Symposium on the *Trajectories of Artificial Satellites determined from Observations*, Paris, 1965.
5. Cunningham, F. G. *Amer. Rocket Soc. J.*, **32**, 1399, 1962.
6. Escobal, P. R. *Amer. Rocket Soc. J.*, **32**, 1939, 1962.
7. Eremenko, R. P. *Bjull. Inst. Teor. Astr.*, **10**, no. 6 (119), 1965.
8. Batrakov, Yu. V. *Bjull. Inst. Teor. Astr.*, 1966 (in press).

### *Effect of air drag*

King-Hele (1, 2, 3) with his colleagues continued a detailed study of the effect of the Earth's atmosphere on the motion of satellites. Cook, King-Hele and Walker (4) and King-Hele (5) at first supposed that the air density at a given height in a spherically symmetric atmosphere varies sinusoidally with the geocentric angular distance from the maximum density direction which occurs almost overhead of the Sun's altitude. They showed how the perigee distance and the orbital period vary with the eccentricity through the satellite's life and how the eccentricity varies with time, and expressed the life-time and the air density at the perigee in terms of the ratio of change of the orbital period. King-Hele and Quinn (6) estimated the life of a satellite of nearly circular orbit. Then Cook, King-Hele, Walker (7) and Cook and King-Hele (8) considered the atmospheric oblateness. Together with Groves (9), Parkyn (10), Lee (11) and Po (12), King-Hele (13) derived formulae for determining the density distribution in a rotating oblate atmosphere. The evidence for the oblateness has been shown by Nigam (14) and Anderson (14a). Allan and Cook (15) showed the symmetry of a geocentric dust belt and the zodiacal light, and Cook (16) studied the effect of aerodynamic lift on satellite orbits. A more generalized theory, in which the effect of luni-solar perturbations and the Earth's oblateness can be expressed in similar form, has been completed.

Fominov (17) considered the problem of determining the parameters of the atmospheric model proposed by himself. He obtained analytical expressions for the perturbation of the elements of a satellite caused by the atmospheric drag for the revolutionary period of the satellite in the case of large orbital eccentricity and also when the oblateness of the atmosphere depends on the altitude. The formulae obtained are applicable for the case when the atmospheric oblateness is considerable, as well as for the case when it is small.

Cook, King-Hele (18) and King-Hele (3a) discussed the day-to-night variation of the air density. Cook and Plimmer (19), Cook (20, 21) and King-Hele (22) considered the oblate rotating atmosphere, King-Hele (23) treated further the effect of a meridional wind on satellite orbits.

Fominov (24) considered the effect of the solar activity, in addition to the diurnal effect on the oblate Earth atmosphere, by distinguishing the pole of the diurnal effect and the pole of the Earth's rotation. He also discussed the variation of density distribution in the atmosphere by the tidal effect of the Moon and the Sun. The orbital evolution due to the tidal dissipation by solid friction of the Earth's mantle has been considered by Kaula (25).

https://doi.org/10.1017/S0251107X00030388 Published online by Cambridge University Press

Cook (26), Smith (27), Jacchia and Slowey (28) and Jacchia (28a) discussed the drag effect and the life-time of a satellite. Katsis (29) followed Brouwer-Hori's method (30) for computing the drag effect. Sehnal and Mills (31) computed the short-period drag perturbations of the orbits of artificial satellites.

Sconzo, Rossoni, Greenfield, Champion (32) determined the atmospheric density from satellite observations. The result shows that the method is sensitive enough to reveal small oscillations of the atmospheric density which are in phase with the decimetric solar radio flux.

Cook and King-Hele (18) observed that dissipative effect, either tidal or due to Poynting-Robertson drag for a micro-satellite, gives a mechanism for permanent or semi-permanent capture leading to secular changes in orbital inclination. They say that this appears to give a partial explanation of the formation and structure of Saturn's rings.

King-Hele (33) with his collaborators determined the upper atmosphere density and its time variation between the recent sunspot maximum and minimum. Cook and Scott (34) derived the exospheric densities near solar minimum, and King-Hele (3a) the semi-annual variation.

## BIBLIOGRAPHY


1. King-Hele, D. G. *Planet. Space Sci.*, **12**, 4835, 1964.
2. King-Hele, D. G. *Theory of Satellite Orbits in an Atmosphere*, Butterworths, 1964, London.
3. King-Hele, D. G. *R. Aircraft Estab. Techn. Rep.*, 66010, 1966.
3a. King-Hele, D. G. *ibid.*, 66076, 1966; *Planet. Space Sci.*, **14**, 863, 1966.
Anderson, A. D. *Planet. Space Sci.*, **14**, 849, 1966.
4. Cook, G. E., King-Hele, D. G., Walker, D. M. C. *Proc. R. Soc.*, **A 257**, 224, 1960.
5. King-Hele, D. G. *Proc. R. Soc.*, **A 267**, 541, 1962; **275**, 357, 1963.
6. King-Hele, D. G., Quinn, E. *Planet. Space Sci.*, **13**, 707, 1965; **14**, 107, 1966.
7. Cook, G. E., King-Hele, D. G., Walker, D. M. C. *Proc. R. Soc.*, **A 264**, 88, 1961.
8. Cook, G. E., King-Hele, D. G. *Proc. R. Soc.*, **A 275**, 357, 1965.
9. Groves, G. V. *Proc. R. Soc.*, **A 254**, 48, 1959; **252**, 16, 1959; *Planet. Space Sci.*, **5**, 314, 1961; *Mathematische Methoden der Himmelsmechanik und Astronautik*, 147, 1966.
10. Parkyn, D. G. *J. geophys. Res.*, **65**, 1, 1960.
11. Lee, V. A. *Amer. Rocket Soc. J.*, **32**, 102, 1962.
12. Po, Nam Tum *Bjull. Inst. Teor. Astr.*, **10**, no. 5 (118), 317 and 324, 1965.
13. King-Hele, D. G. *Planet. Space Sci.*, **11**, 261, 1963; *Ann. Géophys.*, **22**, 40, 1966.
14. Nigam, R. C. *J. geophys. Res.*, **69**, 1361, 1964.
14a. Anderson, A. D. *Nature*, **209**, 656, 1966.
15. Allan, R. R., Cook, G. E. *J. geophys. Res.*, (in press), 1966.
16. Cook, G. E. *Planet. Space Sci.*, **12**, 1009, 1964.
17. Fominov, A. M. *Bjull. Inst. Teor. Astr.*, **9**, no. 3 (106), 185, 1963; *ibid.*, **9**, no. 7 (110), 499, 1964; *ibid.*, **9**, 1966 (in press).
18. Cook, G. E., King-Hele, D. G. *Phil. Trans. R. Soc.*, **A 259**, 33, 1965.
19. Cook, G. E., Plimmer, R. N. A. *Proc. R. Soc.*, **A 258**, 516, 1960.
20. Cook, G. E. *Proc. R. Soc.*, **A 261**, 246, 1961.
21. Cook, G. E. *Planet. Space Sci.*, **13**, 929, 1965.
22. King-Hele, D. G. *Planet. Space Sci.*, **12**, 835, 1964.
23. King-Hele, D. G. *R. Aircraft Estab. Techn. Rep.*, 66010, 1966; *Proc. R. Soc.*, **A 294**, 261, 1966.
24. Fominov, A. M. *Bjull. Inst. Teor. Astr.*, **9**, no. 3 (106), 185; no. 7 (110), 499, 1963.
25. Kaula, W. M. *J. geophys. Res.*, **68**, 4959 and 4967, 1963.
26. Cook, G. E. *Planet. Space Sci.*, **13**, 929, 1965; **15**, 627, 1967.
27. Smith, D. E. *Planet. Space Sci.*, **13**, 1283, 1965; **14**, 455, 1966.
28. Jacchia, L. G., Slowey, J. *Smithson. astrophys. Obs., Spec. Rep.*, no. 136, 1963; no. 242, 1967.
28a. Jacchia, L. G. *ibid.*, no. 184, 1965.
29. Katsis, D. N. *Epimegetoy Ergasterioy Astronomias Panepistemioy Athenon*, 1965.

30. Brouwer, D., Hori, G. *Astr. J.*, **66**, 193 and 264, 1961.
31. Sehnal, L., Mills, S. B. *Smithson. astrophys. Obs., Spec. Rep.*, 1966.
32. Sconzo, P., Rossoni, J. P., Greenfield, R. J., Champion, K. S. *Progress Astronaut. Aeronaut.*, **14**, 157, 1964.
33. King-Hele, D. G., Rees, J. M. *J. atmosph. terr. Phys.*, **25**, 495, 1963.
King-Hele, D. G., Quinn, E. *J. atmosph. terr. Phys.*, **27**, 197, 1965; *Planet. Space Sci.*, **13**, 693, 1965; **14**, 107, 1966.
34. Cook, G. E., Scott, D. E. *R. Aircraft Estab., Techn. Rep.* (to be issued).

## *Rotation of a satellite*

Beletski (1), Beletski and Zonov (2), and Zonov (3) discussed in detail the rotational motion of an Earth satellite along its orbit and classified various kinds of rotational motion. Hagihara (4) studied the rotational motion of an Earth satellite during its flight along its orbit by taking into account of the geomagnetic field, the field of the solar wind of charged particles and the effect of electrostatic and magnetic induction to the body of the satellite. The effect of Earth's oblateness to the rotational motion was studied by Sarychev (5). Morgan (6) studied the effect of librating motion on the orbit of a dumb-bell-shaped satellite. Volkov (7) discussed the translational-rotational motion of a satellite by Hamilton-Jacobi's method and deduced the series representation of the periodic motion in a gravitational field of a sphere, then constructed a general approximate solution in the vicinity of such a periodic translational and rotational motion, which corresponds to a periodic solution of the first sort of Poincaré.

Morrison (8) discussed the damping due to a roll-vee, gyrostabilized system, wherein the pitch axis of the satellite remains perpendicular to the orbital plane. It is shown that, no matter how large the initial local angular velocity of the satellite is, this velocity reaches any given smaller value in a finite time, and how it is possible to obtain a bound on their damping time and how the satellite settles in one of two possible Earth-pointing positions.

Demin (9) studied the satellite problem by means of qualitative methods. In particular the stability in Liapounov's sense and in Kolmogorov-Arnold's sense has been discussed for the case of translational-rotational motion.

Kondurar (10) is continuing his work on the theory of the translational-rotational motion of artificial satellites in the gravitational field of a planet.

### BIBLIOGRAPHY

1. Beletski, V. V. *Iskusstvennye Sputniki Zemli*, **3**, 13, 1959; **6**, 11, 1961; **13**, 53, 1962.
2. Beletski, V. V., Zonov, V. *ibid.*, **7**, 32, 1961.
3. Zonov, V. *ibid.*, **3**, 118, 1959.
4. Hagihara, Y. *Smithson. Contr. Astrophys.*, **5**, 113, 1961.
5. Sarychev, V. A. *Iskusstvennye Sputniki Zemli*, **6**, 3, 1961.
6. Morgan, J. P. *Amer. Rocket Soc. J.*, **31**, 1089, 1961.
7. Velkov, M. S. *Bjull. Inst. Teor. Astr.*, **8**, no. 5 (98), 343, 1962; **9**, no. 9 (105), 120, 144; no. 4 (108), 74, 283, 1963.
8. Morrison, J. A. *Quart. Appl. Math.*, **22**, 148, 1964.
9. Demin, V. G. *Space Research USSR* (Kosm. Issledovanie), **2**, 1964; *Publ. Lumumba Univ.*, Ser. Mechanics, 1966.
10. Kondurar, V. T. *Bjull. Inst. Teor. Astr.*, **10**, no. 9 (122), 582, 1966; *Publ. Pedagogical Inst. Ivanovo*, 1966.

## *Lunar orbiters and artificial satellites of planets*

Kaula (1) computed the life-time of a lunar orbiter by means of the Runge-Kutta method by taking into account the important part of the disturbing function which is of long-periodic Earth effect with respect to both the lunar satellite and the Earth-Moon orbit. He noticed

https://doi.org/10.1017/S0251107X00030388 Published online by Cambridge University Press

the importance of the $J_3$ term of the Moon's gravity field. Chebotarev and Kirpichnikov (2) studied by numerical integration the motion of an artificial lunar satellite and its stability. Forga is carrying out his work on the application of Hill's method on the motion of a lunar satellite. Roy is continuing his attempt to produce an analytical theory of a close artificial lunar satellite taking into account the disturbing effect due to the second harmonics in the Moon's gravitational field, together with the Earth and the solar effects. The secular part and the main periodic part of the second order theory have been derived. The theory is now being programmed for computation.

A particular attention should be paid to the problem of the motion of the lunar satellites, as was pointed out by Kovalevsky (3), for an orbit with small inclination, due to the fact that there is no dominant second harmonics in the lunar gravitational field as in the Earth's field and that the disturbing force produced by other harmonics is comparable in magnitude with the attraction due to the Earth. For moderate inclinations and eccentricities the problem has been tackled by the technique already used for large inclinations and eccentricities by Kovalevsky (4). By discussing the motion of the peri-selenium he showed that the perturbation can be extremely large so that the life-time of the orbit may become short.

Goudas (5) discussed the moments of inertia, the gravity field and the non-homogeneity of the Moon. Poljakchova (6) studied the effect of solar radiation pressure on the motion of an artificial lunar satellite.

Milder (7) estimated the probability of the lunar impact of a lunar orbiter or the escape of a satellite from its primary on the basis of the ensemble averages of statistical mechanics. The restricted three-body configuration is used as a model in calculating the mean orbit life-times between impacts of a lunar satellite in bounded orbits of various energies.

Szebehely and Pierce (8) studied a group of Earth-to-Moon trajectories with consecutive collisions, that is, those which pass through the centres of both the Moon and the Earth, by means of regularization of the equations of motion for a model restricted three-body problem.

Mrs Stellmacher-Amilhat (9) studied the motion of an asteroid during close approach to a planet. She compared various analytical solutions obtained by considering planeto-centric coordinates within the spheres of different radii. Numerical comparison was made also for the orbit of Mariner II. This method permits to define more accurately the idea of the sphere of influence of a planet. A similar problem was discussed by Stiefel (10) who put emphasis on problems of regularization.

Breakwell and Ralph (11), and Wolaver (12) extended the restricted three-body problem for the Earth-Moon system to include the effect of the Sun by means of von Zeipel's method. Breakwell and Perko (13) matched heliocentric ellipses, corrected for the influence of planetary attraction, in the vicinity of a planet with local hyperbolas. It is found that the attraction due to the destination planet alone causes both a displacement of the arrival asymptote and a time-of-arrival correction which consists of two parts: a gross time bias and a local time bias. The latter depending only on the eccentricity of the arrival hyperbola. The computation of fly-by interplanetary trajectories, such as a trajectory from the Earth to Mars via Venus, would be greatly simplified if the planetary attractions could be neglected and the trajectory be idealized as two heliocentric Keplerian orbits joined as one massless planet point.

Dusek (14) studied the motion in the vicinity of the equilibrium points of a generalized restricted three-body problem by including accelerations caused by low-thrust forces in such a way that the conservative character of the problem is preserved, and discussed the stability of the motion by Liapounov's method.

Chebotarev and Bozkova (15) studied the motion of artificial satellites of Mars, Venus and Mercury in the sphere of activity of the respective planet, and in particular the motion of polar satellites.

Kaula (16) recently determined the tesseral harmonic coefficients of the lunar gravitational field by analysing lunar satellite orbit perturbations. Chebotarev and Kirpichnikov (2) computed numerically the orbit of a lunar orbiter. Yarov-Yarovoi (17) invented an analytical theory of the motion of a space vehicle to the Moon. A method was given for finding the polynomials of the lowest power in an auxiliary variable, which represent with a given accuracy the rectangular coordinates, the velocity components and the time in which the orbits pass close to the Earth and the Moon.

Magnaradze (18) studied the motion of a space vehicle with a variable mass during the flight to Venus in the gravitational field of the Sun, Earth, Venus, and Jupiter. The power series in time have been built up and their convergence has been proved for a sufficiently small time interval. To define the coefficients of the series, recurrent relations are derived which may be used for calculations on modern computers. The errors due to the truncating at the first several terms are estimated.

Giacaglia (19) and Oesterwinter (20) studied the motion of a lunar orbiter characterized by a relatively large value of $J_{22}$, the former by taking the lunar equator as reference plane and the latter the ecliptic.

Miss Shute (21) studied the initial conditions for Moon-to-Earth trajectories.

BIBLIOGRAPHY

1. Kaula, W. M. *Trajectories of Artificial Celestial Bodies as Determined from Observations*, La Lune à l'Ère 1966; Proc. Intern. Conf. on Selenodesy, 1966.
2. Chebotarev, G. A., Kirpichnikov, S. N. *Bjull. Inst. Teor. Astr.*, **8**, no. 6 (99), 402, 1962; **10**, no. 2 (115), 109, 1965.
3. Kovalevsky, J. *La Lune à l'Ère Spatiale*, Presses Universitaires de France, p. 140, 1966.
4. Kovalevsky, J. Proc. *IAU Symposium No.* 25, 326, Thessaloniki, 1964.
5. Goudas, C. L. *Icarus*, **3**, 375, 1964; **4**, 218, 1965.
6. Poljakchova, E. N. *Bjull. Inst. Teor. Astr.*, **9**, no. 6 (109), 440, 1964.
7. Milder, D. M. *Astr. J.*, **68**, 724, 1963.
8. Szebehely, V., Pierce, D. A. *Progress Astronaut. Aeronaut.*, **14**, 35, 1964.
9. Stellmacher-Amilhat, I. *Bull. astr.*, (3) **1**, fasc. 3, 1966.
10. Stiefel, E. *Dynamics of Rockets and Satellites*, North Holland Pub. Co., Chap. 4, 1965.
11. Breakwell, J. V., Ralph, P. *Methods in Celestial Mechanics and Astrodynamics*, p. 55, 1966.
12. Wolaver, L. E. *ibid.*, p. 75.
13. Breakwell, J. V., Perko, L. M. *ibid.*, p. 159.
14. Dusek, H. M. *ibid.*, p. 37.
15. Chebotarev, G. A., Bozkova, A. *Bjull. Inst. Teor. Astr.*, **9**, no. 3 (106), 169; no. 6 (109), 388, 1963.
16. Kaula, W. M. Publ. no. 502, Inst. Geophys. Planet. Phys., Univ. Calif., 1966.
17. Yarov-Yarovoi, M. S. *Publ. Sternberg Inst.*, **36**, 1967; *Differential Equations* (USSR), **1**, no. 7, 1965; **1**, no. 9, 1965.
18. Magnaradze, N. S. *Bjull. astr. Obs. Abastumani*, **30**, 1964.
19. Giacaglia, G. E. O. *NASA Techn. Rep.*, X–547–65–218, 1965.
20. Oesterwinter, C. Dissertation, Yale, 1966.
21. Shute, B. E. *Astr. J.*, **71**, 602, 1966. See (**52**) of section on ARTIFICIAL SATELLITES, page 32.

## *Optimization problem*

The problem of moving in a minimum time between two points in a central gravitational field located at set distances from the centre of attraction has been solved by Lebedev and Rumyantsev (1) by assuming that the space vehicle's motor develops an acceleration which is constant in magnitude. The Lagrange method is used leading to the solution of the boundary value problem for a set of differential equations. Kelley (2) used an adjoint technique for

https://doi.org/10.1017/S0251107X00030388 Published online by Cambridge University Press

obtaining the values of the Green function at the terminal points. Jurovics (3, 4) discussed the optimum steering problem for the entry of a multi-stage vehicle into a circular orbit. Kholshevnikov (5) studied the astro-correction of the encounter of two vehicles at a fixed time by means of one-impulse transfer. Novoselov (6) studied the optimal two-impulse transfer between orbits of small inclinations and eccentricities. Novoselov (7) considered the problem of applying the methods of analytical dynamics to the theory of optimal transfers. The problem of the determination of non-planar optimal transfer to a given circular orbit around a central body with spherical symmetry is studied (8). Kirpichnikov (9) is dealing with some problems of the construction of optimal impulse trajectories in the central field with complicated boundary values. Tarushkin (10) proposed a method of successive optimization for solving non-linear problems.

Hiller (11) discussed the so-called Hohmann transfer between non-coplanar circular orbits and also dealt with the transfer between a circular and a non-coplanar elliptic orbits. Recently Hiller (12) considered the transfer between one or two semi-elliptic paths which needs at most three impulses. He assumed the transfer to occur only at the apsides of the ellipses and the whole plane change to take place at coincident apocentres of these elliptic paths, the assumption which had been justified by Barrar (13). Hiller optimized the total characteristic velocity for impulsive transfer between non-coplanar elliptic orbits having a common centre of attraction and collinear major axes in the same sense. He has seen that for three-impulse transfers the optimum mode is to transfer between pericentres of the initial and final elliptic orbits and for two-impulse transfer the optimum mode is to transfer from the pericentre of the inner ellipse to the apocentre of the outer ellipse. Krasinsky (14) studied the optimum transfer between close coplanar Keplerian orbits. Kirpichnikov (15) discussed the cosmic trajectories of minimum time and gave expenditure of mass. Sconzo (16) solved the one impulse rendez-vous problem by using Lambert's theorem and the transition matrix. The 36 elements of the matrix have been computed explicitly.

Hiller (17) studied the optimum impulsive transfers between non-coplanar elliptic and circular orbits and between non-coplanar elliptic orbits with collinear major axes. King-Hele (18) considered the enlargement of elliptic satellite orbits by continuous micro-thrust.

Mission analysis requires the selection of initial conditions and control parameters giving a trajectory that satisfies a set of objectives subject to numerous constraints. In addition to the non-linearity of the dependence of the objectives and constraints on the inputs, their respective numbers may not agree, and the optimization is necessary. Campbell, Moore and Wolf (19) have given a general formulation of the problem which reduces to finding the minimum length of a vector. The method of solution is with iteration. Van Dine, Fimple and Edelbaum (20) gave a new numerical approach to the solution of the non-linear two-point boundary value problem with application to optimum low-thrust space trajectories, consisting of a finite-difference Newton-Raphson algorithm. Munick (21) discussed the problem of finding the thrust programme so that a specified payload is delivered in vertical flight to a desired altitude with the least fuel expenditure. He considered this Goddard problem for a wide class of drags of physical interest. Ehricke (22) studied interplanetary manoeuvers in manned helio-nautical missions. The manoeuvers considered are perihelion brake, off-perihelion acceleration and retro-manoeuvers, and heliocentric planet approach manoeuvers. Goldstein and Seidman (23) considered fuel optimal controls for a ferry vehicle attempting rendez-vous with an orbital satellite under the additional constraint.

## BIBLIOGRAPHY

1. Lebedev, V. N., Rumyantsev, B. N. *Iskusstvennye Sputniki Zemli*, no. 16, 252, 1963.
2. Kelley, J. H. *Amer. Rocket Soc. J.*, **30**, 947, 1960.
3. Jurovics, S. A. *Amer. Rocket Soc. J.*, **31**, 518, 1961.

https://doi.org/10.1017/S0251107X00030388 Published online by Cambridge University Press

4. Jurovics, S. A., McIntyre, J. E. *Amer. Rocket Soc. J.*, **32**, 1354, 1962.
5. Kholshevnikov, K. W. *Bjull. Inst. Teor. Astr.*, **10**, no. 5 (118), 103, 1965.
6. Novoselov, V. S. *Bjull. Inst. Teor. Astr.*, **9**, no. 5 (108), 295, 1963.
7. Novoselov, V. S. *Vestnik Univ. Leningrad*, **13**, 1964.
8. Novoselov, V. S. *Publ. Astr. Obs. Univ. Leningrad*, **20**, 1965; **22**, 1965.
9. Kirpichnikov, S. N. *Vestnik Univ. Leningrad*, no. 1, 1964; no. 7, 1964.
10. Tarushkin, V. T. *Vestnik Univ. Leningrad*, no. 13, 1965.
11. Hiller, H. *Planet. Space Sci.*, **13**, 147, 1233, 1965.
12. Hiller, H. *Planet. Space Sci.*, **14**, 773, 1966.
13. Barrar, R. B. *Astronaut. Acta*, **9**, 1, 1963.
14. Krasinsky, G. A. *Bjull. Inst. Teor. Astr.*, **9**, no. 6 (109), 425, 1964.
15. Kirpichnikov, S. N. *Bjull. Inst. Teor. Astr.*, **10**, no. 1 (114), 27, 1965.
16. Sconzo, P. *Astronaut. Acta*, **9**, 367, 1963.
17. Hiller, H. *Planet. Space Sci.*, **13**, 1233, 1965; **14**, 773, 1966.
18. King-Hele, D. G. Proc. 14th Int. Astronaut. Congr., **4**, 155, Springer, Berlin, 1965.
19. Campbell, J. H., Moore, W. E., Wolf, H. *Methods in Astrodynamics and Celestial Mechanics*, 355, 1966.
20. Van Dine, C. P., Fimple, W. R., Edelbaum, T. N. *ibid.*, 377.
21. Munick, H. *ibid.*, 401.
22. Ehricke, K. A. *ibid.*, 325.
23. Goldstein, A. A., Seidman, T. I. *Mathematische Methoden der Himmelsmechanik und Astronautik*, 257, 1966.

### *Numerical values of the coefficients of geopotential*

New determinations of the harmonic coefficients of the geopotential were made by Kozai (**1**), Smith (**2**), King-Hele (**3**), King-Hele and Cook (**4**), and their collaborators (**5**, **6**, **7**, **8**). Kozai (**1**) used Baker-Nunn camera observations of nine satellites. For odd harmonics King-Hele, Cook and Scott (**6**) analysed the change in the eccentricities of six satellites with orbital inclinations spaced as uniformly as possible between 28° and 96° and found the most satisfactory representation of the potential to be in terms of four coefficients $10^6 J_3 = -2{\cdot}56$, $10^6 J_5 = -0{\cdot}15$, $10^6 J_7 = -0{\cdot}44$, $10^6 J_9 = 0{\cdot}12$. For even zonal harmonics (**4**, **7**) the authors took the inclinations to be higher than 28°. The numerical values are given in Table 1.

**Table 1**

| | Kozai (**1**) | Smith (**2**) | King-Hele *et al.* (**3**, **4**, **5**, **6**, **7**, **8**) |
|---|---|---|---|
| $10^6 J_2$ | 1082·645 | 1082·64 | 1082·64 |
| $10^6 J_4$ | −1·649 | −1·70 | −1·52 |
| $10^6 J_6$ | 0·646 | 0·73 | 0·57 |
| $10^6 J_8$ | −0·270 | −0·46 | 0·44 |
| $10^6 J_{10}$ | −0·054 | −0·17 | — |
| $10^6 J_{12}$ | −0·357 | −0·22 | — |
| $10^6 J_{14}$ | 0·179 | 0·19 | — |
| $10^6 J_3$ | −2·546 | — | −2·56 |
| $10^6 J_5$ | −0·210 | — | −0·15 |
| $10^6 J_7$ | −0·333 | — | −0·44 |
| $10^6 J_9$ | −0·053 | — | 0·12 |
| $10^6 J_{11}$ | 0·302 | — | — |
| $10^6 J_{13}$ | −0·114 | — | — |

https://doi.org/10.1017/S0251107X00030388 Published online by Cambridge University Press

The coefficients of tesseral harmonic expansion of the geopotential are determined by means of photographic observations by Izsak (9) with the notation

$$C_{nm} = N_{nm}\,\bar{C}_{nm}, \quad S_{nm} = N_{nm}\,\bar{S}_{nm},$$

$$N_{n0} = (2n+1)^{1/2}, \quad N_{nm} = \left[\frac{2(2n+1)\cdot(n-m)!}{(n+m)!}\right]^{1/2}, (m \neq 0),$$

The result is shown in Table 2.

**Table 2**

Izsak (9)

| $10^6 . C$ | $10^6 . S$ |
|---|---|
| $\bar{C}_{22} = 1.17$ | $\bar{S}_{22} = -0.95$ |
| $\bar{C}_{31} = 0.81$ | $\bar{S}_{31} = -0.25$ |
| $\bar{C}_{32} = 0.24$ | $\bar{S}_{32} = -0.25$ |
| $\bar{C}_{33} = -0.50$ | $\bar{S}_{33} = 0.93$ |
| $\bar{C}_{41} = -0.18$ | $\bar{S}_{41} = -0.25$ |
| $\bar{C}_{42} = -0.11$ | $\bar{S}_{42} = 0.23$ |
| $\bar{C}_{43} = 0.28$ | $\bar{S}_{43} = -0.08$ |
| $\bar{C}_{44} = -0.08$ | $\bar{S}_{44} = 0.29$ |
| $\bar{C}_{51} = -0.09$ | $\bar{S}_{51} = 0.19$ |
| $\bar{C}_{52} = 0.31$ | $\bar{S}_{52} = -0.50$ |
| $\bar{C}_{53} = -0.72$ | $\bar{S}_{53} = 0.11$ |
| $\bar{C}_{54} = -0.18$ | $\bar{S}_{54} = 0.51$ |
| $\bar{C}_{55} = 0.18$ | $\bar{S}_{55} = -0.42$ |
| $\bar{C}_{61} = -0.01$ | $\bar{S}_{61} = 0.13$ |
| $\bar{C}_{62} = 0.16$ | $\bar{S}_{62} = -0.37$ |
| $\bar{C}_{63} = 0.14$ | $\bar{S}_{63} = -0.17$ |
| $\bar{C}_{64} = -0.20$ | $\bar{S}_{64} = -0.41$ |
| $\bar{C}_{65} = -0.40$ | $\bar{S}_{65} = -0.28$ |
| $\bar{C}_{66} = -0.53$ | $\bar{S}_{66} = -0.41$ |

(six degree solution)

**Table 3**

Guier and Newton (11)

| $10^6 . A$ | $10^6 . B$ |
|---|---|
| $A_2^2 = 7.53$ | $B_2^2 = -3.79$ |
| $A_3^1 = 6.90$ | $B_3^1 = 0.80$ |
| $A_3^2 = 4.56$ | $B_3^2 = -2.54$ |
| $A_3^3 = 2.47$ | $B_3^3 = 3.66$ |
| $A_4^1 = -2.39$ | $B_4^1 = -1.87$ |
| $A_4^2 = 1.77$ | $B_4^2 = 1.88$ |
| $A_4^3 = 3.59$ | $B_4^3 = 0.03$ |
| $A_4^4 = -0.89$ | $B_4^4 = 0.81$ |
| $A_5^1 = 0.64$ | $B_5^1 = -0.78$ |
| $A_5^2 = 1.26$ | $B_5^2 = -1.58$ |
| $A_5^3 = 0.43$ | $B_5^3 = 0.49$ |
| $A_5^4 = -2.29$ | $B_5^4 = -1.22$ |
| $A_5^5 = -0.16$ | $B_5^5 = -3.14$ |
| $A_6^1 = 0.00$ | $B_6^1 = 0.51$ |
| $A_6^2 = -0.82$ | $B_6^2 = -0.79$ |
| $A_6^3 = 2.70$ | $B_6^3 = 0.26$ |
| $A_6^4 = -1.56$ | $B_6^4 = -2.59$ |
| $A_6^5 = -0.90$ | $B_6^5 = -2.60$ |
| $A_6^6 = 0.07$ | $B_6^6 = -1.18$ |

(eight degree solution)

On the other hand, Guier (10), Guier and Newton (11) deduced from the Doppler tracking of five satellites the coefficients of zonal harmonics of odd orders through the ninth, the non-zonal harmonics of orders from the second through the eighth and the sectorial harmonics to the thirteenth order. By combining these values with King-Hele's (3) they computed a shape of the geoid and studied the distribution of the magnitude of the harmonics. They found no apparent relation with the harmonics of the topography and saw that the harmonic coefficients of the geoid are consistent with random density variations with negligible spatial correlation beyond about 0·01 of the Earth's radius, which begin near the top of the mantle and continue to an undetermined depth. Guier and Newton expanded the geopotential in the form

$$\sum_{n,\,m}\left(\frac{R}{r}\right)^{n+1}\left[\frac{(n-m)!}{(n+m)!}\right]^{1/2}(1-x^2)^{m/2}\,\frac{d^m P_n(\cos\theta)}{d\,x^m}\,[A_n^m \cos m\lambda + B_n^m \sin m\lambda]$$

and gave the coefficients $A_n^m$ and $B_n^m$ as in Table 3. Yionoulis (12) computed the coefficients $(n, m) = (13, 12), (13, 13), (15, 13)$.

Kozai (13) discussed all recent determinations of the harmonic coefficients by means of artificial satellites and the future programmes for improving the results and detecting the effects due to Earth tide and any other time variations of the potential to the satellite motion.

A. H. Cook (14) published an account of a comparison between various determinations of the even zonal harmonics of the geopotential from satellite observations by paying particular attention to the problem of estimation that arises for the fact that there are more parameters to determine than the independent data. He (15) discussed also these questions in connection with the motion of the Moon.

Simultaneous observations of artificial satellites facilitate the determination of the correction to station coordinates at the same time as the geopotential. Zhongolovich (16), Kaula (17), and Izsak (19) discussed the problem. Kaula (18) also discussed the Earth mantle, and the tidal dissipation of solid friction, including thermal effect and even a similar effect on the Moon (20).

Zhongolovich (21) studied the general theory of using simultaneous observations of artificial Earth satellites in geodesy and geophysics. Zhongolovich (16) derived the general equations and then the methods of space triangulation (21). The deviation of the geodetic system of coordinates (16) and the satellite positions in geodetic system (22, 23, 24) are determined. The project of the unique world space triangulation is proposed (21). He also dealt with the methods of discussion and the use of INTEROBS observations for determining the atmospheric fluctuations of short periods (22, 25). Batrakov (26) considered the problem of the possibility of using resonant satellites for the determination of the coefficients of the geopotential. He (26a) also developed a general theory of defining the mutual positions of observational stations by simultaneous or nearly simultaneous optical and radio observations of a satellite at several stations.

Kaula (27) reviewed the tesseral harmonic determination from the dynamics of satellite orbits and compared and combined the satellite determination of the Earth's gravity field with other results for geodetic parameters, such as the gravimetric determination. Kaula (28) published his new determination of the 44 tesseral harmonic coefficients, together with 36 station coordinates and 511 orbital elements, from 7234 Baker-Nunn camera determinations of five satellites. Supplementary observations on the accelerations of 24-hour satellites and directions between tracking stations for simultaneous observations are also included in the reduction.

It is noted that the factor GM of the expression for the geopotential, among other constants, has been determined from the radio tracking of Mariner (Venus) and Ranger III–VII lunar missions to be $3{\cdot}986009 \times 10^{14}\ m^3\,s^{-2}$ by Sjogren and Trask (29).

## BIBLIOGRAPHY


1. Kozai, Y. *Publ. astr. Soc. Japan*, **16**, 263, 1964.
2. Smith, D. E. *Planet. Space Sci.*, **13**, 1151, 1965.
3. King-Hele, D. G. *Planet. Space Sci.*, **12**, 835, 1964; *Nature*, **207**, 575, 1965.
4. King-Hele, D. G., Cook, G. E. *Geophys. J.*, **10**, 17, 1965.
5. King-Hele, D. G., Cook, G. E., Rees, J. M. *Geophys. J.*, **8**, 119, 1963.
6. King-Hele, D. G., Cook, G. E., Scott, D. W. *Planet. Space Sci.*, **13**, 1213, 196
7. King-Hele, D. G., Cook, G. E., Scott, D. W. *Planet. Space Sci.*, **14**, 49, 1966.
8. King-Hele, D. G., Cook, G. E., Watson, H. M. *Nature*, **202**, 996, 1964.
9. Izsak, I. G. *J. geophys. Res.*, **69**, 2621, 1964.
10. Guier, W. H. *Nature*, **200**, 124, 1963.
11. Guier, W. H., Newton, R. R. *J. geophys. Res.*, **70**, 4613, 1965.
12. Yionoulis, S. M. *J. geophys. Res.*, **71**, 1768 and 4064, 1966.
13. Kozai, Y. *Space Sci. Rev.*, **5**, 818, 1966.


https://doi.org/10.1017/S0251107X00030388 Published online by Cambridge University Press

14. Cook, A. H. *Geophys. J.*, **10,** 181, 1965.
15. Cook, A. H. *Bull. astr.*, **25,** 33, 1965.
16. Zhongolovich, I. D. *Astr. Zu.*, **41,** no. 1, 156, 1964.
17. Kaula, W. M. Proc. IUTAM Symposium on *Dynamics of Satellites*, at Paris, ed. by M. Roy, Springer, Berlin, 1963; *J. geophys. Res.*, **68,** 473, 1963; *Geophys. J.*, **5,** 104, 1961; *Advance Geophys.*, **9,** 191, 1962; *Dynamics of Rockets and Satellites*, 179, 1965; *Introduction to Space Science*, ed. by W. N. Hess, 297, 1965; Proc. IAG Symposium, *Use of Artificial Satellites for Geodesy*, Athens, 1966; *J. geophys. Res.*, **71,** 4377, 1966.
18. Kaula, W. M. *J. geophys. Res.*, **68,** 4959 and 5183, 1963.
19. Izsak, Y. G. *J. geophys. Res.*, **68,** 4967, 1963; **69,** 2621, 1964; *Trajectories of Artificial Celestial Bodies as Determined by Observations*, 1966.
20. Kaula, W. M. *The Earth-Moon System*, ed. by B. G. Marsden and A. G. W. Cameron, Plenum, New York, 1966; *Revs. Geophys.*, **2,** 661, 1964.
21. Zhongolovich, I. D. *Studia Geoph. et Geodet.*, **9,** Paris, 1965.
22. Zhongolovich, I. D. *Beobachtungen Künstlicher Erdsatelliten*, no. 3, Berlin, 1965; no. 4, Prague, 1966.
23. Zhongolovich, I. D. *Bull. Stations Optical Observ. AES* (USSR), no. 46, 1966.
24. Zhongolovich, I. D. *Bjull. Inst. Teor. Astr.*, **10,** no. 8 (121), 509, 1966.
25. Zhongolovich, I. D. COSPAR Symp., Paris in 1965, Berlin in 1966.
26. Batrakov, Yu. V. Proc. XIV. Intern. Astronaut. Congr., **4,** Paris, 1964.
26a. Batrakov, Yu. V. *Astr. Zu.*, **42,** no. 1, 195, 1965.
27. Kaula, W. M. Proc. IAG Symposium, *Use of Artificial Satellites for Geodesy*, Athens, 1966; *J. geophys. Res.*, **71,** 5303, 1966; Publ. no. 508, Inst. Geophys. Planet. Phys., Univ. Calif., 1966; Proc. *IAU Symposium No.* 21, *Astronomical Constants, Bull. astr.*, **25,** 21, 1965.
28. Kaula, W. M. Publ. no. 487, Inst. Geophys. Planet. Phys., Univ. Calif., 1966.
29. Sjogren, W. L., Trask, D. W. *J. Spacecraft*, **2,** 689, 1965.

## THREE-BODY PROBLEM: REGULARIZATION

Lemaître's transformation (**1**) for the regularization of the three-body problem has been worked out by Deprit and Delie (**2**), Deprit and Roels (**3**). Deprit and Broucke (**4**) applied Lemaître's regularizing transformation to the restricted problem of three bodies in the form convenient for numerical work on electronic computers. Broucke (**5**) generalized the regularizing transformation in the form

$$Z = \tfrac{1}{4}\ (\zeta^n + \zeta^{-n}), \qquad Z = \tfrac{1}{2}\ \cos n\,\zeta,$$

where $n$ is a finite non-zero real number and $\zeta$, $Z$ are complex variables. The first form reduces to Birkhoff's for $n = 1$ and to Lemaître's or to Arenstorf's (**6**) for $n = 2$. The second form reduces to Thiele's for $n = 2$. Broucke deduced the formulae through conformal mapping in the general case and formulated the canonical equations of motion.

Sconzo (**7**) carried out the formal series expansion up to the eighteenth order terms of the solution of the three-body problem when the Sundman type variable is used. In particular, Levi-Civita's variable $u$ defined by $dt = V\,du$, where $V$ is the total potential, is used throughout the expansion. This expansion will be obtained by means of iterative procedure similar to that in the method of Peano-Picard for constructing the solution of a system of differential equations.

Szebehely and Giacaglia (**8, 9**) presented the equations of motion of the restricted three-body problem by allowing non-zero eccentricity of Jupiter's orbit with the true anomaly of the primaries as independent variable and of dimensionless pulsating coordinates as dependent variables, first introduced by Scheibner in 1866 and later by Nechvile. The short- and long-period effects of the eccentricity are discussed in connection with the generalized Jacobi integral. The similarity between the equations for the circular and for the elliptic cases permits the introduction of regularizing transformations by following the methods applicable for the circular case. The major effect of the eccentricity is that the regularized equations are in the form of integro-differential equations. Szebehely (10) then expanded the terms appearing in the regularized equations according to the powers of a dimensionless parameter associated with the mass-ratio of the primaries.

Szebehely (11) attempted to give the power series solution of the restricted three-body problem by using the Thiele-Burrau regularization transformation. Since real singularities are eliminated in this way, the only remaining singularities are on the imaginary regularized time axis. They may be eliminated by Poincaré's time transformation and the general solution of the restricted problem may be represented in convergent Taylor series.

According to Sundman it is known that, if the total angular momentum of the three bodies is not zero, the coordinates and the time are expressed by analytic functions of the regularizing variable $\tau$ in the strip of the complex plane $\tau$ defined by $|\Re\mathrm{e}\,\tau| < \infty$, $|\Im\mathrm{m}\,\tau| < \Omega$, where $\Omega$ depends on the masses and the initial condition, that the coordinates and the time are expanded in convergent Taylor series about any real value of $\tau$ within a circle of radius larger than or equal to $\Omega$, and that the series can be continued analytically.

Cesco (12, 13) has shown by an example that Borel's integral method of summability of divergent series, conveniently adapted for numerical computation, can be used for this purpose even outside the circle of convergence of the Taylor series solution.

Brumberg (14) has numerically applied Mittag-Leffler's theorem on the expansion of an analytic function defined by a Taylor series in series of polynomials. The infinite strip of Sundman is contained in Mittag-Leffler's star region so that the series of polynomials are convergent for any real value of $\tau$ and accordingly of $t$.

Lóper García (15) improved and extended the classical ephemeris computed by Zumkley (16) of a planar problem with equal masses. Cesco (17) applied this method to find the quasi-asymptotic solution of a hyperbolic-elliptic type of the three-body problem. Cesco (18) derived the differential equations for the Keplerian elements in the $n$-body problem and solved without referring to the method of variation of elements. Cesco says that this straightforward derivation simplifies and completes the results obtained by Chazy (19).

Kustaanheimo (20), Kustaanheimo and Stiefel (21) transformed by conformal mapping the Keplerian motion in the three-dimensional space to that in the four-dimensional manifold in which the equations of motion reduce to linear differential equations with constant coefficients completely regular at the attracting centres. The theory is based on spinor analysis developed by Kustaanheimo himself (22) and Kustaanheimo and Nuoto (23) in real matrix form. They tried to generalize to higher dimensions by applying Hurwitz-Radón theorem for quadratic forms.

Rössler (24) applied this Kustaanheimo-Stiefel transformation for computing the perturbation by combining with the three-dimensionally generalized form of Levi-Civita's regularization. Numerical experiment has been made and it is shown that the regularized computation is in many cases, particularly for orbits of high eccentricity, superior to classical methods. The disturbing function is expanded in terms of eccentric anomalies after Hansen, and the theory is applied to Vesta and compared with Leveau's classical work (25).

Further Stiefel and Waldvogel (26) generalized Birkhoff's regularizing transformation to three dimensions along the line of Kustaanheimo and Stiefel. A Kustaanheimo and Stiefel transformation is combined with two inversions. A mapping can be constructed which makes possible a simultaneous regularization of both singularities in the three-dimensional restricted three-body problem. An extension to the three-dimensional elliptic restricted three-body problem is suggested. Stiefel (27) discussed some methods for coordinate perturbation with regularized coordinates. Waldvogel (28) showed the connection between regularization and the theory of functions of complex variables.

https://doi.org/10.1017/S0251107X00030388 Published online by Cambridge University Press

A four-dimensional spinor regularization was given by Kustaanheimo (**29**) by basing on the Lorenz-invariant spinor square root of a four-dimensional vector in the Minkowski space time. He (**30**) also gave the spinor form of the energetic identities which he treated in vector form in 1963.

Belorizky (**31**) proved once that the regularizing variable $\omega$ of Sundman can not in the whole $\omega$-plane make the coordinates of the three bodies holomorphic functions of $\omega$ in the triangular equilibrium solution of Lagrange. Now he (**32**) extended it to the general three-body problem.

Recently Pierce (**33**) obtained a solution in the form of trigonometric series of Levi-Civita's regularized equations by the method of general perturbations. The solution shows some of the characteristics of typical lunar or planetary theory. Then he computed four different orbits both circular and collision orbits without restriction on the eccentricity. He has seen the improvements of long-term accuracy by replacing the osculating reference orbit with a mean reference orbit.

## BIBLIOGRAPHY


1. Lemaître, G. *Bull. Cl. Sci. Acad. Belg.*, **38**, 582, 1218, 1952; **40**, 759, 1954; *Vistas in Astronomy*, **1**, 207, 1955.
2. Deprit, A., Delie, A. *Ann. Soc. Sci. Bruxelles*, **75**, 5, 1961; *Bull. Cl. Sci. Acad. Belg.*, **48**, 652, 1962; *Archive Rat. Mech. Analysis*, **12**, 325, 1963.
3. Deprit, A., Roels, J. *Bull. Cl. Sci. Acad. Belg.*, **48**, 432, 1962.
4. Deprit, A., Broucke, R. *Icarus*, **2**, 207, 1963.
5. Broucke, R. *Icarus*, **4**, 8, 1965.
6. Arenstorf, R. F. *Astr. J.*, **68**, 548, 1963.
7. Sconzo, P. unpublished yet.
8. Szebehely, V., Giacaglia, G. E. O. *Astr. J.*, **69**, 230, 1964.
9. Szebehely, V. *Mathematische Methoden der Himmelsmechanik und Astronautik*, 21, 1966.
10. Szebehely, V. *Astr. J.*, **69**, 309, 1964.
11. Szebehely, V., Pierce, D. A. *Astr. J.*, **71**, 886, 1966.
12. Cesco, R. P. *Obs. astr. La Plata, ser. Astr.*, **25**, no. 4, 1963.
13. Cesco, R. P. *Obs. astr. La Plata, ser. Astr.*, **31**, no. 14, 1965.
14. Brumberg, V. A. *Bjull. Inst. Teor. Astr.*, **9**, no. 4 (107), 234, 1963.
15. Lóper García, F. *Commission Sci. Res. Province of Buenos Aires*, La Plata, no. 21, 1966.
16. Zumkley, J. *Astr. Nachr.*, **272**, 66, 1941.
17. Cesco, R. P. *Obs. astr. La Plata, ser. Astr.*, (in press).
18. Cesco, R. P. *Obs. astr. La Plata, ser. Astr.*, (in press).
19. Chazy, J. *Mécanique Céleste*, Presses Universitaires, Paris, 221, 1953.
20. Kustaanheimo, P. *Ann. Univ. Turkuens*, A. I, 73, 1964 = *Publ. astr. Obs. Helsinki*, no. 102, 1964.
21. Kustaanheimo, P., Stiefel, E. *J. reine angew. Math.*, **218**, 204, 1965 = *Publ. Astr. Obs. Helsinki*, no. 110, 1965.
22. Kustaanheimo, P. Lectures on Celestial Mechanics I, *Publ. Astr. Obs. Helsinki*, no. 117, 1966.
23. Kustaanheimo, P., Nuoto, V. S. *Publ. Astr. Obs. Helsinki*, no. 109, 1965.
24. Rössler, M. *Z. angew. Math. Phys.*, **16**, 395, 1965.
25. Rössler, M. Dissertation, Techn. Hochsch. Zürich, 1966.
26. Stiefel, E., Waldvogel, J. *C. R. Acad. Sci.*, Paris, **260**, 805, 1965.
27. Stiefel, E. *Mathematische Methoden der Himmelsmechanik und Astronautik*, 87, 1966.
28. Waldvogel, J. Dissertation, Techn. Hochsch. Zürich, 1966.
    Waldvogel, J. *Mathematische Methoden der Himmelsmechanik und Astronautik*, 33, 1966.
29. Kustaanheimo, P. *Comm. Soc. Sci. Fenn., Phys.-Math.*, **30**–9, 1965.
30. Kustaanheimo, P. *Mathematische Methoden der Himmelsmechanik und Astronautik*, 333, 1966.
31. Belorizky, D. *Bull. astr.*, **16**, 327, 1952.
32. Belorizky, D. *C. R. Acad. Sci.*, Paris, **262**, 1133, 1966.
33. Pierce, D. A. *Astr. J.*, **71**, 545 and 562, 1966.


https://doi.org/10.1017/S0251107X00030388 Published online by Cambridge University Press

THREE-BODY PROBLEM: NUMERICAL SURVEY

Extensive numerical works have been carried out in search for periodic orbits not only around the equilibrium points and the masses but also remote from these singularities in the restricted three-body problem due to the recent development of electronic computers which facilitate the numerical treatment in a great deal.

Szebehely and Williams (**1**) published tables showing pertinent information regarding the collinear equilibrium points of the restricted three-body problem. General applications to problems of cosmogony, stellar dynamics, space science are pointed out by Szebehely (**2**). He (**3**) also reviewed the linear and non-linear aspects of the problem of motion around the equilibrium points with special reference to the restricted problem.

Danby (**4**) discussed the variational stability of the triangular equilibrium points in the elliptic restricted problem by taking the mass ratio $\mu$ and the eccentricity $e$ of the primary's orbit as two parameters, and found the region in the $(\mu, e)$-plane in which there is stability. For the general problem of three bodies with masses $m_1, m_2, m_3$ it is known that this configuration is stable if $(m_2m_3 + m_3m_1 + m_1m_2)/(m_1 + m_2 + m_3)^2$ is less than some quantity depending only on $e$. Grebenikov (**5**) discussed the same problem on the stability in the sense of Liapounov. Bennett (**6**) computed the characteristic exponents of the same problem, and refined and extended Danby's transition curve for the stability and instability regions by the use of the symmetry character of the Hamiltonian systems. In the unstable region not considered by Danby it is found that the instabilities are due to three different types of the characteristic roots. The curves separating these three regions are extensions of the curves for transition between the stable and unstable regions. For the collinear points there is no value of $\mu$ and $e$ for which variational stability exists. Bernstein and Ellis (**7**), by a method similar to the one by Danby and Bennett, gave a criterion for the boundedness of all solutions of a linearized non-homogeneous equations with periodic coefficients for the motion of an infinitesimal mass in the neighbourhood of the triangular equilibrium points. Szebehely (**8**) discussed the effect of the Coriolis term on the stability characteristics of the triangular equilibrium points. It is known that the stability of these points is due to the Coriolis term, which when neglected gives unstable solutions. The relation between the critical mass-ratio at which instability sets in and the magnitude of the Coriolis force is discussed.

Rabe (**9**) is continuing his research on the restricted Trojan problem. He takes the rather accurately determined long-period solutions of the restricted problem as reference or intermediate orbits in his work for the existence of additional classes of periodic librations about the triangular equilibrium points (**9**) in the fundamental planes of the circular and the elliptic problems (**10, 11**) as well as in the three-dimensional space. For the plane restricted problem a third order stability analysis of the most general non-periodic Trojan librations incorporating long- and short-period terms has been undertaken also on the basis of some selected reference solutions of long period. Certain stability limits were found to exist in the form of an upper $e_{max}$ for the heliocentric eccentricity $e$ of such Trojans. This limit $e_{max}$ decreases with the increasing size of the basic libration of long period. The approximate results are $e_{max} = 0{\cdot}19$ for every small basic amplitude, $e_{max} = 0{\cdot}16$ for a reference libration extending over almost 21° in longitude, and $e_{max} = 0{\cdot}08$ for a basic libration of 43° total amplitude. Since $e_{max}$ seems to decrease roughly in proportion to the second power of the increasing libration amplitude, these findings tend to explain rather well the actually observed maximum eccentricities and librations of the Trojan asteroids. As a product of this stability study the existence of several families of commensurability-related periodic solutions incorporating short-period oscillations has been also studied, from the occurrence of certain integral values of the characteristic exponents as a function of the amplitude of these superposed short-period terms (**12**).

https://doi.org/10.1017/S0251107X00030388 Published online by Cambridge University Press

Another series of works have been published by Deprit and his colleagues. Deprit and Price (13) started the integration of the differential equations by referring to the method of Steffensen (14), in which the coordinates are expanded in powers of time, and computed the characteristic exponents by the variational equations. Deprit (15), Deprit and Henrard (16) computed symmetric orbits which are doubly asymptotic to one of the three collinear equilibrium points. Numerical interpolation gave thirty-five special values of the mass-ratio which give rise to doubly asymptotic orbits: three for the $L_1$-point, two for the $L_2$-point, and thirty for the $L_3$-point. Two of these values are of particular interest since they are near the mass-ratio of the Earth-Moon system. Then Deprit (17) and Deprit and Delie (18) studied numerically the motion in the vicinity of the triangular equilibrium points. It has been seen that short- and long-period elliptic oscillations of the first order theory can be extended to two families of periodic motions except for a denumerable set of critical mass-ratios for which the long-period librations are in resonance with the short-period oscillations. Thus Brown's conjecture is justified after Rabe on the horse-shoe orbits which go through orbits doubly asymptotic to the $L_1$, $L_2$, $L_3$-points successively.

Deprit (19) has shown further that Routh's criterion on the existence of periodic solutions around $L_4$ is valid only in the first order, by expanding the analytical relation between the mass-ratio and the lower bound of the orbital parameter in power series up to the fourteenth order, and that the limiting orbits are of the stable type for mass-ratios as high as 0·044, the orbits evolving from infinitesimal ellipses into large-sized asymmetrical ovals. Deprit and Palmore (20) studied that analytic continuation of the family of short-period orbits around $L_4$ and saw that the characteristic exponents are of the stable type, in contradiction to Goodrich's conclusion (21).

Roels has studied the resonance effect in the vicinity of the triangular equilibrium points (22), in particular, those periodic orbits with periods twice that of Jupiter (23). An expansion of the solution in Fourier series is suggested, which is not of d'Alembert characteristic and which can be obtained by leaving some parameters at first arbitrary and then adjusted at the next approximation. The behaviour of the families of orbits is given in the neighbourhood of the critical mass-ratio. Delie (24) has given the d'Alembert series for the long-period solutions around those equilibrium points. Roels (25) gave a new method which enables to discover families of periodic orbits for all mass-ratios even for those discrete values for which the classical series expansions do not exist. A general solution in the vicinity of the collinear equilibrium points is discussed by Henrard (26). Henrard and Rom are applying the same technique for normalizing the problem at $L_4$.

Deprit, Henrard, Rom (27) carried out Birkhoff's (28) normalization about a stable equilibrium for a conservative Hamiltonian system with two degrees of freedom by building up explicitly the necessary canonical transformation by the method of indeterminate coefficients. The normalizing canonical transformation expresses the Cartesian phase variables in a form of double Fourier series in two angular coordinates whose coefficients are power series in the square roots of two action momenta. In the normalized part of the transformed Hamiltonian the angular coordinates are ignorable. They implemented such a Birkhoff's normalization on an electronic computer. The characteristic exponents along the singular families of long and short period orbits around $L_4$ are developed as power series of the orbital parameter. Then the procedure was applied to the equilateral triangular centre of libration for the Earth-Moon system up to order 13. Breakwell and Pringle (29) also had computed the same problem up to order four with only partial normalization. The question on resonance has been treated by Gustavson (30).

Goodrich (31) studied particularly the short-period periodic orbits around the triangular equilibrium points, in contrast to Rabe's work for long-period periodic orbits. Power series expansions are used and the coefficients are determined by recurrence formulae. Goodrich found two types of short-period orbits. The Jacobi constant increases as the orbital size increases. The Jacobi constant is smaller and the initial velocity is larger for a short-period orbit compared with a long-period orbit. For small values of the orbital parameter representing the deviation from the equilibrium point the orbit begins with elliptical (type I), and then for a certain value of the orbital parameter there occur two branches, type I and type II, of which the latter type orbits are asymmetrical and cut the JS-axis. Later for a larger value of the orbital parameter the type I orbits can cross the JS-axis.

These years are flourishing in the survey of families of periodic orbits by means of powerful electronic computers. Bartlett (32) computed periodic solutions with Thiele's variable and by means of a modified Runge-Kutta method, in particular asymmetric periodic orbits in the restricted problem with two equal masses for which Strömgren and his colleagues have made very extensive study. Bartlett found seven new classes different from Strömgren's. Hénon (34), after the manner in which Hénon and Heiles (33) studied the third integral in the dynamical problem of the galaxy, obtained periodic orbits which cut twice the straight line JS joining the two finite masses. Hénon's procedure is to study the behaviour of the successive intersection of a trajectory with the surface of section. Thus Hénon found twenty-one classes, of which six are new. He discussed the stability character of the invariant points, which represent periodic orbits, for the transformation of a point on the surface of section to its consecutive intersection of the trajectory with the surface. G. Zech of the Astronomisches Recheninstitut is studying several families of periodic orbits of the restricted three-body problem by numerical integration.

Goudas (35) computed numerically nine doubly-symmetric periodic orbits in three dimensions and examined their stability through the eigen-values of the Jacobian. Goudas called the least unstable orbits to be quasi-stable, for which all eigen-values are unity. He showed that the mass-ratio $\mu$ has only a quantitative effect on the problem such that it does not produce any new type of periodic orbits, that almost all periodic orbits which are highly inclined to the plane of motion of the two primaries are very unstable, and that all periodic orbits going round the straight line equilibrium points are very unstable whatever their inclinations are and in general that the families of periodic orbits change in a more or less regular way as $\mu$ varies. The discussion is greatly facilitated by the assumption of the doubly-symmetric character.

Then Hénon (36) started an exhaustive and systematic explorative study of the periodic and non-periodic orbits of the problem for two equal masses. He found three types of orbits, quasi-periodic, semi-ergodic and escape orbits. For high values of Jacobi's constant $C$ all orbits in the vicinity of the main bodies are quasi-periodic. As $C$ decreases, the region occupied by the semi-ergodic orbits appears and extends. When $C < 3{\cdot}4568\ldots$, Jacobi's limit opens and the semi-ergodic orbits become escape orbits, at the same time the region occupied by quasi-periodic orbits shrinks and eventually vanishes. The behaviour of small island sub-regions is quite interesting. This elegant work of Hénon is promising an extensive theory on the behaviour of various types of orbits as $C$ varies and as the period passes through rational and irrational numbers. A work of Wintner discussing the genealogy of various classes of Strömgren's periodic orbits is reminded.

Hénon and Heiles (33) studied by numerical experiment the problem of the existence of the third integral and found that the third integral exists only for a limited range of initial conditions. A trajectory is integrated to the next point at which the trajectory cuts a fixed surface. An invariant point for such a transformation to the next intersection corresponds to a periodic trajectory. The transformed points are seen to cluster in some limited regions, which they call islands. The set is dense everywhere but the islands do not cover the whole area but leave a so-called sea between islands in which the ergodic trajectory is dense everywhere. Barbanis (37) studied the behaviour of the hodographs of galactic orbits by means of a formal third integral. Contopoulos (**48**) gave tables of this third integral. Contopoulos (**39**) classified the integrals of motion into isolating, quasi-isolating and ergodic, and proved by means of von Zeipel's method that near any given Hamiltonian in the form of a series there is a separable Hamiltonian.

Contopoulos (**40**), Contopoulos and Montsoulas (**41**) discussed the general case of resonance in an axially symmetric potential field in which the unperturbed frequencies in two directions are in a rational ratio $p/q$. It has been seen that the general form of the third integral is not valid but a new integral is found which can be used as a third integral. For $p + q < 4$ the case shows quite a peculiar character. In particular Contopoulos discussed the case $p = q$ and the role of small divisors in a third integral. Contopoulos and Woltjer (**42**) discussed the third integral in the case of non-smooth potentials. Contopoulos (**43**) studied by numerical process the periodic orbits. Periodic orbits in the vicinity of a periodic orbit, which oscillates $p$ times along the $x$-axis and $q$ times along the $y$-axis, are called tube orbits, which can be studied by the third integral. He discussed the transition from a tube orbit to an ordinary box orbit. Barbanis (**44**) studied the transition from the case of the isolating third integral to that of the non-isolating third integral by numerical experiment with a potential similar to Contopoulos's (**40**) or to Hénon-Heiles's (**33**). He has seen that the third integral ceases to be isolating for a large range of initial conditions with increasing total energy, but that even for energies higher than the energy of escape there is a set of orbits which do not escape to infinity, indicating the existence of a very nearly isolating integral. Anderle (**45**) discussed a special form of the third integral.

Contopoulos (**38**) applied the idea to the restricted three-body problem and also in the elliptic problem in the form of series. Numerical integration by Bozis (**46**) indicates the existence of a new non-isolating integral besides the isolated Jacobi integral in the restricted three-body problem. Bozis transformed the equations of motion by Birkhoff's regularizing transformation in order to include collision and near-collision orbits. With various values of the initial conditions he computed various orbits. The boundaries and the invariant curves are given analytically in the regularized system for the two-body problem where the two integrals are known to exist. He found the boundaries of the orbit by using a series-type integral of Contopoulos.

Barbanis (**44**) studied the behaviour and topology of the family of periodic orbits of the problem. Hénon's papers (**34**, **36**) on the numerical exploration of the restricted three-body problem have been discussed by Bozis and Szebehely (**47**).

## BIBLIOGRAPHY


1. Szebehely, V., Williams, C. A. *Astr. J.*, **69**, 460, 1964.
2. Szebehely, V. *Space Sci. Rev.*, **2**, 219, 1963.
3. Szebehely, V. *Methods in Astrodynamics and Celestial Mechanics*, 3, 1966.
4. Danby, J. M. A. *Astr. J.*, **69**, 165 and 294, 1964.
5. Grebenikov, E. A. *Astr. Zu.*, **41**, no. 3, 567, 1964.
6. Bennett, A. *Icarus*, **4**, 177, 1965.
7. Bernstein, I. S., Ellis, J. *Icarus*, **5**, 299, 1966.
8. Szebehely, V. Intern. Congress of Mathematicians, Moscow, 1966.
9. Rabe, E. Lectures in Applied Mathematics, Vol. 6, 79, Amer. Math. Soc., 1966.
10. Rabe, E. *ibid.*, 102.
11. Rabe, E. Proc. *IAU Symposium No.* 25, 176, Thessaloniki, 1964.
12. Rabe, E. *Astr. J.*, **70**, 329 and 687, 1965.
13. Deprit, A., Price, J. F. *Astr. J.*, **70**, 836, 1965.
14. Steffensen, J. F. *K. Danske Videnskab. Selskab., Math. Fys. Medd.*, **30**, no. 18, 1956.
15. Deprit, A. *Icarus*, **4**, 273, 1965.
16. Deprit, A., Henrard, J. *Astr. J.*, **70**, 271, 1965.


https://doi.org/10.1017/S0251107X00030388 Published online by Cambridge University Press

17. Deprit, A. Boeing Sci. Res. Lab., D1-82-0335, 1963.
18. Deprit, A., Delie, A. *Icarus*, **4**, 242, 1965.
19. Deprit, A. *Astr. J.*, **71**, 77, 1966.
20. Deprit, A., Palmore, J. *Astr. J.*, **71**, 94, 1966.
21. Goodrich, E. F. *NASA Techn. Note*, X-643-65-420, 1965.
22. Roels, J. Dissertation, Univ. Louvain, 1965.
23. Roels, J. *Bull. astr.*, (3) **1**, fasc. 3, 1966.
24. Delie, A. Dissertation, Univ. Louvain, 1966.
25. Roels, J. *Bull. astr.*, (3) **1**, fasc. 2, 67, 1966.
26. Henrard, J. Dissertation, Univ. Louvain, 1966.
27. Deprit, A., Henrard, J., Rom, A. R. M. Boeing Sci. Res. Lab., D1-82-0556, 1966; Boeing Sci. Res. Lab., D1-82-0564, 1966; *Icarus*, **5**, 1966 (in press).
28. Birkhoff, G. D. *Dynamical Systems*, Amer. Math. Soc., 1927.
29. Breakwell, J. V., Pringle, R. *Methods in Celestial Mechanics and Astrodynamics*, 55, 1966.
30. Gustavson, F. G. *Astr. J.*, **71**, 670, 1966.
31. Goodrich, E. F. *Astr. J.*, **71**, 88, 1966.
32. Bartlett, J. H. *Publ. Medd. Københavns Obs.*, no. 179, 1964.
33. Hénon, M., Heiles, C. *Astr. J.*, **69**, 73, 1964.
34. Hénon, M. *Ann. Astrophys.*, **28**, 499 and 992, 1965.
35. Goudas, C. L. *Icarus*, **2**, 1, 1963.
36. Hénon, M. *Bull. astr.*, (3) **1**, fasc. 1, 57, 1966; fasc. 2, 49, 1966.
37. Barbanis, B. *Astr. J.*, **70**, 285, 1965.
38. Contopoulos, G. *Astrophys. J.*, **142**, 802, 1965.
39. Contopoulos, G. *Astrophys. J.*, **138**, 1297, 1963.
40. Contopoulos, G. *Astr. J.*, **68**, 763, 1963; Proc. *IAU Symposium No.* 25, 3, Thessaloniki, 1964.
41. Contopoulos, G., Montsoulas, M. *Astr. J.*, **70**, 817, 1965; **71**, 687, 1966.
42. Contopoulos, G., Woltjer, L. *Astrophys. J.*, **140**, 1106, 1964.
43. Contopoulos, G. *Astr. J.*, **70**, 526, 1965.
44. Barbanis, B. *Astr. J.*, **71**, 415, 1966; Proc. *IAU Symposium No.* 25, 19, Thessaloniki, 1964.
45. Anderle, P. *Bull. astr. Inst. Csl.*, **17**, no. 4, 166 and 169.
46. Bozis, G. *Astr. J.*, **71**, 404, 1966.
47. Bozis, G., Szebehely, V. Proc. *IAU Symposium No.* 25, 163, Thessaloniki, 1964.
48. Contopoulos, G. *Astrophys. J.*, **145**, 109, 1966; *Astrophys. J. Suppl.*, **13**, no. 122, 1966.

## THREE-BODY PROBLEM: ANALYTICAL THEORY

Nahon (1) derived a simplified form of the differential equations for the planar three-body problem when the position variable is the virial of Sundman defined as the moment of inertia of the three masses with respect to their centre of mass. D'Ambly (41) discussed the collinear central figure of the $n$-body problem.

Pius (2) obtained the bounds of the variation of the semi-major axes in the restricted three-body problem for small values of the disturbing mass.

Jefferys (3) proved the existence of doubly-symmetric periodic orbits in the three-dimensional restricted problem by the method of analytic continuation for sufficiently small mass-ratios. Krasinsky (4) studied the closed path of double collision with the larger mass in the plane circular restricted problem and obtained the existence of some types of paths both symmetric and asymmetric with respect to the $x$-axis by the method of a small parameter.

Richards (5) showed the asymptotic instability of the triangular equilibrium points, not by means of the linearized system of variational equations of motion but by means of the Jacobi integral only, by discussing the variation of the potential and the behaviour of the zero-velocity curves. Lanzano (6) studied the periodic motions around one of the primaries in the restricted problem, which can be generated from circular Keplerian orbits and are valid for any mass-ratio, by extending the procedure of Siegel for the lunar theory. The Fourier series representation is obtained, in which the coefficients are expanded in powers of the parameter related to the orbital period, and is shown to be convergent for small values of the period and for any value of the mass-ratio $\mu$ of the primaries. On the other hand, Arenstorf (**7**) found periodic solutions representing the analytic continuation of a Keplerian elliptic motion by a more general periodicity condition than Poincaré's, and proved the existence of such periodic solutions as holomorphic functions of the mass-ratio $\mu$ and the parameter $\epsilon$ which represents the deviation of the complex radius vector from the generating orbit. Danby (**8**) discussed the inclusion of extra factors, such as the variation of the law of attraction, the non-sphericity of the bodies, and the addition of a fourth distant body. The possible existence of such solutions is considered and where they exist, their stability is studied. Omarov (**9**) solved the restricted problem with variable mass of the primaries.

Choudhry (**10**) proved the existence of analytic continuation and its parametric representation for $\mu \neq 0$ corresponding to the circular orbits for $\mu = 0$ in the restricted problem in a three-dimensional coordinate system by referring to Birkhoff's work on the three-body problem. Barrar (**11**) proved the existence of periodic solutions of the second sort of Poincaré in the restricted problem.

Jefferys (**12**) has shown the existence of a class of inclined periodic solutions of the restricted problem, which form a continuous sequence with eccentricity as parameter and a discrete sequence with mean motion as parameter. They exist only for special values of the inclination and if the ratio of the mean motions to that of Jupiter is sufficiently small or sufficiently great. These are Poincaré's periodic solutions of the third sort.

Grémillard (**13**) published in a series of papers the existence of periodic solutions of the third sort of Poincaré by revising the former work of von Zeipel. He proved the existence of periodic solutions in which two eccentricities are zero at the initial epoch but are expanded in powers of a parameter $\rho$ for $p - q$ even, where $n/n' = p/q$ is the ratio of mutually prime integers, and the existence of periodic solutions in which the initial eccentricities are not zero but inclinations are small for $p - q$ odd and greater than unity. He also proved von Zeipel's result that there does not exist any solution other than those for which $\bar{\lambda} = r\,(\pi/2)$ for $p - q$ odd and $\bar{\lambda} = r\,\pi$ for $p - q$ even ($r = 0, \pm 1, \pm 2, \ldots$), where $\bar{\lambda} = p\lambda_0' - q\lambda_0$ with the initial mean longitudes $\lambda_0$ and $\lambda_0'$. He is now constructing numerical examples for the case $p - q = 2$. Choudhry (**14**) showed the existence of periodic solutions of the third sort in the elliptic restricted problem and proved the stability of the generating solution.

Hénon is now studying periodic orbits of the second species of Poincaré which exist in the restricted problem when the second mass vanishes and which are characterized by a series of encounters between the second and the third masses.

Alekseev (**15**) estimated by Merman's method (**16**) the perturbation of a hyperbolic motion in the three-body problem basing on Merman's theorems in the theory of perturbed motions.

Pius (**17**) applied the theory of symmetric periodic orbits of the Schwarzschild type as a first approximation to the motion of Hecuba. Barrar (**18**) gave a new proof for the existence of periodic orbits which are closed after several revolutions after the fashion of Moser (**19**). Moser referred to the invariant curves of area-preserving mappings of a torus according to Birkhoff.

Petrovskaya (**20**) constructed a periodic solution of the first sort in the plane restricted problem in the form of power series in $\mu$ and $\beta^{1/2}$ where $\mu$ denotes the mass-ratio and $\beta$ the ratio of the radii of the undisturbed orbit of the infinitesimal mass and of the relative orbits of the primaries. She found that the series are convergent for $0 \leqslant |\mu| \leqslant 0{\cdot}001$, $0 \leqslant |\beta| \leqslant 0{\cdot}4480$ for the inferior motion of the infinitesimal mass and $0{\cdot}26 \leqslant |\beta| \leqslant 0{\cdot}42785$ for the exterior motion.

Eckstein, Shi and Kevorkian (**21**) have studied the motion of a close satellite to the smaller of the two finite masses by taking as the independent variable, instead of the time, the angle between the radius vector of the satellite and the instantaneous node of its orbit around the primary. The equations of motion first given by Struble (23) are solved by a generalized asymptotic expansion procedure involving three variables to the second order as regards the smaller of the two finite masses. The three variables differ each other by one order of magnitude in the small parameter, and give the measure of the periods of the node and the pericentre. It is shown that rather large amplitude oscillations in the eccentricity, inclination and pericentre occur over the longest of these three periods for a moderate value of the inclination. They (22) demonstrated that the energy integral or any known exact integral of motion can be used to determine certain higher-order terms in the time history of the satellite motion after the geometry of the orbit is obtained. They showed how the loss of accuracy due to the presence of secular and long-period terms in such cases is to be overcome.

Szebehely, Pierce, Standish (24) computed by using Birkhoff's regularization a family of orbits with the property that they are non-periodic and collide with both primaries. Szebehely (25) discussed the problem of capture and of so-called 'swing-by' orbits and outlined the application of Tisserand's criterion for identifying comets to a new problem of space research.

Szebehely (26) reviewed recent results on the restricted problem as an irreducible dynamical system with two degrees of freedom.

Shibahara and Yoshida (27) sharpened the theorem of Birkhoff and Merman (28) in the general three-body problem. Alekseev (29) discussed with examples the interchange of a hyperbolic motion and an elliptic motion, and he (30) derived a criterion for hyperbolic and hyperbolic-elliptic motions according to the idea of Chazy and Merman.

Arenstorf (31) expanded the solution of the differential equations in the perturbation theory as functions of the initial values from which useful estimates and a sufficiently detailed description can be obtained by comparing with the unperturbed solutions so as to enable solution of the boundary-value problem for the perturbed system, and applied to the periodic solutions in the restricted three-body problem.

Conley (32) obtained a denumerably infinite number of long-periodic solutions of the planar restricted problem on the basis of Poincaré's fixed point theorem for torus mappings.

Schubart and Stumpff (33) developed an $n$-body programme of high accuracy for the calculation of disturbed ephemerides of planets and comets, and tested it successively by the high-speed computers at Heidelberg and Darmstadt. Schubart (34) succeeded in rediscovering the lost comet Tempel-Tuttle 1866 I by identifying it with 1965 *i* according to a calculation of the perturbations of 1866 I on the basis of this $n$-body problem. By means of another programme which allows generally the calculation of the orbits of a system of numerous mass points Wielen (35) studied the motion in a fictitious open cluster consisting of 100 stars. He considered three models differing in the distributions of the initial velocities and of masses. The analytical and statistical properties of stellar clusters known from stellar dynamics, such as time of relaxation and rate of escape, have been verified.

The $n$-body problem when $n$ is sufficiently great is now being attacked by means of computers in view of application to stellar dynamics for unveiling the global characteristics by examining the behaviour of each individual constituent. Indeed, Miller (36) has shown that stellar dynamical systems possess the macroscopic irreversibility characteristic of statistical mechanical systems by means of a computer calculation of an $n$-body system. Aarseth (37) studied numerically the evolution of an isolated $n$-body system with $n = 100$, in order to determine the dynamical age of the system. The equipartition effect leads to pronounced mass segregation with the formation of a dense nucleus of heavy members and an extended halo of mainly light bodies. Von Hoerner (38), by taking $n = 16$ and $n = 25$ for globular clusters, found that within two to four times of relaxation all clusters with different starting conditions approach an isothermal polytrope and a Maxwellian velocity distribution, and that in much longer times the deviations from such a configuration become important. The virial theorem and Chandrasekhar's time (39) of relaxation are approximately verified. Miller (40) took $n = 4, 8, 12, 16, 24, 32$ and observed that two similar systems evolve simultaneously and the separation of their representative points in phase space grows exponentially with time because of encounter effects. The time constant of this exponential growth is short, about four times the mean time between binary collisions of one particle divided by the number $n$. The $n$-body systems with inverse-square law forces are seen to behave as tightly coupled systems.

Aarseth (42) further studied the advanced stages of dynamical evolution. The overall relaxation time is in good agreement with that predicted by Chandrasekhar's theory for centrally concentrated clusters but the mean relaxation time for each mass group is well correlated with a pronounced mass segregation, the heavy members forming a dense nucleus. Aarseth does not find any approach to the equipartition of kinetic energy for the clusters as a whole. The total velocity distribution develops an excess at high and low velocities as compared with the Maxwellian. He found some evidence for selective escape among light members in isolated clusters and the total mass loss to be greater than in the case of an equal mass distribution. The dispersion of the binding energy increases with time and may be used to show the dynamical age of simulated clusters.

## BIBLIOGRAPHY


1. Nahon, F. *C. R. Acad. Sci.*, Paris, **258**, 815, 1964.
2. Pius, L. Yu. *Bjull. Inst. Teor. Astr.*, **11**, no. 2 (125), 1967 (in press).
3. Jefferys, W. H. *Astr. J.*, **70**, 393, 1965.
4. Krasinsky, G. A. *Bjull. Inst. Teor. Astr.*, **9**, no. 2 (105), 154, 1963.
5. Richards, P. B. *Icarus*, **1**, 334, 1963.
6. Lanzano, P. *Icarus*, **2**, 364, 1963.
7. Arenstorf, R. F. *Amer. J. Math.*, **85**, 27, 1963.
8. Danby, J. M. A. *Astr. J.*, **70**, 181, 1965.
9. Omarov, T. B. *Astr. Zu.*, **41**, no. 1, 170, 1964.
10. Choudhry, R. K. *J. Indian Math. Soc.*, **28**, 119, 1964.
11. Barrar, R. B. *Astr. J.*, **70**, 3, 1965.
12. Jefferys, W. H. *Astr. J.*, **71**, 99 and 566, 1966.
13. Grémillard, J. *C. R. Acad. Sci.*, Paris, **234**, 2339, 1952; **236**, 49, 1953; **239**, 153, 1954; **244**, 1011, 1957; **247**, 1307, 1958; **251**, 2121, 1960; **252**, 1424, 1961; *Bull. astr.*, **24**, 483, 1965.
14. Choudhry, R. K. *Bjull. Inst. Teor. Astr.*, **10**, no. 8 (121), 523, 1966.
15. Alekseev, V. M. *Astr. Zu.*, **38**, no. 2, 325; no. 4, 726; no. 6, 1099, 1961; **39**, no. 4, 724, 1962.
16. Merman, G. A. *Bjull. Inst. Teor. Astr.*, **6**, no. 2 (75), 73, 1955.
17. Pius, L. Yu. *Bjull. Inst. Teor. Astr.*, **8**, no. 1 (94), 11, 1961.
18. Barrar, R. B. *Math. Annalen*, **160**, 363, 1965.
19. Moser, J. *Math. Annalen*, **126**, 325, 1953; *Nachr. Akad. Wiss. Göttingen, Math.-Phys. Kl.*, **1962**, 1.
20. Petrovskaya, M. S. *Bjull. Inst. Teor. Astr.*, **9**, no. 2 (105), 85, 108, 1963; **10**, no. 6 (119), 1965.
21. Eckstein, M. C., Shi, Y. Y., Kevorkian, J. *Astr. J.*, **71**, 248, 1966; *Douglas Report*, no. 3078, 1964; *Methods in Astrodynamics and Celestial Mechanics*, 183, 1966.
22. Eckstein, M. C., Shi, Y. Y., Kevorkian, J. *Astr. J.*, **71**, 301, 1966; Proc. *IAU Symposium No.* 25, 291, Thessaloniki, 1964.
    Kevorkian, J. *Astr. J.*, **71**, 878, 1966.
23. Struble, R. A. *J. Math. Analysis Appl.*, **1**, 1960.
24. Szebehely, V., Pierce, D., Standish, M. *Celestial Mechanics and Astrodynamics*, **35**, 1964.

25. Szebehely, V. Conference on the Exploration of Mars and Venus, NASA–VPI, VIII 1, 1965.
26. Szebehely, V. *Lectures in Applied Mathematics*, **5**, 150, 1966, Amer. Math. Soc.
27. Shibahara, R., Yoshida, J. *Publ. astr. Soc. Japan*, **15**, 87, 1963.
28. Merman, G. A. *Astr. Zu.*, **30**, no. 3, 332, 1953; *Bjull. Inst. Teor. Astr.*, **6**, no. 2 (75), 69, 73, 1955.
cf., Khilmi, G. F., *Qualitative Methods in the Many-Body Problem*, Gordon-Breach, New York, 1961.
29. Alekseev, V. M. *Astr. Zu.*, **38**, no. 2, 325; no. 4, 726; no. 6, 1099, 1961; **39**, no. 4, 724; no. 6, 1102, 1962.
30. Alekseev, V. M. *Vestnik Moskov. Univ.*, no. 1, 67, 1961 = *Amer. Rocket Soc. J.*, **32**, 1488, 1962.
31. Arenstorf, R. F. *Amer. J. Math.*, **85**, 27, 1963; *Mathematische Methoden der Himmelsmechanik und Astronautik*, 75, 1966.
32. Conley, C. C. *Comm. Pure Appl. Math.*, **16**, 449, 1963.
33. Schubart, J., Stumpff, P. *Astr. Rechen-Inst. Veröff.*, **18**, 1966.
34. Schubart, J. *Sterne und Weltraum*, **4**, 277, 1965.
35. Wielen, R. Dissertation, Heidelberg, 1965.
36. Miller, R. H. Proc. *IAU Symposium No.* 25, 137, Thessaloniki, 1964.
37. Aarseth, S. J. *ibid.*, 141; *Mon. Not. R. astr. Soc.*, **126**, 223, 1963.
38. von Hoerner, S. *ibid.*, 146; *Z. Astrophys.*, **50**, 184, 1960; **57**, 47, 1963.
39. Chandrasekhar, S. *Principles of Stellar Dynamics*, Dover Pub., New York, 1960.
40. Miller, R. H. *Astrophys. J.*, **140**, 250, 1964.
41. D'Ambly, C.-G. *Astr. Nachr.*, **288**, 81, 1965.
42. Aarseth, S. J. *Mon. Not. R. astr. Soc.*, **132**, 35, 1966.

## PERIODICITY, ERGODICITY, STABILITY

Nahon (1) studied by referring to the virial a class of potential which is a generalization of radial potential and which admits trajectories on the equi-potential surfaces with constant velocity. In some cases (2) the Lagrangian function shows that all trajectories of a given energy have damping oscillations around a unique iso-kinetic trajectory of the same energy. Nahon gave a procedure of constructing all fields which admit equi-potential orbits. Further Nahon (3), by limiting to conservative systems of two degrees of freedom, defined a field of directions and gave the necessary and sufficient condition for an initial field to be permanent. He (4) then, starting from particular solutions of the Hamilton-Jacobi equation, deduced their asymptotic property and showed the invariance with respect to Birkhoff's conformal transformation. The study is intended to draw conclusions on stellar dynamics. Mrs Losco found by this method force fields in which level lines are spirals (4a). In such cases a quantitative study can be made as to the asymptotic behaviour of trajectories for $t \to +\infty$, independently of the knowledge of the first integral, either isolating or non-isolating. Miss Ghertzman (5) studied the motion of a particle in a force field with axial symmetry as a function of a parameter which represents, if it is not small, the $J_2$ term of the potential of an oblate planet. She followed numerically the solution for various values of the parameter and found its critical value which limits the two regions of different characteristics of motion, i.e. revolution and libration, of the mean anomaly.

Moser (6) discussed the combination tones for Duffing's problem with forcing terms by referring to Kolmogorov and Arnold's theory, and Struble and Yionouli (7) the general perturbation of the harmonically forced Duffing's equation, and Struble (8) the oscillations of a pendulum under parametric excitation. Hale (9) discussed the successive approximations of the solution of differential equations containing a small parameter. Harvey (10) and Villari (11) showed the existence of periodic solutions of the same type of differential equations with forced terms, and Ezeilo the existence of almost-periodic solutions of a certain type of differential

https://doi.org/10.1017/S0251107X00030388 Published online by Cambridge University Press

equations with a dissipative term. Silnikov (12) studied the case of the existence of a denumerable number of periodic solutions. Knobloch (13) presented after Wazewsky a new method of approximation for periodic solutions of a non-linear differential equation of the second order.

Brumberg (14) discussed the solution in the form of series of polynomials in the three-body problem according to the idea of Mittag-Leffler (15), Painlevé (16), Goursat (17), Belorizky (18). Merman (19) studied the asymptotic solutions of a canonical system of one degree of freedom in the case in which the characteristic exponents are zero, by referring to the indices of a singular point. He (20) proved the existence of unstable regions of periodic solutions of canonical systems in the near-resonance cases.

Moser is working on the convergence proof of the expansions for quasi-periodic motions. Moser (21) showed that an expansion of the motion in trigonometric series is convergent provided that the frequencies are rationally independent, and accordingly succeeded in the rigorous existence proof of quasi-periodic solutions and justified the series expansions commonly used in celestial mechanics if properly interpreted in his own sense. Jefferys and Moser (22) applied these results to the three-body problem and established the existence of inclined quasi-periodic motions with three frequencies (two mean motions and a nodal precession) in the unstable case where the inclination $I$ exceeds $I_0$ such that $\sin^2 I_0 = 3/5$. Also they found inclined quasi-periodic motions with four independent frequencies for smaller inclination. These results hold for small masses or in the lunar case.

Krasinski (23) proved the existence of quasi-periodic solutions close to circular orbits in the plane $n$-body problem.

The stability of a Hamiltonian system of differential equations, especially a system of linear canonical differential equations with periodic coefficients, has been studied recently by Moser (24), Gelfand and Lidskii (25), and Segal (26) with the variational equations. Glimm (27), by extending the idea of Kolmogorov, Moser and Arnold (28), discussed the formal stability of a Hamiltonian system. Salvadori (29) studied the stability of an equilibrium in the critical case according to Liapounov. Pliss (30) showed that the stability problem of the trivial solution of the system

$$\frac{dx}{dt} = Ax + p(x, y), \qquad \frac{dy}{dt} = By + q(x, y)$$

in $n$-dimensional vector notation is reduced to that for the existence and construction of an $n$-dimensional vector function $p^*(x)$ such that the above stability problem is equivalent to the stability problem of the trivial solution of a system

$$\frac{dx}{dt} = Ax + p^*(x).$$

The construction of Liapounov's functionals (31, 32, 33), by means of which the stability statement is obtained, has been described by Brayton and Moser (34) with application to electric circuit net works. By extending this idea to partial differential equations for a non-linear mixed initial boundary value problem Brayton and Miranker (35) constructed Liapounov's functionals and showed that complete stability follows from the existence of a Liapounov functional, and gave a number of examples. Yoshizawa (36) constructed Liapounov's functional, and showed the existence of a bounded solution and discussed the asymptotic behaviour of the solution of a system of differential equations. He also obtained theorems on the extreme stability and almost-periodic solutions of functional-differential equations after Hale (37) and La Salle (38). Halkin (39) extended a theorem of Liapounov on the closeness and connectivity of a set. Dearman and Le May (40) made a survey of various techniques for generating Liapounov functionals.

Fomin (41) studied the perturbation method in the theory of dynamical stability of systems with distributed parameters by reducing the stability problem to the study of linear Hamiltonian systems in separable Hilbert space. Almkvist (42) discussed the stability of linear differential equations with periodic coefficients in Hilbert space by means of the perturbation method. Ura and Kimura (43) studied the stability in the topological dynamics of Gottschalk and Hedlund (44) on the basis of Zubov's condition for the stability of a closed invariant set. Ura (45) studied the stability, relative stability and saddle points on the flow, that is, the trajectories outside a closed invariant set. Schwartz (46) considered a compact analytical mapping of Banach space and Conley (47) applied the idea to an area-preserving disk mapping into itself associated with the satellite problem in an axially symmetric field. Minty (48), Browder (49) and Shinprot (50) provèd the fixed point theorem in Hilbert space.

Moser (51) and Arnold (52) studied the area-preserving mapping of a torus onto itself by developing the idea of Birkhoff. The results assert the existence of curves near the circle $r =$ constant, which are invariant under the mapping in polar coordinates

$$r' = r + \mu F(r, \theta), \qquad \theta' = \theta + \beta + \mu\,[\alpha(r) + G(r, \theta)],$$
$$a \leqslant r \leqslant b, \qquad b - a \geqslant 1, \qquad 0 < \mu \leqslant 1, \qquad \beta = \text{constant},$$

where $F$ and $G$ are $p \geqslant 4$ times continuously differentiable. The invariant curves exist not for every rotation number (53) $\omega$, but only for those $\omega$ which lie in the interval $\alpha(a) + \epsilon < (\omega - \beta)/\mu < \alpha(b) - \omega$ and which satisfy the infinitely many inequalities $|\,n\,\omega - 2\,\pi\,m\,| \geqslant \mu\,\epsilon\,n^{-(1+\sigma)}$ for some $\sigma > 0$ and for every non-zero integers $m$ and $n$. The theorem is applied by Jefferys (54) with numerical computation of periodic orbits in the restricted three-body problem. By judging from the behaviour of the trajectories in the neighbourhood of an invariant point Jefferys studied the stability and the ergodicity of the motion and the transition between stability and instability.

A motion near a periodic solution is characterized by the eigen-values of the linear terms of the differential equations in local coordinates. Diliberto (55) has shown that when these local coordinates have purely imaginary characteristic roots the possibility of stability exists and that when these roots are commensurable with the frequency of the periodic solution the system is in general unstable. It was believed that there were an infinite set of algebraic conditions necessary for formal stability. Diliberto (56) reduced these conditions to two for a Hamiltonian system with two degrees of freedom.

Kyner (70) studied the relationships among the classical Delaunay theory, the Diliberto periodic surface theory and the Krylov-Bogoliubov method of averaging, and saw that the first two methods produce the same second order approximations and that there exists a formal expansion of a family of periodic two-surfaces if a monotonicity condition is satisfied.

Diliberto's theory (57, 58) of periodic surfaces has been applied to the satellite orbit problem. It was shown by Haseltine (59) in the case of satellite motion that the process of iteration did not break down at higher approximations and the dynamical system was formally stable. In the case of incommensurable periods there always exist formal expansions and no non-trivial example of convergent expansions. For the case of commensurable periods there is an algebraic difficulty. Diliberto (55) recently solved this difficulty in two cases and he (56) proved that the doubly-periodic expansions in the case of commensurable periods are possible for satellite motion around an oblate Earth with even zonal harmonics.

The ergodic theorem (65) has been discussed recently by several authors, such as Dowker (60), Brunk (61), Blum and Hanson (62), Chacon and Ornstein (63). Furstenberg (64) studied the strict ergodicity and transformations in almost-periodic processes.

Let $\mathfrak{A}$ be a diffeomorphism (52) of a plane ring which conserves the area, and $\gamma$ be a simple curve in the ring which is not homologous to zero. Then Poincaré's theorem tells

https://doi.org/10.1017/S0251107X00030388 Published online by Cambridge University Press

that the curves $\gamma$ and $\mathfrak{A}\gamma$ have at least two common points. Arnold (**66**) calls the operation $\mathfrak{A}: \Omega \rightarrow \Omega$ is globaly canonical if it is homotopic to the identity and

$$\oint_{\gamma} p\mathrm{d}q = \oint_{\mathfrak{A}\gamma} p\mathrm{d}q. \qquad (p\mathrm{d}q = p_1\mathrm{d}q_1 + \ldots + p_n\mathrm{d}q_n),$$

for each closed curve $\gamma \subset \Omega$. Let $P(x) = p(\mathfrak{A}x)$, $Q(x) = q(\mathfrak{A}x)$. The operation $\mathfrak{A}$ is globaly canonical if and only if

$$\mathfrak{A}(x) = \int_{x_0}^{x} (Q - q)\,\mathrm{d}P + (p - P)\,\mathrm{d}q$$

defines a monovalent function $\mathfrak{A}(x)$. Let $T$ be a torus $p = 0$ in $\Omega$ and $\mathfrak{A}T$ be the image of $T$ by a globaly canonical operation $\mathfrak{A}$. The tori $T$ and $\mathfrak{A}T$ have at least $2^n$ common points provided that $\mathfrak{A}T$ satisfies $p = p(q)$, $|\,\mathrm{d}p\,/\,\mathrm{d}q\,| < \infty$. Arnold proved that, if $\mathfrak{A}$ is a globaly canonical operation sufficiently near $\mathfrak{A}_0 : p,\ q \rightarrow p,\ q + \omega(p)$, such that for $\det \left|\dfrac{\partial\omega}{\partial p}\right| \neq 0$ there exists a point $p_0$ for which $\omega_i(p_0)$ are commensurable, i.e., $\omega_i(p_0) = 2\pi\, m_i/N$, with integers $N$ and $m_i$, $i = 1, 2, \ldots, n$, then the operation $\mathfrak{A}^N$ has at least $2^n$ fixed points in the neighbourhood of the torus $p = p_0$.

Arnold (**67**), by extending his former theory (**68**), considered the $k$-dimensional vector differential equations containing a small parameter $\epsilon$ and with an analytic $\omega(I)$

$$\dot{\varphi} = \omega(I) + \epsilon f(I, \varphi),$$
$$\dot{I} = \epsilon F(I, \varphi),$$

where the components of $\omega$ are non-commensurable and $f$, $F$ are analytic and periodic with period $2\pi$ in the angular variable $\varphi$, such that $|\,\mathfrak{Im}\,\varphi\,| < \varrho$, in comparison with

$$\dot{\mathcal{J}} = \epsilon \bar{F}(\mathcal{J}), \qquad \bar{F}(\mathcal{J}) = (2\pi)^{-k} \int\!\!\int F(\mathcal{J}, \varphi)\,\mathrm{d}\varphi.$$

He proved for $k = 2$, but not yet for $k > 2$, that with positive constants $C$ and $C'$ we have

$$C\sqrt{\epsilon} < |\,I(t) - \mathcal{J}(t)\,| < C'\sqrt{\epsilon}\,(\log 1/\epsilon)^2$$

for $0 \leqslant t \leqslant 1/\epsilon$.

Arnold (**69**) discussed stationary motions and their stability on the modern version of differential geometry of Lie groups of infinite dimensions, in contrast to the groups of diffeomorphism, and applied it to hydrodynamics of perfect fluid.

## BIBLIOGRAPHY


1. Nahon, F. *C. R. Acad. Sci.*, Paris, **257A,** 2403, 1963; **259A,** 65, 1964.
2. Nahon, F. Proc. *IAU Symposium No.* 25, 43, Thessaloniki, 1964.
3. Nahon, F. *C. R. Acad. Sci.*, Paris, **262A,** 669, 1966.
4. Nahon, F. *ibid.*, **262A,** 1428, 1966.

4a. cf., Szebehely, V. Proc. *IAU Symposium No.* 25, 34, 1964.

5. Ghertzman, L. *Bull. astr.*, (3) **1**, fasc. 3, 1966.
6. Moser, J. *Comm. Pure Appl. Math.*, **18,** 167, 1965; Intern. Symposium on Nonlinear Diff. Eqs. and Nonlinear Mech., Acad. Press, New York, 71, 1963.
7. Struble, R. A., Yionouli, S. M. *Archive Rat. Mech. Analysis*, **9,** 422, 1962.
8. Struble, R. A. *Quart. Appl. Math.*, **21,** 121, 1963.
9. Hale, J. K. *Contr. Diff. Eqs.*, **1,** 215, 1963.
10. Harvey, C. A. *Contr. Diff. Eqs.*, **1,** 425, 1963.
11. Villari, G. *Annali Mat. Pura Appl.*, **65,** 153, 389, 1964; **69,** 171, 191, 1965.

12. Silnikov, L. P. *Doklady Mat.*, **6**, 163, 1965.
13. Knobloch, H. W. *Math. Zeitschr.*, **82**, 177, 1963; *Michigan Math. J.*, **9**, 303, 1962.
14. Brumberg, V. A. *Bjull. Inst. Teor. Astr.*, **9**, no. 4 (107), 234, 1963.
15. Mittag-Leffler, G. *Acta Math.*, **23**, 43, 1900; **24**, 183, 205, 1901; **26**, 353, 1902; **29**, 101, 1905; **42**, 285, 1920.
16. Painlevé, P. *Leçons sur la Théorie Analytique des Equations Différentielles*, Hermann, Paris, 1895; *Bull. astr.*, **15**, 81, 1898.
17. Goursat, E. *Bull. Sci. Math.*, (2) **27**, 226, 1903.
18. Belorizky, D. *J. Observateurs*, **16**, 109, 149, 189, 1933; *Bull. Sci. Math.*, (2) **62**, 271, 1938.
19. Merman, G. A. *Bjull. Inst. Teor. Astr.*, **9**, no. 6 (109), 394, 1964.
20. Merman, G. A. *ibid.*, **11**, no. 4 (127), 1967 (in press).
21. Moser, J. *Math. Annalen*, 1967 (in press); *SIAM Rev.*, **8**, 145, 1966.
22. Jefferys, W. H., Moser, J. *Astr. J.*, **71**, 568, 1966.
23. Krasinski, G. A. *Bjull. Inst. Teor. Astr.*, **11**, no. 2 (125), 1967. (in press).
24. Moser, J. *Comm. Pure Appl. Math.*, **9**, 673, 1956; **11**, 81, 257, 1958; *Nachr. Akad. Wiss. Göttingen*, Math-Phys. Kl., **1955**, 87; **1962**, 1.
25. Gelfand, I. M., Lidskii, V. B. *Uspekhi Mat. Nauk*, **10**, no. 1 (63), 3, 1955 = *Amer. Math. Soc. Transl.*, (2), **8**, 143, 1958.
26. Segal, M. *Comm. Pure Appl. Math.*, **17**, 401, 1964.
27. Glimm, J. *Comm. Pure Appl. Math.*, **17**, 509, 1964.
28. Arnold, V. I. *Uspekhi Mat. Nauk*, **18**, 13, 91, 1963; = *Soviet Math. Survey*, **18**, no. 5, 9; no. 6, 85, 1963; *Doklady Mat.*, Akad. Nauk, **156**, 9, 1964.
cf. Janet, M. Proc. *IAU Symposium No.* 25, 33, Thessaloniki, 1964.
29. Salvadori, L. *Annali Math. Pura Appl.*, **69**, 1, 1965.
30. Pliss, V. A. *Doklady Mat.*, Akad. Nauk, **154**, 1044, 1964.
31. La Salle, J., Lefschetz, S. *Stability of Liapounov's Direct Method with Applications*, Acad. Press, New York, 1961.
32. Hahn, W. *Theorie und Anwendung der direkten Methode von Liapunov*, Springer, Berlin, 1959.
33. Cesari, L. *Asymptotic Behavior and Stability Problems in Ordinary Differential Equations*, Springer, Berlin, 1959.
34. Brayton, R. K., Moser, J. *Quart. J. Appl. Math.*, **22**, 9, 81, 1964.
35. Brayton, R. K., Miranker, W. L. *Archive Rat. Mech. Analysis*, **17**, 358, 1964.
36. Yoshizawa, T. *Mem. Coll. Sci.*, Univ. Kyoto, **33**, 301, 1960; *Contr. Diff. Eqs.*, **1**, 371, 1963; *Archive Rat. Mech. Analysis*, **6**, 409, 1960; **17**, 148, 1964.
37. Hale, J. K. *Contr. Diff. Eqs.*, **1**, 401, 1963; *Archive Rat. Mech. Analysis*, **15**, 289, 1964.
38. La Salle, J. P. *Annals of Math.*, **65**, 571, 1957.
39. Halkin, H. *Archive Rat. Mech. Analysis*, **17**, 272, 1964; *J. Math. Analysis Appl.*, **7**, 272, 1964; **10**, 325, 1965; *J. Analyse Math.*, **12**, 1, 1964.
40. Dearman, C. C., Le May, A. R. *Mathematische Methoden der Himmelsmechanik und Astronautik*, 213, 1966.
41. Fomin, V. M. *Doklady Mat.*, Akad. Nauk, **163**, 830, 1965.
42. Almkvist, G. *Pacific J. Math.*, **16**, 383, 1966.
43. Ura, T., Kimura, I. *Proc. Japan Acad.*, **40**, 703, 1964.
44. Gottschalk, W., Hedlund, G. Topological Dynamics, Amer. Math. Soc., 1955.
45. Ura, T. *Contr. Diff. Eqs.*, **3**, 249, 1965.
46. Schwartz, J. *Comm. Pure Appl. Math.*, **16**, 25, 1963.
47. Conley, C. C. *Comm. Pure Appl. Math.*, **17**, 237, 1964.
48. Minty, G. J. *Duke Math.* J., **29**, 341, 1962.
49. Browder, F. E. *Duke Math. J.*, **30**, 557, 1963.
50. Shinprot, M. *Archive Rat. Mech. Analysis*, **17**, 225, 1964.
51. Moser, J. *Nachr. Akad. Wiss. Göttingen*, Math. Phys. Kl., **1962**, no. 1. Cf., (**24**).
52. Arnold, V. I. *Uspekhi Mat.*, Akad. Nauk, **18**, no. 6, 91, 1963.
53. Birkhoff, G. D. *Acta Math.*, **43**, 1, 1920; *Annali Scuola Norm. Sup. Pisa*, (2) **4**, 267, 1935; **5**, 1, 1936 = *Collec. Math. Pap.*, II, 111, 466 and 668, Amer. Math. Soc., 1950.

https://doi.org/10.1017/S0251107X00030388 Published online by Cambridge University Press

**54.** Jefferys, W. H. *Astr. J.*, **71,** 306, 1966.
**55.** Diliberto, S. P. *Techn. Rep.*, Contract Nour 222 (88) and NR 041–255, 1966.
**56.** Diliberto, S. P. *Astr. J.*, **71,** 246, 1966.
**57.** Diliberto, S. P., Kyner, W. T., Freund, R. B. *Astr. J.*, **66,** 118, 1961.
**58.** Diliberto, S. P. *Naval Ordnance Lab. Rep.*, 6445, 1959.
**59.** Haseltine, W. R. *Astr. J.*, **68,** 522, 1963.
**60.** Dowker, Y. N. *Duke Math. J.*, **14,** 105, 1947.
**61.** Brunk, H. D. *Trans. Amer. Math. Soc.*, **78,** 482, 1955.
**62.** Blum, J. R., Hanson, D. L. *Proc. Amer. Math. Soc.*, **16,** 413, 1965.
**63.** Chacon, R. V., Ornstein, D. S. *Illinois J. Math.*, **4,** 153, 1960.
**64.** Furstenberg, H. *Amer. J. Math.*, **83,** 573, 1961.
**65.** Yosida, K. *Functional Analysis*, Springer, Berlin, 1966.
**66.** Arnold, V. I. *C. R. Acad. Sci.*, Paris, **261,** 3719, 1965.
**67.** Arnold, V. I. *Doklady Akad. Nauk*, **161,** no. 1, 9, 1965.
**68.** Arnold, V. I. *Uspekhi Mat. Nauk*, **18,** no. 5 (113), 13; no. 6 (114), 91, 1963. Cf., (28).
**69.** Arnold, V. I. *C. R. Acad. Sci.*, Paris, **260,** 5668, 1965; **261,** 17, 1965; *Ann. Inst. Fourier*, Univ. Grenoble, **16,** 319, 1966.
**70.** Kyner, W. T. Proc. *IAU Symposium No.* 25, 26, Thessaloniki, 1964.

Y. HAGIHARA
*President of the Commission*