I-SUBSEMIGROUPS AND α-MONOMORPHISMS

Dedicated to the memory of Hanna Neumann

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1. Introduction

To each idempotent v of a semigroup T, there is associated, in a natural way, a subsemigroup T_v of T. The subsemigroup T_v is simply the collection of all elements of T for which v acts as a two-sided identity. We refer to such a subsemigroup as an I-subsemigroup of T. We first establish some elementary properties of these subsemigroups with no restrictions on the semigroup in which they are contained. Then we turn our attention to the semigroup of all continuous selfmaps of a topological space. The I-subsemigroups of these semigroups are investigated in some detail and so are the α-monomorphisms [3, p. 518] from one such semigroup into another. Among other things, a relationship is established between I-subsemigroups and α-monomorphisms. An analogous theory exists for semigroups of closed selfmaps on topological spaces. A number of results are listed for these semigroups with the proofs often deleted since, in many cases, the situation is much the same as for semigroups of continuous functions.

2. Some elementary properties of I-subsemigroups

Let T be any semigroup and let v be any idempotent of T. It is an easy matter to show that $T_v = \{a \in T : av = va = a\}$ is a subsemigroup of T and is, in fact, the largest subsemigroup for which v acts as a two-sided identity. As we mentioned before, we refer to T_v as an I-subsemigroup of T. Clifford and Preston observe in [1, p. 22] that vT consists of all elements of T for which v is a left identity and, of course, Tv consists of all elements of T for which v is a right identity. Consequently

$$T_v = vT \cap Tv = vTv.$$

Moreover, vT is just the principal right ideal of T generated by v and, similarly, Tv is the principal left ideal which is generated by v. Thus, each I-subsemigroup of T is the intersection of the principal right ideal and the principal left ideal which are generated by some idempotent of T.

Our first result gives a sufficient condition that one I-subsemigroup be embeddable in another.

LEMMA 2.1. *Let v and w be any two idempotents of a semigroup T and suppose that $wv = v$. Then the mapping which sends $a \in T_v$ into aw is a monomorphism from T_v into T_w.*

PROOF. First, we note that if $a \in T_v$, then $av = va = a$ and we have

(2.1.1) $$w(aw) = w(va)w = (wv)aw = vaw = aw = (aw)w.$$

Thus, w is a two-sided identity for aw and we have shown that the mapping ϕ defined $\phi(a) = aw$ for $a \in T_v$ is a mapping from T_v into T_w.
Using (2.1.1) again, we get

$$\phi(a)\phi(b) = awbw = a(w(bw)) = a((bw)w) = ab(ww) = abw = \phi(ab)$$

for all $a, b \in T_v$. To complete the proof, we need only show that ϕ is injective. Suppose $aw = bw$. Then

$$a = av = awv = bwv = bv = b.$$

PROPOSITION 2.2. *Let v and w be any two idempotents of a semigroup T. Then the mapping which sends $a \in T_v$ into aw is an isomorphism from T_v onto T_w if and only if $vw = w$ and $wv = v$.*

PROOF. (Sufficiency) Define $\phi(a) = aw$ and $\psi(b) = bv$ for $a \in T_v$ and $b \in T_w$. Then

$$(\phi \circ \psi)(b) = bvw = bw = b.$$

In a similar manner, $(\psi \circ \phi)\ (a) = a$ for each $a \in T_v$. These facts, together with the previous lemma imply that ϕ is an isomorphism from T_v onto T_w.

(Necessity) Now suppose the mapping ϕ defined by $\phi(a) = aw$ is an isomorphism from T_v onto T_w. Since v is the identity of T_v and w is the identity of T_w, we immediately get

(2.2.1) $$vw = \phi(v) = w.$$

We must show that $wv = v$ as well. First, we show that $wv \in T_v$. It is immediate that $(wv)v = wv$. Moreover, from (2.2.1), we get $v(wv) = (vw)v = wv$. Thus v is a two-sided identity for vw. That is to say, $vw \in T_v$. Now since ϕ is injective and $\phi(v) = w$, it is sufficient to show that $\phi(wv) = w$ in order to complete the proof. We again use (2.2.1) and get

$$\phi(wv) = wvw = ww = w.$$

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We eventually want to investigate in some detail the I-subsemigroups of semigroups of continuous selfmaps. As we mentioned before, there is a close connection between the I-subsemigroups of these semigroups and certain monomorphisms that we have considered previously. These are the α-monomorphisms which were introduced in [3]. In the course of these investigations two new classes of topological spaces will emerge naturally and in order to have at our disposal some helpful information about them we define them and discuss a few elementary facts concerning them in the next section.

3. RS^* spaces and conformable spaces

An S^*-space was defined in [6] to be any T_1 space X with the property that if $F \subset X$ is closed and $p \in X - F$, then there exists a continuous selfmap f of X and a point $q \in X$ such that $f(x) = q$ for $x \in F$ and $f(p) \neq q$. This is equivalent to requiring that point inverses form a basis for the closed subsets of X. It was shown in [6] that every completely regular Hausdorff space which contains an arc is an S^*-space and also that every 0-dimensional Hausdorff space is an S^*-space. By a 0 dimensional space, we mean one whose topology has a basis of sets which are both closed and open. Our next definition singles out a subclass of the class of S^*-spaces.

DEFINITION 3.1. A space X is an RS^*-space if each retract of X (i.e., subspace which is the range of an idempotent continuous selfmap) is an S^*-space.

Since every space is the range of its identity map, it is immediate that every RS^*-space is also an S^*-space. The next result shows that RS^*-spaces are fairly abundant.

THEOREM 3.2. *The class of RS^*-spaces includes all 0-dimensional Hausdorff spaces as well as all completely regular, Hausdorff, locally arcwise connected spaces.*

PROOF. By Theorem 2 of [6, p. 296], every 0-dimensional Hausdorff space is an S^*-space. This, together with the fact that the property of being 0-dimensional and Hausdorff is hereditary, implies that every 0-dimensional Hausdorff space is an RS^*-space.

Now suppose X is completely regular, Hausdorff and locally arcwise connected and let V be the range of an idempotent continuous selfmap v of X. We must show that V is an S^*-space in the topology it inherits from X. Let $\{H_\alpha : a \in \Lambda\}$ denote the collection of all components of X. Then each H_α is open (as well as closed) and arcwise connected. Thus $v(H_\alpha)$ is arcwise connected and it follows that $v(H_\alpha)$ is either a point or contains an arc for each $\alpha \in \Lambda$. We consider two cases:

Case 1: Some $v(H_\alpha)$ contains an arc.

Case 2: No $v(H_\alpha)$ contains an arc.

If Case 1 holds, the conclusion is immediate since in that case, V is a completely regular Hausdorff space which contains an arc and by Theorem 3 of [6, p. 296], every such space is an S^*-space. Suppose, on the other hand, Case 2 holds and let p be any point of V. We show that p is isolated in V. Let H_{α_p} be the component to which p belongs. Since Case 2 holds, $v(H_{\alpha_p})$ is a point and since v is idempotent, $v(p) = p$. Thus $v(x) = p$ for each $x \in H_{\alpha_p}$. Then for any $q \in H_{\alpha_p} - \{p\}$, we have $v(q) \neq q$. This implies that $q \notin V$ since v is identity on V. Hence $V \cap H_{\alpha_p} = \{p\}$ and since H_{α_p} is open in X, it follows that p is isolated in V. Thus, we see that when Case 2 holds, V is discrete and is therefore an S^*-space.

It is not difficult to produce an example of an S^*-space which is not an RS^*-space. In [2], de Groot proves the existence of 1-dimensional subspaces of the Euclidean plane with the property that the only continuous selfmaps are the constant maps and the identity function. None of these spaces are S^*-spaces. Denote any one of them by H and let X be the free union of H and an arc. By Theorem 3 of [3, p. 296], X is an S^*-space. However, X is not an RS^*-space since H is a retract of X and is not an S^*-space.

DEFINITION 3.3. A topological space X is said to be conformable if it is a first countable S^*-space and for each pair of compact, countable subspaces A and B with both having exactly one imit point, there exists a continuous selfmap f of X mapping A into B such that $B - f(A)$ is finite.

Here, a locally Euclidean space will be a Hausdorff space with the property that each point belongs to a neighborhood which is homeomorphic to some Euclidean N-space and we make no requirement that the dimensions of all these neighborhoods be the same.

THEOREM 3.4. *Every locally Euclidean space is conformable.*

PROOF. It is evident that a locally Euclidean space X is first countable. Furthermore it is completely regular (in fact it is locally compact) and contains an arc and, as we noted previously, such spaces are S^*-spaces. Now let A and B be two compact, countable subspaces of X and suppose each has exactly one limit point. Denote the limit point of A by p and the limit point of B by q. Since X is locally Euclidean, there exists a neighborhood H of q which is homeomorphic to one of the Euclidean N-spaces and since B is compact, $B - H$ is finite. Now let f be any bijection from A onto $B \cap H$ which carries the point p onto q. Then f is a continuous mapping from A into H. We let αX denote any compactification of X and note that since A is compact, the function f maps a closed subset of the normal space αX into H which is an absolute retract for normal spaces. Thus, f can be extended to a continuous function which maps αX into H. The restriction

g of this function to X is a continuous selfmap of X with the property that $B - g(A)$ is finite and this proves that X is conformable.

THEOREM 3.5. *Every 0-dimensional metric space is conformable.*

PROOF. Let X be any such space. As we noted previously, any 0-dimensional Hausdorff space is an S^*-space and, of course, X is first countable. Now let A and B be compact, countable subspaces of X with unique limit points p and q respectively. Denote the points of $A - \{p\}$ by $\{a_n\}_{n=1}^{\infty}$ and inductively choose a sequence $\{G_n\}_{n=1}^{\infty}$ of mutually disjoint clopen (i.e., both closed and open) subsets of X with finite diameter such that for each n, $G_n \cap A = \{a_n\}$ and the sequence $\{d_n\}_{n=1}^{\infty}$ converges to zero where d_n is the diameter of G_n. Denote the points of $B - \{q\}$ by $\{b_n\}_{n=1}^{\infty}$ and define a selfmap f of X by

$$f(x) = b_n \text{ for } x \in G_n$$

and

$$f(x) = g \text{ for } x \in X - \cup \{G_n\}_{n=1}^{\infty}.$$

It is immediate that f is continuous at each point of each G_n. Let

$$H = (\cup \{G_n\}_{n=1}^{\infty}) \cup \{p\}.$$

Since the sequence $\{d_n\}_{n=1}^{\infty}$ of diameters converges to zero, it follows that H is closed and this implies that f is continuous at every point of $X - H$. We have yet to check continuity at the point p. Let V be any neighborhood of the point q. Then $b_n \in V$ for n larger than some N. Now choose a neighborhood W of p such that $W \cap G_n = \emptyset$ for $n \leqq N$. It follows that $f(W) \subset V$ and hence that f is continuous. This completes the proof since $f(A) = B$.

THEOREM 3.6. *Let X be a compact metric space with an infinite number of isolated points. Then X is conformable if and only if it is 0-dimensional.*

PROOF. Sufficiency follows immediately from the previous theorem. Now suppose that X is conformable. Since X is compact and has an infinite number of isolated points, there exists a sequence $\{b_n\}_{n=1}^{\infty}$ converging to a point q where each b_n is isolated in X. Let $B = (\cup \{b_n\}_{n=1}^{\infty}) \cup \{q\}$ and let V be any connected subset of X. Assume V has more than one point. Then V contains a sequence $\{a_n\}_{n=1}^{\infty}$ which converges to some point $p \in V$. Let $A = (\cup \{a_n\}_{n=1}^{\infty}) \cup \{p\}$ and since X is conformable there exists a continuous selfmap f of X such that $B - f(A)$ is finite. Hence $f(V)$ contains isolated points which is a contradiction since V is connected. Thus, the only connected subsets of X are the singletons. Since any compact totally disconnected space is 0-dimensional, the proof is complete.

4. α-homomorphisms revisited

In our discussion of semigroups of continuous functions, we will need some results which can be proven for rather general semigroups of functions. It seems to be convenient to collect them in a separate section. We begin by recalling three definitions from [3].

DEFINITION 4.1. Let X be any set and let $F(X)$ be the semigroup, under composition, of all selfmaps of X. An α-semigroup is any subsemigroup of $F(X)$ which contains the identity function and all the constant functions.

DEFINITION 4.2. Let S be an arbitrary semigroup. A Z-subsemigroup of S is any subsemigroup T with identity v such that if $vz = z$ for any left zero z of S, then $z \in T$.

DEFINITION 4.3. A homomorphism from a semigroup S into a semigroup T is an α-homomorphism if the image of S is a Z-subsemigroup of T. If the α-homomorphism is injective, we refer to it as an α-monomorphism.

In our first result of this section, we prove a lemma which will play a fundamental role in our discussion of semigroups of continuous functions and also in our discussion of semigroups of closed functions. Part of the result was almost proved in [3] as Theorem (2.3) but not quite. Theorem (2.3) of that paper tells us that if ϕ is an α-monomorphism from $\alpha(X)$ into $\alpha(Y)$ where the latter are α-semigroups of functions on X and Y respectively, then there exists a function j from X into Y and a function k from Y into X such that

(i) $$k \circ j = i_X, \text{ the identity function on } X$$

and

(ii) $$\phi(f) = j \circ f \circ k \text{ for all } f \in \alpha(X).$$

We use this to prove the

FUNDAMENTAL LEMMA (4.4.) *Let $\alpha(X)$ and $\alpha(Y)$ be any two α-semigroups of functions on the sets X and Y respectively and let ϕ be an α-monomorphism from $\alpha(X)$ into $\alpha(Y)$. Then there exists an idempotent element v of the semigroup $\alpha(Y)$ and a bijection h from the range V of v onto X such that the following three conditions are satisfied*:

(4.4.1) $$\phi(f) = h^{-1} \circ f \circ h \circ v \textit{ for each } f \in \alpha(X),$$

(4.4.2) $$h^{-1}(f^{-1}(x)) = V \cap (\phi(f))^{-1}(h^{-1}(x)) \textit{ for each } x \in X \textit{ and } f \in \alpha(X),$$

(4.4.3) $$h^{-1}(f(X)) = \phi(f)(V) \textit{ for each } f \in \alpha(X).$$

Moreover, the bijection h and the idempotent v are unique in the sense that if k and w are any two functions such that $\phi(f) = k^{-1} \circ f \circ k \circ w$ *for each* $f \in \alpha(X)$, *then* $k = h$ *and* $w = v$.

PROOF. From (ii), we conclude the existence of a function j from X into Y and a function k from Y into X such that $\phi(f) = j \circ f \circ k$ for all $f \in \alpha(X)$. Furthermore, it is immediate from (i) that j is injective and we set $h = j^{-1}$. Since $\phi(i_X) = j \circ k = h^{-1} \circ k$ and i_X is idempotent, then $h^{-1} \circ k$ is an idempotent element of $\alpha(Y)$. We put $h^{-1} \circ k = v$ and get $k = h \circ v$. It follows that $\phi(f) = h^{-1} \circ f \circ h \circ v$ for all f in $\alpha(X)$. Moreover, since $h = j^{-1}$, the domain $\mathfrak{D}(h)$ of h is the range $\mathfrak{R}(j)$ of j. Thus,

$$\mathfrak{D}(h) = \mathfrak{R}(j) = \mathfrak{R}(j \circ k) = \mathfrak{R}(v) = V$$

and since j is injective from X onto V, the function h must be injective from V onto X.

Now we verify (4.4.2). For any point $a \in X$, the symbol $\langle a \rangle$ will denote the constant function which maps all X into a. The same sort of notation will be used to denote constant functions on Y but it will be clear from context whether the domain of the function is X or Y. We will need to use the fact that

$$\phi\langle a \rangle = \langle h^{-1}(a) \rangle \tag{4.4.4}$$

for each $a \in X$. This follows immediately from (4.4.1). Now suppose that $b \in h^{-1}(f^{-1}(x))$. Then $b \in V$ and $b = h^{-1}(a)$ for some $a \in f^{-1}(x)$. Thus, $f(a) = x$ which implies that $f \circ \langle a \rangle = \langle x \rangle$. Using this and (4.4.4), we get

$$\begin{aligned} \langle h^{-1}(x) \rangle &= \phi\langle x \rangle = \phi(f \circ \langle a \rangle) \\ &= \phi(f) \circ \phi\langle a \rangle = \phi(f) \circ \langle h^{-1}(a) \rangle \end{aligned}$$

$$\phi(f) \circ \langle b \rangle = \langle \phi(f)(b) \rangle.$$

It follows from this that $\phi(f)(b) = h^{-1}(x)$ and this implies that

$$b \in V \cap (\phi(f))\ ^{1}(h^{-1}(x)).$$

To prove the reverse inclusion, suppose that

$$c \in V \cap (\phi(f))^{-1}(h^{-1}(x)).$$

Then $\phi(f)(c) = h^{-1}(x)$ and since c belongs to the domain of h, we may write $c = h^{-1}(h(c))$. Using these facts and also (4.4.4), we get

$$\begin{aligned} \phi\langle x \rangle = \langle h^{-1}(x) \rangle &= \langle \phi(f)(c) \rangle \\ &= \langle \phi(f)(h^{-1}(h(c))) \rangle = \phi(f) \circ \langle h^{-1}(h(c)) \rangle \\ &= \phi(f) \circ \phi\langle h(c) \rangle = \phi\langle f(h(c)) \rangle. \end{aligned}$$

Since ϕ is injective, this implies that $\langle x\rangle = \langle f(h(c))\rangle$ which, in turn, implies that $f(h(c)) = x$. Thus, $c \in h^{-1}(f^{-1}(x))$ and (4.4.2) has been verified. Since the verification of (4.4.3) follows by using similar techniques, we omit it.

We conclude this section with a result which was essentially proved in [3] although it is stated here in considerably more generality. The result shows that in many cases one really gains very little by considering α-homomorphisms in place of α-monomorphisms since in these cases α-homomorphisms are either injective or they map everything onto a single left zero.

DEFINITION 4.5. A semigroup $\delta(X)$ of selfmaps of X is said to be doubly transitive if for points $p, q, x, y \in X$ with $p \neq q$, then there exists a function f in $\delta(X)$ with $f(p) = x$ and $f(q) = y$.

THEOREM 4.6. *Let $\alpha(X)$ be any doubly transitive α-semigroup and let ϕ be any α-homomorphism from $\alpha(X)$ into an α-semigroup $\alpha(Y)$. Then either ϕ is injective or it maps everything onto a single left zero of $\alpha(Y)$.*

PROOF. Suppose that ϕ does not map everything onto a single left zero of $\alpha(Y)$. It readily follows from this that $\phi(i_X)$ is not a left zero where i_X is the identity of $\alpha(X)$. Since the left zeros coincide with the constant functions, this simply means that $\phi(i_X)$ is a nonconstant idempotent and hence has at least two different elements p and q in its range. Then $\phi(i_X)$ is an identity for the constant functions $\langle p\rangle$ and $\langle q\rangle$ and since ϕ is an α-homomorphism, it follows that both $\langle p\rangle$ and $\langle q\rangle$ must belong to $\phi(\alpha(X))$. One now completes the proof exactly as in the proof of Theorem 4.1 of [3, p. 523].

5. Semigroups of continuous functions

We are now in a position to examine in some detail the I-subsemigroups and α-monomorphisms of semigroups of continuous functions. The symbol $S(X)$ denotes the semigroup, under composition, of all continuous selfmaps of the topological space X. Our first result shows, among other things that each I-subsemigroup of $S(X)$ is isomorphic to $S(Y)$ for an appropriately chosen space Y.

THEOREM 5.1. *Let v be any idempotent of $S(X)$ with range V. Then the mapping which sends f in $S(V)$ into $f\circ v$ is an isomorphism from $S(V)$ onto the I-subsemigroup $S(X)_v$.*

PROOF. Since v is idempotent, it is the identity map on V. Thus,

$$v\circ(f\circ v) = f\circ v = (f\circ v)\circ v$$

for each $f \in S(V)$ which implies that $f\circ v \in S(X)_v$. Hence, the mapping ϕ defined by $\phi(f) = f\circ v$ does indeed map $S(V)$ into $S(X)_v$. The fact that v is the identity on V also implies that

$$f \circ v \circ g \circ v = f \circ g \circ v$$

for all $f, g \in S(V)$ which means that ϕ is a homomorphism. It is immediate that it is injective. To show that it maps $S(V)$ onto $S(X)_v$, let $g \in S(X)_v$ be given. Then by definition,

$$g \circ v = v \circ g = g.$$

It readily follows that g maps V into V. In fact, it follows that g maps all of X into V. Let f be the restriction of g to V. Then $g \in S(V)$ and one easily shows that $\phi(f) = g$.

Our next result involves S^*-spaces and RS^*-spaces which were discussed in Section 3. It gives algebraic conditions on the semigroups which are both necessary and sufficient for embedding an S^*-space into an RS^*-space as a retract.

THEOREM 5.2. *Let X be an S^*-space and let Y be an RS^*-space. Then $S(X)$ is isomorphic to an I-subsemigroup of $S(Y)$ if and only if X is homeomorphic to a retract of Y.*

PROOF. First, suppose X is homomorphic to a retract V of Y. Then $S(X)$ is isomorphic to $S(V)$ and by Theorem (5.1), $S(V)$ is isomorphic to an I-subsemigroup of $S(Y)$.

Now suppose that $S(X)$ is isomorphic to an I-subsemigroup $S(Y)_v$ of $S(Y)$ and denote the range of v by V. By Theorem (5.1), $S(Y)_v$ is isomorphic to $S(V)$ and hence, $S(X)$ is isomorphic to $S(V)$. Since both X and V are S^*-spaces, it follows from Theorem 1 of [6, p. 295] that X is homeomorphic to V.

Now we begin our discussion of α-monomorphisms for semigroups of continuous functions and we eventually relate these to the I-subsemigroups of such semigroups. The next result follows very quickly from the Fundamental Lemma.

THEOREM 5.3. *Let X be an S^*-space, let Y be a T_1 space and let ϕ be an α-monomorphism from $S(X)$ into $S(Y)$. Then there exists a unique idempotent v of $S(Y)$ and a unique continuous bijection h from the range V of v onto X such that*

$$\phi(f) = h^{-1} \circ f \circ h \circ v \tag{5.3.1}$$

for each f in $S(X)$.

PROOF. Since $S(X)$ and $S(Y)$ are both α-semigroups, the existence of a unique idempotent v of $S(Y)$ and a unique bijection h from V onto X satisfying (5.3.1) follows immediately from the Fundamental Lemma (4.4). The fact that h is continuous follows just about as quickly. As we observed in Section 3, preimages of points under continuous selfmaps form a basis for the closed subsets of any

S^*-space. This, together with (4.4.2) and the fact that Y is a T_1 space guarantees us that h is continuous.

COROLLARY 5.4. *Let X be an S^*-space and Y a T_1 space and suppose that there exists an α-monomorphism from $S(X)$ into $S(Y)$. Then X is a continuous image of Y.*

PROOF. In the previous theorem, $h \circ v$ maps Y continuously onto X.

It is appropriate to remark at this point that Theorem (5.3) does not completely determine the α-monomorphisms from $S(X)$ into $S(Y)$. It tells us that each α-monomorphism uniquely determines a pair of functions v and h satisfying certain conditions which include (5.3.1). However, there are such functions which do not result in an α-monomorphism from $S(X)$ into $S(Y)$. To be somewhat more specific, it is not difficult to produce an S^*-space X, a T_1 space Y, a continuous idempotent v on Y and a continuous bijection h from V onto X such that the mapping ϕ defined by (5.3.1) is not an α-monomorphism. The difficulty is that even though f is continuous on X, $\phi(f)$ may not be continuous on Y. If, however one takes h to map V homeomorphically onto X rather than just continuously, the resulting function ϕ will map $S(X)$ into $S(Y)$ and it is routine to show that it is an α-monomorphism. Consequently, if we could show that the continuous bijection h of Theorem (5.3) must actually be a homeomorphism, then the α-monomorphisms would be completely determined. The next example shows however, that we cannot hope to do this is general.

EXAMPLE 5.5. Let X be any nondiscrete space and let Y be the free union of a collection $\{Z_\alpha: \alpha \in \Delta\}$ of spaces where card $\Lambda =$ card X. Choose $p_\alpha \in Z_\alpha$ for each α and define $v(z) = p_\alpha$ for each $z \in Z_\alpha$. Then v is a continuous idempotent on Y and its range V is discrete. Since card $V =$ card X, we can choose a bijection h from V onto X. It will be continuous but not a homeomorphism since X is not discrete. Now define a mapping ϕ by

$$\phi(f) = h^{-1} \circ f \circ h \circ v.$$

The assertion is that ϕ is an α-monomorphism from $S(X)$ into $S(Y)$. To see this, factor ϕ into two mappings θ and ψ. That is, define

$$\psi(f) = h^{-1} \circ f \circ h$$

for each f in $S(X)$ and define

$$\theta(g) = g \circ v$$

for each $g \in S(V)$. Since V is discrete every selfmap of V is continuous so ψ does map $S(X)$ into $S(V)$. Moreover, Theorem (5.1) assures us that θ maps $S(V)$ into $S(Y)$. Consequently, $\phi = \theta \circ \psi$ maps $S(X)$ into $S(Y)$. One then verifies easily that ϕ is an α-monomorphism.

The next several results are concerned with cases where the bijection h of Theorem (5.3) must necessarily be a homeomorphism. In such cases, of course, we will have completely determined the α-monomorphisms. The first result follows quickly from Theorem (5.3).

THEOREM 5.6. *Let X be a Hausdorff S^*-space and let Y be a compact T_1 space. Then for each α-monomorphism ϕ from $S(X)$ into $S(Y)$ there exists a unique idempotent v of $S(Y)$ and a unique homeomorphism h from the range V of v onto X such that*

$$\phi(f) = h^{-1} \circ f \circ h \circ v \tag{5.6.1}$$

for each f in $S(X)$.

PROOF. The existence (and uniqueness) of the continuous idempotent v and the continuous bijection h is an immediate consequence of Theorem (5.3). But since V is the continuous image of Y under v it must also be compact and since X is Hausdorff, the continuous bijection h must actually be a homeomorphism.

The next result shows that there are a number of instances where Y is not compact but h must still be a homeomorphism.

THEOREM 5.7. *Let X be any conformable space and let Y be any first countable T_1 space which is not the union of an infinite number of mutually disjoint nonempty open subsets. Then for each α-monomorphism ϕ from $S(X)$ into $S(Y)$, there exists a unique idempotent v of $S(Y)$ and a unique homeomorphism h from the range V of v onto X such that*

$$\phi(f) = h^{-1} \circ f \circ h \circ v \tag{5.7.1}$$

for each f in $S(X)$.

PROOF. The existence, uniqueness and continuity of the functions v and h follow immediately from Theorem (5.3). To complete the proof, we need only show that h^{-1} is continuous and since X is T_1, we can do this using sequences. Suppose $\{a_n\}_{n=1}^{\infty}$ is a sequence in X which converges to a point $p \in X$. We must show that $\lim h^{-1}(a_n) = h^{-1}(p)$. Now since Y is not the union of an infinite family of mutually disjoint nonempty open subsets any discrete continuous image of Y must be finite. Consequently, the subspace V must either be finite or it contains a limit point q. In the former case, it follows immediately that h is a homeomorphism so we may assume that V has a limit point q. Furthermore, since Y is first countable, there exists an infinite sequence of distinct points in V converging to q. In other words, V contains a compact, countable subset B with exactly one limit point q. Since h is continuous and injective, $h(B)$ is a compact, countable subset of X with exactly one limit point $h(q)$. Denote the points of the sequence $\{a_n\}_{n=1}^{\infty}$ together with its limit point by A. Then A is also a compact, countable subset of X with exactly one limit point and since X is conformable, there exists an element f in $S(X)$ mapping $h(B)$ into A such that $A - f(h(B))$ is finite. Thus, there is a positive integer N such that $a_n \in f(h(B))$ for each $n > N$. For each such n, choose $b_n \in B$ such that

$$f(h(b_n)) = a_n \tag{5.7.2}$$

then $\{b_n\}_{n=1}^{\infty}$ is an infinite sequence of distinct points of B and must therefore converge to the unique limit point q of B. Finally, since v is idempotent, it is the identity map on V and hence on B. It follows from this and (5.7.1) and (5.7.2) that

$$h^{-1}(a_n) = h^{-1} \circ f \circ h \circ v(b_n) = \phi(f)(b_n) \tag{5.7.3}$$

for $n > N$. Since $\phi(f)$ is continuous and $\{b_n\}_{n=1}^{\infty}$ converges to q, the sequence $\{\phi(f)(b_n)\}_{n=1}^{\infty}$ must converge to $\phi(f)(q)$.

But

$$\phi(f)(q) = h^{-1} \circ f \circ h \circ v(q) = h^{-1}(f(h(q))) = h^{-1}(p). \tag{5.7.4}$$

We have used the fact that f maps the unique limit point $h(q)$ of $h(B)$ into the unique limit point p of A. This is a consequence of the fact that $A - f(h(B))$ is finite. Thus, (5.7.3) and (5.7.4) together imply that $\lim h^{-1}(a_n) = h^{-1}(p)$ and hence that h^{-1} is continuous. This completes the proof.

We remark that in view of Example (5.5), one cannot hope to prove Theorem (5.7) if one deletes the requirement that Y must not be the union of an infinite number of mutually disjoint nonempty open subsets.

The next several results relate α-monomorphisms to I-subsemigroups.

THEOREM 5.8. *Let X be a Hausdorff S^*-space and let Y be a compact T_1 space. Then a monomorphism ϕ from $S(X)$ into $S(Y)$ is an α-monomorphism if and only if the image of $S(X)$ under ϕ is an I-subsemigroup of $S(Y)$.*

PROOF. (Sufficiency). Suppose ϕ is a monomorphism which maps $S(X)$ onto some I-subsemigroup $S(Y)_v$ of $S(Y)$. In order to conclude that ϕ is an α-monomorphism, we must show that $S(Y)_v$ is a Z-subsemigroup of $S(Y)$. But this is an immediate consequence of Definition (4.2) and the fact that $S(Y)_v$ is the family of all functions in $S(Y)$ for which v serves as a two-sided identity.

(Necessity). Now let ϕ be an α-monomorphism from $S(X)$ into $S(Y)$. According to Theorem (5.6) there exists an idempotent v of $S(Y)$ and a homeomorphism h from the range V of v onto X such that

$$\phi(f) = h^{-1} \circ f \circ h \circ v \tag{5.8.1}$$

for each f in $S(X)$. Now the mapping ψ which sends $f \in S(X)$ into $h^{-1} \circ f \circ h$ is an isomorphism from $S(X)$ onto $S(V)$ and the mapping θ which sends $g \in S(V)$ into $g \circ v$ is, according to Theorem (5.1), an isomorphism from $S(V)$ onto $S(Y)_v$. Thu $\phi = \theta \circ \psi$ is a monomorphism from $S(X)$ onto $S(Y)_v$ and the proof is complete.

The proof of the following result is deleted since it is identical to the proof of the preceeding theorem with the exception that one appeals to Theorem (5.7) in place of Theorem (5.6).

THEOREM 5.9. *Let X be any conformable space and let Y be any first countable T_1 space which is not the union of an infinite number of mutually disjoint nonempty open subsets. Then a monomorphism ϕ from $S(X)$ into $S(Y)$ is an α-monomorphism if and only if the image of $S(X)$ under ϕ is an I-subsemigroup of $S(Y)$.*

The next result follows immediately from Theorems (5.8) and (5.2) and the one after that follows immediately from Theorem (5.9) and (5.2).

THEOREM 5.10. *Let X be a Hausdorfi S*-space and let Y be a compact RS*-space. Then the following statements are equivalent.*

(5.10.1) *There exists an α-monomorphism from $S(X)$ into $S(Y)$.*

(5.10.2) *$S(X)$ is isomorphic to an I-subsemigroup of $S(Y)$.*

(5.10.3) *X is homeomorphic to a retact of Y.*

THEOREM 5.11. *Let X be any conformable space and let Y be any first countable RS*-space which is not the union of an infinite number of mutually disjoint nonempty open subsets. Then the following statements are equivalent.*

(5.11.1) *There exists an α-monomorphism from $S(X)$ into $S(Y)$.*

(5.11.2) *$S(X)$ is isomorphic to an I-subsemigroup of $S(Y)$.*

(5.11.3) *X is homeomorphic to a retract of Y.*

REMARK: Theorem (4.6) tells us that if an α-semigroup is doubly transitive then those α-homomorphisms mapping it into another α-semigroup are of a rather trivial nature if they are not injective. Those topological spaces X for which $S(X)$ is doubly transitive are called V-spaces in [3]. Among other things, it is shown that every completely regular, Hausdorff, arcwise connected space is a V-space as well as every 0-dimensional Hausdorff space. For such spaces, nothing it lost by restricting one's attention to α-monomorphisms rather than to α-homomorphisms.

6. Semigroups of a-monomorphisms

In this section we prove that the composition of any two α-monomorphisms is an α-monomorphism and we use the results of the previous section to describe the semigroup of all α-monomorphisms on various semigroups of continuous functions.

LEMMA 6.1. *Let S, T and L be arbitrary semigroups, let ϕ be an α-monomorphism from S into T and let θ be an α-monomorphism from T into L. Then $\theta \circ \phi$ is an α-monomorphism from S into L.*

PROOF. The existence of ϕ and θ implies that both S and T have identities and we denote these by i and j respectively. Now suppose that

$$((\theta \circ \phi))(i))z = z \tag{6.1.1}$$

for some left zero z of L. We must show that $z \in (\theta \circ \phi)(S)$. First of all, we have $j\phi(i) = \phi(i)$ which implies that

$$\theta(j)\theta(\phi(i)) = \theta(\phi(i)). \tag{6.1.2}$$

This, together with (6.1.1) results in

$$(\theta(j))z = z \tag{6.1.3}$$

and since θ is an α-monomorphism, it follows that $\theta(a) = z$ for some $a \in T$. Moreover, since z is a left zero of $\theta(T)$ and θ is an isomorphism from T onto $\theta(T)$, we conclude that a is a left zero of T. Next, we show that

$$(\phi(i))a = a \tag{6.1.4}$$

Since θ is injective and $\theta(a) = z$, this can be accomplished by showing that $\theta((\phi(i))a) = z$. We use (6.1.1) and get

$$\theta((\phi(i))a) = \theta(\phi(i))\theta(a) = (\theta(\phi(i)))z = z.$$

Thus, (6.1.4) is verified and since ϕ is an α-monomorphism, it follows that $\phi(b) = a$ for some $b \in S$. Then $(\theta \circ \phi)(b) = z$ and the proof is complete.

It follows from Lemma (6.1) that the collection of all α-monomorphisms mapping a semigroup T with identity into itself is a semigroup under composition. Moreover, there is no possibility of this collection being empty since every automorphism of T is an α-monomorphism. We look a bit closer at the semigroup of all α-monomorphisms on $S(X)$. First, let X be an arbitrary topological space and let $E(X)$ denote the collection of all pairs (v, h) where v is an idempotent continuous selfmap of X and h is a homeomorphism whose domain is the range V of v and whose range is all of X. Let (v, h) and (w, k) be two such pairs. It is a routine matter to show that $h^{-1} \circ w \circ h \circ v$ is an idempotent element of $S(X)$ whose range is $h^{-1}(W)$ where W is the range of w. Since the domain of $k \circ h$ is also $h^{-1}(W)$ and its range is X, the pair

$$(h^{-1} \circ w \circ j \circ v,\ k \circ h)$$

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belongs to $E(X)$. Thus the multiplication defined by

$$(v, h)(w, k) = (h^{-1} \circ w \circ h \circ v,\ k \circ h)$$

is an associative binary operation on $E(X)$ and from now on when we speak of $E(X)$ we mean the semigroup with this multiplication.

THEOREM 6.2. *Let X be a compact Hausdorff S^*-space. Then the semigroup of all α-monomorphisms mapping $S(X)$ into $S(X)$ is isomorphic to the semigroup $E(X)$.*

PROOF. Let T denote the semigroup of all α-monomorphisms mapping $S(X)$ into $S(X)$ and let ϕ be an element of T. According to Theorem (5.6), there exists a unique idempotent v in $S(X)$ and a unique homeomorphism h from the range V of v onto X such that

$$\phi(f) = h^{-1} \circ f \circ h \circ v \tag{6.2.1}$$

for each $f \in S(X)$. We define a mapping Φ from T into $E(X)$ by

$$\Phi(\phi) = (v, h). \tag{6.2.2}$$

Suppose that $\Phi(\phi) = (v, h)$ and $\Phi(\theta) = (w, k)$. This means that (6.2.1) holds for each f in $S(X)$ and also that

$$\theta(f) = k^{-1} \circ f \circ k \circ w \tag{6.2.3}$$

for each f in $S(X)$. It follows from (6.2.2) and (6.2.3) that

$$(\phi \circ \theta)(f) = (k \circ h)^{-1} \circ f \circ (k \circ h) \circ (h^{-1} \circ w \circ h \circ v) \tag{6.2.4}$$

and this implies that

$$\Phi(\phi \circ \theta) = (h^{-1} \circ w \circ h \circ v,\ k \circ h) \tag{6.2.5}$$

thus,

$$\Phi(\phi \circ \theta) = (h^{-1} \circ w \circ h \circ v,\ k \circ h) = (v, h)(w, k) = \Phi(\phi)\Phi(\theta).$$

This shows that Φ is a homomorphism. It is immediate that Φ is injective and it is onto since for any pair $(u, t) \in E(X)$, the mapping which sends f into $t^{-1} \circ f \circ t \circ u$ is an α-monomorphism σ and $\Phi(\sigma) = (u, t)$.

The following result is proved just as we proved the previous one with the exception that Theorem (5.7) is used in place of Theorem (5.6).

THEOREM 6.3. *Let X be any conformable space which is not the union of an infinite number of mutually disjoint nonempty open subsets. Then the* *semigroup of all α-monomorphisms mapping $S(X)$ into $S(X)$ is isomorphic to the semigroup $E(X)$.*

REMARKS: For any space X, the mapping ϕ which sends $(v, h) \in E(X)$ into h^{-1} is a homomorphism from $E(X)$ onto the semigroup of all homeomorphisms mapping X *into* X which, of course, is a subsemigroup of $S(X)$ and it contains the group $G(X)$ of units of $S(X)$. The group $G(X)$ is simply the group of all homeomorphisms mapping X *onto* X. Thus, for any space X satisfying the hypothesis of either of the previous two theorems, the semigroup of all α-monomorphisms mapping $S(X)$ into $S(X)$ maps homomorphically onto a subsemigroup of $S(X)$ which contains the group $G(X)$.

The semigroup $E(X)$ is generally not a group. Its identity is (i_X, i_X) where i_X is the identity map on X. One easily shows that an element (v, h) in $E(X$, has an inverse if and only if $v = i_X$ and h is a homeomorphism from X onto X. Thus, the group of units of $E(X)$ is

$$\{(i_X, h): h \text{ is a homeomorphism from } X \text{ onto } X\}$$

and the mapping ϕ which sends (v, h) into h^{-1} maps this group isomorphically onto the group $G(X)$ of units of $S(X)$.

It is not all difficult to find examples where $E(X)$ is not a group. The closed unit interval J is one such example. Define

$$v(x) = x \text{ for } 0 \leqq x \leqq \tfrac{1}{2}$$

$$v(x) = \tfrac{1}{2} \text{ for } \tfrac{1}{2} < x \leqq 1$$

$$h(x) = 2x \text{ for } 0 \leqq x \leqq \tfrac{1}{2}.$$

Then $(v, h) \in E(J)$ and is not a unit. $E(R)$ where R is the space of real number is, however, a group and is therefore isomorphic to the group of all homeomorphisms mapping R onto R. To see this, suppose that $(v, h) \in E(R)$. Then v is an idempotent continuous selfmap of R and h maps its range V homeomorphically onto R. Now the range of a continuous idempotent in any Hausdorff space is closed and the only closed subspace of R which is homeomorphic to R is R itself. This forces V to be identical to R. Thus, $v = i_X$ and h is a homeomorphism from R onto R. This means, as we observed previously, that (v, h) has an inverse. One more observation: the space R is, in view of Theorem (3.5), a conformable space and it is certainly not the union of an infinite number of mutually disjoint nonempty open subsets. Thus, R satisfies the hypothesis of Theorem (6.3), so that the semigroup of all α-monomorphisms mapping $S(R)$ into $S(R)$ is isomorphic to $E(R)$ and is therefore a group. All this is just another way of saying that each α-monomorphism from $S(R)$ into $S(R)$ is actually an automorphism. This also follows directly from Theorems (3.5) and (5.7), and the fact that the only closed subset of R which is homeomorphic to R is R itself.

7. Semigroups of closed functions

By a *closed* function mapping a topological space X into itself we mean any function f mapping X into X such that $f(H)$ is closed whenever H is closed. We stress the fact that here a closed function is not assumed to be continuous. The family of all closed selfmaps of X is a semigroup under composition and we denote it by $\Gamma(X)$. Such semigroups and also related semigroups have been investigated in [4], [7], [8] and [9]. In this section, we obtain results for $\Gamma(X)$ which are analogous to various results obtained for $S(X)$ in the preceeding two sections. In most cases, the proofs are extremely similar to those given for $S(X)$ and we will often simply make a remark to that effect when we feel that it is appropriate. The first result is a prime example as its proof follows just as the proof of Theorem (5.1).

THEOREM 7.1. *Let X be any topological space and let v be any idempotent of $\Gamma(X)$ with range V. Then the mapping which sends f in $\Gamma(V)$ into $f\circ v$ is an isomorphism from $\Gamma(V)$ onto the I-subsemigroup $\Gamma(X)_v$.*

Before we get the analogue to Theorem (5.2), we need a lemma.

LEMMA 7.2. *Let X be a T_1 space. A subspace V of X is the range of an idempotent in $\Gamma(X)$ if and only if V is closed.*

PROOF. It is immediate that if V is the range of an idempotent closed selfmap (in fact, any closed selfmap will suffice) then V is closed. On the hand if V is closed, we choose any point $p\in V$ and define

$$v(x) = x \text{ for } x\in V$$

$$v(x) = p \text{ for } x\in X-V.$$

Then for any closed subset H of X, we have

$$v(H) = v(H\cap V)\cup v(H-V).$$

Now $v(H\cap V) = H\cap V$ which is closed and $v(H-V)$ is either $\{p\}$ or empty. In either event, $v(H)$ is closed.

THEOREM 7.3. *Let X and Y be T_1 spaces. Then $\Gamma(X)$ is isomorphic to an I-subsemigroup of $\Gamma(Y)$ if and only if X is homeomorphic to a closed subspace of Y.*

PROOF. First, suppose that X is homomorphic to a closed subset V of Y. Then $\Gamma(X)$ is isomorphic to $\Gamma(V)$. By the previous lemma, there is an idempotent $v\in\Gamma(Y)$ whose range is V and hence, by Theorem (7.1), $\Gamma(V)$ is isomorphic to $\Gamma(Y)_v$. Thus, $\Gamma(X)$ is also isomorphic to $\Gamma(Y)_v$.

Now suppose that $\Gamma(X)$ is isomorphic to some $\Gamma(Y)_v$. We use Theorem (7.1) to conclude that $\Gamma(X)$ and $\Gamma(V)$ are isomorphic where, as before, V is the range of v. Then V is closed and by Theorem (2.10) of [4, p. 512], it is homeomorphic to X.

THEOREM 7.4. *Let X and Y be two T_1 spaces. Then for any α-monomorphism ϕ from $\Gamma(X)$ into $\Gamma(Y)$, there exists a unique idempotent v of $\Gamma(Y)$ and a unique continuous bijection h from the range V of v onto X such that*

$$\phi(f) = h^{-1} \circ f \circ h \circ v \tag{7.4.1}$$

for each f in $\Gamma(X)$.

PROOF. Since X and Y are T_1 spaces both $\Gamma(X)$ and $\Gamma(Y)$ are α-semigroups so we may appeal to the Fundamental Lemma (4.4) just as in the proof of Theorem (5.3). In this case also, the only thing we really need to do is to show that h is continuous. Let H be any closed subset of X. According to Lemma (7.2), there is a function f (in fact, an idempotent) in $\Gamma(X)$ such that $H = f(X)$. Then we use (4.4.3) to get

$$h^{-1}(H) = h^{-1}(f(X)) = \phi(f)(V).$$

Now V is closed in Y since it is the range of the closed idempotent v. Thus $\phi(f)(V)$ is closed since $\phi(f)$ is a closed function.

COROLLARY 7.5. *Let X and Y be T_1 spaces and suppose that there exists an α-monomorphism from $\Gamma(X)$ into $\Gamma(Y)$. Then X is a continuous image of Y.*

The proof of the next result follows immediately from Theorem (7.4). Just as Theorem (5.6) follows from Theorem (5.3).

THEOREM 7.6. *Let X be a Hausdorff space and let Y be a compact T_1 space. Then for any α-monomorphism ϕ from $\Gamma(X)$ into $\Gamma(Y)$, there exists a unique idempotent v of $\Gamma(Y)$ and a unique homeomorphism h from the range V of v onto X such that*

$$\phi(f) = h^{-1} \circ f \circ h \circ v \tag{7.6.1}$$

or each f in $\Gamma(X)$.

THEOREM 7.7. *Let X be Hausdorff and let Y be a compact T_1 space. Then a monomorphism ϕ from $\Gamma(X)$ into $\Gamma(Y)$ is an α-monomorphism if and only if the image of $\Gamma(X)$ under ϕ is an I-subsemigroup of $\Gamma(Y)$.*

The proof is analogous to the proof of Theorem (5.8) using Theorems (7.6) and (7.1) in place of Theorems (5.6) and (5.1).

Theorem (7.7) then combines with Theorem (7.3) to give

THEOREM 7.8. *Let X be Hausdorff and let Y be a compact T_1 space. Then the following statements are equivalent.*

(7.8.1) *There exists an α-monomorphism from $\Gamma(X)$ into $\Gamma(Y)$.*

(7.8.2) *$\Gamma(X)$ is isomorphic to an I-subsemigroup of $\Gamma(Y)$.*

(7.8.3) *X is homeomorphic to a closed subspace of Y.*

Now let X be any space and let $H(X)$ denote the collection of all pairs (v, h) where v is an idempotent closed selfmap and h is a homeomorphism whose domain is the range of v and whose range is all of X. Define the product of two such pairs (just as in the case for continuous functions) by

$$(v, h)(w, k) = (h^{-1} \circ w \circ h \circ v,\ k \circ h).$$

Then $H(X)$ is a semigroup with this multiplication and we can now state the analogue of Theorem (6.2) for semigroups of closed functions.

THEOREM 7.9. *Let X be a compact Hausdorff space. Then the semigroup of all α-monomorphisms mapping $\Gamma(X)$ into $\Gamma(X)$ is isomorphic to the semigroup $H(X)$.*

The proof is analogous to the proof of Theorem (6.2) using, of course, Theorem (7.6) in place of Theorem (5.6).

REMARK: In a T_1 space X, every selfmap with finite range is a closed function. It readily follows from this that $\Gamma(X)$ is doubly transitive so Theorem (4.6) implies that if X and Y are any two T_1 spaces, then any nonconstant α-homomorphism from $\Gamma(X)$ into $\Gamma(Y)$ must necessarily be injective.

8. Homomorphisms which are not α-homomorphisms

We have devoted a considerable amount of space to discussing α-homomorphisms (or somewhat more accurately, α-monomorphisms) so it seems appropriate to devote a little space to the discussion of some "natural" homomorphisms which are not α-homomorphisms. We begin with what is perhaps the simplest example.

EXAMPLE 8.1. Let X be any topological space and let Y be any space with more than one point. Choose any nonconstant idempotent v of $S(Y)$ (the identity will suffice) and map all of $S(X)$ into v. Choose any point a in the range of v. Then $v \circ \langle a \rangle = \langle a \rangle$ but the constant function $\langle a \rangle$ does not belong to the image of $S(X)$ under the homomorphism. Consequently, it is not an α-homomorphism. In exactly the same manner, one constructs homomorphisms from $\Gamma(X)$ into $\Gamma(Y)$ which are not α-homomorphisms.

EXAMPLE 8.2. Let X be any topological space and let Y denote the discrete space whose elements are all the function in $S(X)$. Then $S(Y)$ consists of *all* selfmaps of Y and we define a mapping ϕ from $S(X)$ into $S(Y)$ as follows:

$$(\phi(f))(g) = f \circ g$$

for all $f \in S(X)$ and $g \in Y(= S(X))$. It is well known that ϕ is a monomorphism from $S(X)$ into $\Gamma(Y)$. However, ϕ is not an α-monomorphism. First of all, $\phi(i_X) = i_Y$ where i_X and i_Y denote the identity functions on X and Y respectively. Then in order for ϕ to be an α-monomorphism, every left zero would have to belong to $\phi(S(X))$. But there are many left zeros which do not satisfy this. In fact, the only ones which do are those constant functions on Y which map everything onto some element of Y which, itself, is a constant function on X. So, in order to get a left zero of $S(Y)$ which is not in $\phi(S(X))$, one need only choose any nonconstant function f in $S(X)$ and then take that selfmap of Y which maps everything into that point. The latter discussion carries through entirely for the semigroups $\Gamma(X)$ and $\Gamma(Y)$.

EXAMPLE 8.3. Let X be any completely regular noncompact Hausdorff space and let βX denote its Stone-Čech compactification. By the well known property of βX, each f in $S(X)$ has a unique extension to a function f^* in $S(\beta X)$. We define $\phi(f) = f^*$. Since $f^* \circ g^*$ and $(f \circ g)^*$ agree on the dense subspace X, it follows that $f^* \circ g^* = (f \circ g)^*$. Thus, ϕ is a homomorphism and it is immediate that it is injective. Let i be the identity on X and j the identity on βX. Then $\phi(i) = j$ and to show that ϕ is not an α-monomorphism, it is sufficient to produce a left zero of $S(\beta X)$ which is not in $\phi(S(X))$. To do this, choose any point in $\beta X - X$ and take the constant function which maps everything into that point.

One can combine the techniques in the previous two examples and prove that any semigroup can be embedded in $S(Y)$ where Y is a suitably chosen compact Hausdorff 0-dimensional space (and hence an S^*-space — even an RS^*-space). This was also observed in [5, p. 110]. One first adjoins an identity to T (if T does not already have one) getting a semigroup T^1. Then one embeds T^1 into $S(T^1)$ as in Example (8.2). T^1 is assumed to have the discrete topology. Finally, one embeds $S(T^1)$ into $S(\beta T^1)$ as in Example (8.3). It follows, in particular, that for any noncompact Hausdorff S^*-space X, there exists a compact Hausdorff 0-dimensional space Y such that $S(X)$ can be embedded in $S(Y)$. This, of course, can never be done with an α-monomorphism in view of Corollary (5.4). As for semigroups of closed functions, the previous devices allow us to take any noncompact T_1 space X and embed $\Gamma(X)$ into $S(\beta Y)$ where Y is $\Gamma(X)$ topologized with the discrete topology. However, every continuous selfmap on a compact Hausdorff space is also a closed selfmap so $S(\beta Y)$ is a subsemigroup of $\Gamma(\beta Y)$. Thus $\Gamma(X)$ can be embedded in the semigroup of all closed selfmaps of a compact

Hausdorff space. Corollary (7.5) tells us that this cannot be done with an α-monomorphism.

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