Appendix C
Fermionic coherent states

In this appendix we derive the field representation for fermion operators. In the Bose case the field representation was just the coordinate representation which is also much used in quantum mechanics. For Fermi operators the analog leads to the so-called Grassmann variables. This means that the Fermi operator fields $\hat{\psi}(x)$ will be represented by 'numbers' $\psi(x)$, which have to be *anticommuting*. As this might not be so familiar, we shall first describe how this works.

Consider the quantum Fermi operators satisfying the commutation relations

$$\{\hat{a}_k, \hat{a}_l\} = 0,\ \{\hat{a}_k^\dagger, \hat{a}_l^\dagger\} = 0,\ \{\hat{a}_k, \hat{a}_l^\dagger\} = \delta_{kl}, \tag{C.1}$$

where $\{A, B\} = AB + BA$. In the following we shall consider a finite number n of such operators, $k = 1, 2, \ldots, n$. (In the continuum limit of a fermionic lattice field theory $n \to \infty$.) It is sometimes convenient to use the $2n$ equivalent Hermitian operators

$$\hat{a}_k^1 = (\hat{a}_k + \hat{a}_k^\dagger)/\sqrt{2},\ \hat{a}_k^2 = (\hat{a}_k - \hat{a}_k^\dagger)/i\sqrt{2}, \tag{C.2}$$

with the commutation relations

$$\{\hat{a}_k^p, \hat{a}_l^q\} = \delta_{pq}\delta_{kl},\ \ p, q = 1, 2. \tag{C.3}$$

The non-Hermitian operators are used more often.

It is clarifying to look at a representation in Hilbert space. For $n = 1$ we have the 'no-quantum state' $|0\rangle$ which is by definition the eigenstate of $\hat{a}$ with eigenvalue 0, $\hat{a}|0\rangle = 0$, and the one-quantum state $|1\rangle$ obtained from $|0\rangle$ by the application of $\hat{a}^\dagger$, $|1\rangle = \hat{a}^\dagger|0\rangle$. Further application of $\hat{a}^\dagger$ on $|0\rangle$ gives zero, since $(\hat{a}^\dagger)^2 = 0$ because of (C.1) (note that $|1\rangle$ is the 'no-quantum state' for $\hat{a}^\dagger$). So a pair of Fermi operators ($\hat{a}$, $\hat{a}^\dagger$) can be

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represented in a simple two-dimensional Hilbert space,

$$|0\rangle \rightarrow \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \ |1\rangle \rightarrow \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \ \hat{a} \rightarrow \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \ \hat{a}^\dagger \rightarrow \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}. \tag{C.4}$$

For $n > 1$ we can take a tensor product of these representations. A basis in Hilbert space is provided by

$$|k_1 \cdots k_p\rangle = \hat{a}^\dagger_{k_1} \cdots \hat{a}^\dagger_{k_p}|0\rangle, \quad p = 1, \ldots, n\,, \tag{C.5}$$

with the properties

$$\sum_{p=0}^{n} \frac{1}{p!} \sum_{k_1 \cdots k_p} |k_1 \cdots k_p\rangle\langle k_1 \cdots k_p| = 1, \tag{C.6}$$

$$\langle k_1 \cdots k_p | l_1 \cdots l_q\rangle = \delta_{pq} \delta^{k_1 \cdots k_p}_{l_1 \cdots l_q}, \tag{C.7}$$

where

$$\delta^{k_1 \cdots k_p}_{l_1 \cdots l_q} = \sum_{\text{perm}\, \pi} (-1)^\pi \delta^{k_1}_{\pi l_1} \cdots \delta^{k_p}_{\pi l_p}. \tag{C.8}$$

An arbitrary state $|\psi\rangle$ can be written as†

$$|\psi\rangle = \psi(\hat{a}^\dagger)|0\rangle, \tag{C.9}$$

$$\psi(\hat{a}^\dagger) = \sum_{p=0}^{n} \frac{1}{p!} \psi_{k_1 \cdots k_p} \hat{a}^\dagger_{k_1} \cdots \hat{a}^\dagger_{k_p}, \tag{C.10}$$

where $\psi_{k_1 \cdots k_p}$ is totally antisymmetric in $k_1 \cdots k_p$, and we sum over repeated indices unless indicated otherwise. An arbitrary operator $\hat{A}$ can be written as

$$\hat{A} = \sum_{pq} \frac{1}{p!q!} A_{k_1 \cdots k_p,\, l_1 \cdots l_q} \hat{a}^\dagger_{k_1} \cdots \hat{a}^\dagger_{k_p} \hat{a}_{l_q} \cdots \hat{a}_{l_1}, \tag{C.11}$$

where all creation operators are ordered to the left of all annihilation operators. This is called the *normal ordered* form of $\hat{A}$. A familiar example is the number operator

$$\hat{N} = \hat{a}^\dagger_k \hat{a}_k, \tag{C.12}$$

which has eigenvectors $|k_1 \cdots k_p\rangle$ with eigenvalue p. Note that $A_{k_1 \cdots k_p,\, l_1 \cdots l_q}$ is in general not equal to $\langle k_1 \cdots k_p|\hat{A}|l_1 \cdots l_q\rangle$. Note also that the coefficients $A_{k_1 \cdots k_p,\, l_1 \cdots l_q}$ may themselves be elements of a

† Recall that repeated indices are summed, i.e. $\psi_{k_1 \cdots k_p} \hat{a}^\dagger_{k_1} \cdots \hat{a}^\dagger_{k_p} = \sum_{k_1=1}^{n} \cdots \sum_{k_p=1}^{n} \psi_{k_1 \cdots k_p} \hat{a}^\dagger_{k_1} \cdots \hat{a}^\dagger_{k_p}$.

Grassmann algebra, e.g. $\hat{A} = c_k^+ \hat{a}_k + \hat{a}_k^\dagger c_k$, with anticommuting c and c^+.

Suppose now that there are eigenstates $|a\rangle$ of the $\hat{a}_k$ with eigenvalue a_k. Then it follows that the a_k have to be anticommuting:

$$a_k a_l = -a_l a_k. \tag{C.13}$$

To see this, assume

$$\hat{a}_k a_l = \epsilon a_l \hat{a}_k, \tag{C.14}$$

with ϵ some number $\neq 0$. Then

$$\begin{aligned}\hat{a}_k \hat{a}_l |a\rangle = \hat{a}_k a_l |a\rangle &= \epsilon a_l \hat{a}_k |a\rangle = \epsilon a_l a_k |a\rangle \\ &= -\hat{a}_l \hat{a}_k |a\rangle = -\epsilon a_k a_l |a\rangle.\end{aligned} \tag{C.15}$$

Hence (C.13) has to hold. The a_k cannot be ordinary numbers. Assuming $a_k|a\rangle = +|a\rangle a_k$ leads to

$$\begin{aligned}\hat{a}_k a_l |a\rangle = \hat{a}_k |a\rangle a_l = a_k |a\rangle a_l &= a_k a_l |a\rangle \\ &= \epsilon a_l \hat{a}_k |a\rangle = \epsilon a_l a_k |a\rangle,\end{aligned} \tag{C.16}$$

and it follows that

$$\epsilon = -1. \tag{C.17}$$

So the 'numbers' a_k have to anticommute with the fermionic operators as well.

We also introduce independent conjugate anticommuting a_k^+, assume these to anticommute with the a_k and the Fermi operators, and impose the usual rules of Hermitian conjugation,

$$\hat{a}_k \xrightarrow{\dagger} \hat{a}_k^\dagger, \;\; a_k \xrightarrow{\dagger} a_k^+, \;\; |a\rangle \xrightarrow{\dagger} \langle a|, \;\; \langle a|\hat{a}_k^\dagger = \langle a|a_k^+, \tag{C.18}$$

$$a_k a_l \xrightarrow{\dagger} a_l^+ a_k^+, \;\; \{a_k^+, a_l^+\} = 0. \tag{C.19}$$

The anticommuting a_k^+ are on the same footing as the a_k.

The a_k and a_k^+ together with the unit element 1 generate a *Grassmann algebra*. An arbitrary element f of this algebra has the form

$$\begin{aligned}f(a^+, a) = f_{0,0} + f_{k,0} a_k^+ + f_{0,l} a_l + \frac{1}{2!} f_{k_1 k_2, 0} a_{k_1}^+ a_{k_2}^+ \\ + f_{k,l} a_k^+ a_l + \cdots + f_{1\cdots n, 1\cdots n} a_1^+ \cdots a_n^+ a_n \cdots a_1,\end{aligned} \tag{C.20}$$

where the f's are complex numbers.

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We have extended Hilbert space into a vector space over the elements of a Grassmann algebra. The a_k and a_k^+ are called Grassmann variables and $f(a^+, a)$ is called a function of the Grassmann variables. This nomenclature could be somewhat misleading – the generators a_k and a_k^+ are fixed objects and it is only the indices 'k' and '+' that vary. However, we will also be using other generators b_k, b_k^+, c_k, ..., and so effectively we draw elements from a Grassmann algebra with an infinite number of generators. It is straightforward to construct a matrix representation of these generators, but this does not seem to be useful because the rules above are sufficient for our derivations.

We now express the $|a\rangle$ in terms of the basis vectors (C.5). The state $|a\rangle$ is given by

$$|a\rangle = e^{-a_k \hat{a}_k^\dagger}|0\rangle. \tag{C.21}$$

Indeed, since $(a_k)^2=0$,

$$e^{-a_k \hat{a}_k^\dagger} = \prod_k e^{-a_k \hat{a}_k^\dagger} = \prod_k (1 - a_k \hat{a}_k^\dagger), \tag{C.22}$$

and using $\hat{a}_k(1 - a_k \hat{a}_k^\dagger)|0\rangle = a_k \hat{a}_k \hat{a}_k^\dagger|0\rangle = a_k|0\rangle$ (no summation over k) gives

$$\begin{aligned}\hat{a}_k|a\rangle &= \left[\prod_{l\neq k}(1 - a_l \hat{a}_l^\dagger)\right] \hat{a}_k(1 - a_k \hat{a}_k^\dagger)|0\rangle = \left[\prod_{l\neq k}(1 - a_l \hat{a}_l^\dagger)\right] a_k|0\rangle \\ &= a_k|a\rangle.\end{aligned} \tag{C.23}$$

Note that a_k *commutes* with pairs of fermion objects, e.g. $[a_k, a_l\hat{a}_m^\dagger] = 0$. Two states $|a\rangle$ and $|b\rangle$ have the inner product

$$\begin{aligned}\langle a|b\rangle &= \langle 0|(1 - \hat{a}_1 a_1^+) \cdots (1 - \hat{a}_n a_n^+)(1 - b_n \hat{a}_n^\dagger) \cdots (1 - b_1 \hat{a}_1^\dagger)|0\rangle \\ &= \prod_k (1 + a_k^+ b_k) \\ &= e^{a^+ b},\end{aligned} \tag{C.24}$$

where

$$a^+ b \equiv a_k^+ b_k. \tag{C.25}$$

We would like a completeness relation of the form

$$\hat{1} = \int da^+ \, da \, \frac{|a\rangle\langle a|}{\langle a|a\rangle}. \tag{C.26}$$

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For $n = 1$ this relation reads

$$\begin{aligned}\hat{1} &= |0\rangle\langle 0| + \hat{a}^\dagger|0\rangle\langle 0|\hat{a} \\ &= \int da^+\, da\, (1 - a^+ a)(1 - a\hat{a}^\dagger)|0\rangle\langle 0|(1 - \hat{a}a^+) \\ &= \int da^+\, da\, [\,(1 - a^+ a)|0\rangle\langle 0| - a\hat{a}^\dagger|0\rangle\langle 0| \\ &\qquad + a^+|0\rangle\langle 0|\hat{a} + aa^+\hat{a}^\dagger|0\rangle\langle 0|\hat{a}\,], \end{aligned} \tag{C.27}$$

which is satisfied if we define the Berezin 'integral':

$$\int da = 0, \ \int da^+ = 0, \ \int da\, a = 1, \ \int da^+\, a^+ = 1, \tag{C.28}$$

where da and da^+ are taken anticommuting. For general n we define

$$da = da_1 \cdots da_n, \ \ da^+ = da_n^+ \cdots da_1^+, \tag{C.29}$$

$$\int da_k = 0, \ \int da_k\, a_k = 1, \int da_k^+ = 0, \ \int da_k^+\, a_k^+ = 1 \tag{C.30}$$

(no summation over k; anticommuting da's and da^+'s). The integral sign symbolizes Grassmannian integration, which has some similarities to ordinary integration (and differentiation, see (C.42)). Cumbersome checking of minus signs can be avoided by combining every da_k with da_k^+ into commuting pairs, as in the notation

$$da^+\, da \equiv \prod_{k=1}^{n} da_k^+\, da_k, \tag{C.31}$$

which we shall use in the following. Similar pairing will be done repeatedly in the following.

We check the completeness relation (C.26) for general n by verifying that it gives the right answer for an arbitrary inner product $\langle\psi|\phi\rangle$. Multiplying (C.9) by (C.26), we get

$$|\psi\rangle = \int da^+\, da\, e^{-a^+ a}\, \psi(a^+)|a\rangle, \tag{C.32}$$

$$\psi(a^+) = \langle a|\psi\rangle = \sum_p \frac{1}{p!}\psi_{k_1 \cdots k_p} a_{k_1}^+ \cdots a_{k_p}^+. \tag{C.33}$$

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The inner product takes the form

$$
\begin{aligned}
\langle\psi|\phi\rangle &= \int da^+\, da\, e^{-a^+a}\, \psi(a^+)^\dagger \phi(a^+) \\
&= \sum_{pq} \frac{1}{p!q!}\, \psi^*_{k_1\cdots k_p} \phi_{l_1\cdots l_q} \\
&\quad \times \int da^+\, da\, e^{-a^+a}\, a_{k_p}\cdots a_{k_1} a^+_{l_1}\cdots a^+_{l_q}. \qquad \text{(C.34)}
\end{aligned}
$$

By (C.28) the integral is non-zero only if $p = q$ and $(k_1, \ldots, k_p) = (l_1, \ldots, l_p)$ up to a permutation,

$$
\begin{aligned}
\int da^+\, da\, e^{-a^+a} a_{k_p}\cdots a_{k_1} a^+_{k_1}\cdots a^+_{k_p} &= \prod_{l\neq k_1,\cdots,k_p} \int da^+_l\, da_l\, e^{-a^+_l a_l} \\
&\quad \times \prod_{m=k_1,\cdots,k_p} \int da^+_m\, da_m\, a_m a^+_m \\
&= 1, \qquad \text{(C.35)}
\end{aligned}
$$

and

$$
\int da^+\, da\, e^{-a^+a} a_{k_p}\cdots a_{l_1} a^+_{k_1}\cdots a^+_{l_q} = \delta_{pq}\, \delta^{k_1\cdots k_p}_{l_1\cdots l_q}. \qquad \text{(C.36)}
$$

Hence, (C.34) gives

$$
\langle\psi|\phi\rangle = \sum_p \frac{1}{p!} \psi^*_{k_1\cdots k_p} \phi_{k_1\cdots k_p}, \qquad \text{(C.37)}
$$

which is the right answer. Therefore (C.26) is correct for general n.

The connection between Grassmannian integration and differentiation can be seen as follows. Left and right differentiation can be defined by looking at terms linear in a translation over fermion b_k,

$$
\begin{aligned}
f(a+b) &= f(a) + b_k f^{\mathrm{L}}_k(a) + \tfrac{1}{2} b_k b_l f^{\mathrm{L}}_{kl}(a) + \cdots \qquad \text{(C.38)} \\
&= f(a) + f^{\mathrm{R}}_k(a) b_k + \tfrac{1}{2} f^{\mathrm{R}}_{kl}(a) b_k b_l + \cdots, \qquad \text{(C.39)}
\end{aligned}
$$

which suggest

$$
\frac{\partial}{\partial a_k} f(a) := f^{\mathrm{L}}_k(a), \qquad \text{(C.40)}
$$

$$
f(a)\, \frac{\overleftarrow{\partial}}{\partial a_k} := f^{\mathrm{R}}_k(a) \qquad \text{(C.41)}
$$

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(the extension to functions of both a and a^+ is obvious). It follows that 'integration' is left differentiation:

$$\int da_k\, f(a) = \frac{\partial}{\partial a_k} f(a). \tag{C.42}$$

We shall now derive some further important properties of Grassmannian integration. Let $f(a^+, a)$ be an arbitrary element of the Grassmann algebra of the form by (C.20). Then

$$\int da^+\, da\, f(a^+, a) = f_{1\cdots n, 1\cdots n}. \tag{C.43}$$

It follows that the integration is translation invariant,

$$\int da^+\, da\, f(a^+ + b^+, a + b) = \int da^+\, da\, f(a^+, a). \tag{C.44}$$

Furthermore, for an arbitrary matrix M,

$$\begin{aligned} \int da^+\, da\, e^{-a^+ M a} &= \int da^+\, da\, \frac{(-1)^n}{n!} (a^+ M a)^n \\ &= \int da^+\, da\, \frac{1}{n!} M_{k_1 l_1} \cdots M_{k_n l_n} a_{l_1} a^+_{k_1} \cdots a_{l_n} a^+_{k_n} \\ &= \frac{1}{n!} M_{k_1 l_1} \cdots M_{k_n l_n} \delta^{k_1 \cdots k_n}_{l_1 \cdots l_n}. \end{aligned} \tag{C.45}$$

Using the identity

$$\epsilon_{k_1 \cdots k_n} \epsilon_{l_1 \cdots l_n} = \delta^{k_1 \cdots k_n}_{l_1 \cdots l_n}, \tag{C.46}$$

where $\epsilon_{k_1 \cdots k_n}$ is the n-dimensional ϵ tensor (with $\epsilon_{1 \cdots n} = +1$) we obtain the formula

$$\int da^+\, da\, e^{-a^+ M a} = \det M, \tag{C.47}$$

since

$$\det M = M_{1 l_1} \cdots M_{n l_n} \epsilon_{l_1 \cdots l_n}. \tag{C.48}$$

The more general formula

$$\int da^+\, da\, e^{-a^+ M a + a^+ b + b^+ a} = \det M\ e^{b^+ M^{-1} b}. \tag{C.49}$$

follows from the translation invariance (C.44) by making the translation $a+ \to a^+ + b^+ M^{-1}$, $a \to a + M^{-1} a$. Note that (C.49) remains well defined if $\det M \to 0$.

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We can interpret (C.44) as a translation invariance of the fermionic 'measure',

$$da^+ = d(a^+ + b^+), \;\; da = d(a + b). \tag{C.50}$$

A linear multiplicative transformation of variables

$$a_k \to T_{kl} a_l, \;\; a_k^+ \to a_l^+ S_{lk}, \tag{C.51}$$

has the effect

$$d(a^+ S) = (\det S)^{-1} \, da^+, \;\; d(Ta) = (\det T)^{-1} \, da, \tag{C.52}$$

i.e.

$$\int da^+ \, da \, f(a^+ S, Ta) = \det(ST) \int da^+ \, da \, f(a^+, a). \tag{C.53}$$

This follows easily from (C.43) and (C.48). According to (C.52), the fermionic measure transforms inversely to the bosonic measure dx: $d(Tx) = \det T \, dx$.

We note in passing the formula

$$\int da \, e^{-\frac{1}{2} a^{\mathrm{T}} M a} = \pm \sqrt{\det M}, \tag{C.54}$$

where T denotes transposition and M is an antisymmetric matrix (in this case only the antisymmetric part of M contributes anyway). This formula follows from (C.47), by making the transformation of variables

$$\begin{pmatrix} a_k \\ a_k^+ \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} b_k \\ c_k \end{pmatrix}, \tag{C.55}$$

which leads to

$$\det M = (-1)^{n/2} \int db \, e^{-\frac{1}{2} a^{\mathrm{T}} M a} \int dc \, e^{\frac{1}{2} b^{\mathrm{T}} M b}, \tag{C.56}$$

where we assumed n to be even (otherwise $\det M = 0$). As is obvious from the left-hand side of (C.54), the square root of the determinant of an antisymmetric matrix is multilinear in its matrix elements. It is called a Pfaffian.

States $|\psi\rangle$ are represented by Grassmann wavefunctions $\psi(a^+)$ depending only on the a_k^+ (cf. (C.33)). The representatives of operators $\hat{A}$ depend in general also on the a_k:

$$\langle a | \hat{A} | a \rangle =: A(a^+, a). \tag{C.57}$$

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In the normal ordered form (C.11), $A(a^+, a)$ is obtained from $\hat{A}$ by replacing everywhere the operators by their Grassmann representative, keeping the same order, and multiplying by $e^{a^+ a}$:

$$A(a^+, a) = e^{a^+ a} \sum_{pq} \frac{1}{p!q!} A_{k_1 \cdots k_p,\, l_1 \cdots l_q} a^+_{k_1} \cdots a^+_{k_p} a_{l_q} \cdots a_{l_1}. \qquad \text{(C.58)}$$

(The $e^{a^+ a}$ just comes from the normalization factor $\langle a|a\rangle$.)

It is now straightforward to derive the following rules:

$$\begin{aligned} A\psi(a^+) &:= \langle a|\hat{A}|\psi\rangle \\ &= \int db^+\, db\, e^{-b^+ b} A(a^+, b)\psi(b^+), \end{aligned} \qquad \text{(C.59)}$$

$$\begin{aligned} AB(a^+, a) &:= \langle a|\hat{A}\hat{B}|a\rangle \\ &= \int db^+\, db\, e^{-b^+ b} A(a^+, b)B(b^+, a), \end{aligned} \qquad \text{(C.60)}$$

$$\begin{aligned} \hat{A} &= A(\hat{a}^\dagger, \hat{a}),\ \ \hat{B} = B(\hat{a}^\dagger),\ \ \hat{C} = C(\hat{a}) \\ \Rightarrow BAC(a^+, a) &= B(a^+)A(a^+, a)C(a). \end{aligned} \qquad \text{(C.61)}$$

A useful identity is

$$\hat{A} = \exp\left[\hat{a}^\dagger_k M_{kl} \hat{a}_l\right] \Rightarrow A(a^+, a) = \exp\left[a^+_k (e^M)_{kl} a_l\right], \qquad \text{(C.62)}$$

This identity can be derived with well-known differentiation/integration tricks. Let $F(t)$ be given by

$$F(t) = \langle a|e^{t\hat{a}^\dagger M \hat{a}}|a\rangle. \qquad \text{(C.63)}$$

To compute $F(1) = A(a^+, a)$ we differentiate with respect to t and subsequently integrate, with the initial condition $F(0) = \exp(a^+ a)$. Differentiation gives

$$F'(t) = \langle a|\hat{a}^\dagger M \hat{a}\, e^{t\hat{a}^\dagger M \hat{a}}|a\rangle = a^+_k M_{kl} \langle a|\hat{a}_l e^{t\hat{a}^\dagger M \hat{a}}|a\rangle. \qquad \text{(C.64)}$$

The $\hat{a}_l$ needs to be pulled trough the exponential so that we can use $\hat{a}_l|0\rangle = a_l|0\rangle$. For this we use a similar differentiation trick:

$$\begin{aligned} \hat{G}_l(t) &\equiv e^{-t\hat{a}^\dagger M \hat{a}} \hat{a}_l e^{t\hat{a}^\dagger M \hat{a}}, \\ \hat{G}'_l(t) &= e^{-t\hat{a}^\dagger M \hat{a}} [\hat{a}_l, \hat{a}^\dagger M \hat{a}]\, e^{t\hat{a}^\dagger M \hat{a}} = M_{lm} \hat{G}_m(t), \quad \hat{G}(0) = \hat{a}_l, \\ \hat{G}_l(t) &= (e^{tM})_{lm} \hat{a}_m, \\ \hat{a}_l e^{t\hat{a}^\dagger M \hat{a}} &= e^{t\hat{a}^\dagger M \hat{a}} (e^{tM})_{lm} \hat{a}_m. \end{aligned} \qquad \text{(C.65)}$$

The differential equation for $F(t)$ now reads

$$F'(t) = a^+_k M_{kl} \langle a|e^{t\hat{a}^\dagger M \hat{a}}|a\rangle (e^{tM})_{lm} a_m = (a^+ e^{tM} a)'\, F(t), \qquad \text{(C.66)}$$

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with the solution

$$F(t) = \exp(a^+ e^{tM} a), \quad A(a^+, a) = F(1) = \exp(a^+ e^M a). \tag{C.67}$$

Next we derive an important formula for the trace of a fermionic operator. It is usually sufficient to consider only even operators, i.e. operators containing only terms with an even number of fermionic operators or fermionic variables. Such $\hat{A}$ and also their representative $A(a^+, a)$ commute with arbitrary anticommuting numbers, for example $A(a^+, b)c_k = +c_k A(a^+, b)$. The formula reads

$$\mathrm{Tr}\, \hat{A} = \int da^+ \, da \, e^{-a^+ a} \, A(a^+, -a), \tag{C.68}$$

for even $\hat{A}$. This trace formula can be derived as follows:

$$\begin{aligned}
\mathrm{Tr}\, \hat{A} &= \sum_{p=0}^{n} \frac{1}{p!} \sum_{k_1 \cdots k_p} \langle k_1 \cdots k_p | \hat{A} | k_1 \cdots k_p \rangle \\
&= \int (da^+ \, da) \, (db^+ \, db) \, e^{-a^+ a - b^+ b} \\
&\quad \sum_p \frac{1}{p!} \sum_{k_1 \cdots k_p} \langle k_1 \cdots k_p | a \rangle \langle a | \hat{A} | b \rangle \langle b | k_1 \cdots k_p \rangle \\
&= \int (da^+ \, da) \, (db^+ \, db) \, e^{-a^+ a - b^+ b} \sum_p \frac{1}{p!} a_{k_p} \cdots a_{k_1} A(a^+, b) b^+_{k_1} \cdots b^+_{k_p} \\
&= \int (da^+ \, da) \, (db^+ \, db) \, e^{-a^+ a - b^+ b} \sum_p \frac{1}{p!} a_{k_p} \cdots a_{k_1} b^+_{k_1} \cdots b^+_{k_p} A(a^+, b) \\
&= \int (da^+ \, da) \, (db^+ \, db) \, e^{-a^+ a - b^+ b} \, e^{a_k b^+_k} \, A(a^+, b) \\
&= (-1)^n \int (da^+ \, db) \, e^{a^+ b} \, A(a^+, b) \\
&= \int (da^+ \, db) \, e^{-a^+ b} A(a^+, -b),
\end{aligned} \tag{C.69}$$

which is the desired result. We integrated over a and b^+ using $(da^+ \, da) \times (db^+ \, db) = (-1)^n (da^+ \, db) \, (db^+ \, da)$ and (C.49). In the last line we made the substitution $b \to -b$ using (C.52).

We note furthermore that omitting the minus sign from $A(a^+, -a)$ in (C.68) leads to

$$\int da^+ \, da \, e^{-a^+ a} A(a^+, a) = \mathrm{Tr}(-1)^{\hat{N}} \hat{A}, \tag{C.70}$$

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where $\hat{N}$ is the fermion-number operator (C.12). This formula can be derived from the trace formula (C.68), the operator-product rule (C.60), with $B = \exp(i\pi\hat{N})$, and the application

$$\hat{B} = e^{i\pi\hat{a}^\dagger\hat{a}} \rightarrow B(a^+, a) = e^{-a^+ a} \tag{C.71}$$

of the rule (C.62).

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