RESIDUAL FINITENESS OF PERMUTATIONAL PRODUCTS

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Conditions sufficient to guarantee that a generalized free product of two residually finite groups A and B is again residually finite have been given by Baumslag [1]. We here show the same conditions guarantee that a certain permutational product of A and B is also residually finite.

Given two groups $(A, +)$ and $(B, \cdot)$, the set $A \cup B$ is the *amalgam* of these groups if and only if $A \cap B = H$ is a subgroup of both A and B and, for all $h, h_1 \in H$, $h+h_1 = h \cdot h_1$. The common subgroup H is called the *amalgamated subgroup*. The notation $A \cup B|H = \mathfrak{A}$ will be used to denote the amalgam.

A group G is *residually finite* if G contains a set of normal subgroups $\{N_\lambda | \lambda \in \Lambda\}$ (called a *filter*) such that each G/N_λ is finite and

$$\cap \{N_\lambda | \lambda \in \Lambda\} = \{1\}.$$

If H is a subgroup of A and $\{A_\lambda\}$ a filter of A, let $\{A_\lambda\} \cap H$ denote the set of distinct $A_\lambda \cap H$. If A and B are subgroups of the residually finite group G with filter $\{G_\lambda\}$ and $A \cap B = H$, then note that $\{G_\lambda\} \cap A = \{A_\lambda\}$ and $\{G_\lambda\} \cap B = \{B_\lambda\}$ are filters of A and B such that

$$\{A_\lambda\} \cap H = \{B_\lambda\} \cap H.$$

A *transversal* of the subgroup H of A in the group A is a complete set of (left)coset representatives of H in A.

We here prove

THEOREM 1. *Suppose $\mathfrak{A} = A \cup B|H$ is an amalgam of residually finite groups A and B with filters $\{A_i | i \in I\}$ and $\{B_j | j \in J\}$ respectively, such that*

$$\text{(i)} \qquad \{A_i\} \cap H = \{B_j\} \cap H.$$

If, in addition, there exist transversals S and T of H in A and B respectively, such that SA_i/A_i and TB_j/B_j are transversals of HA_i/A_i in A/A_i and HB_j/B_j in B/B_j for all $i \in I$, $j \in J$, then the permutational product constructed using the transversals S and T is residually finite.

THEOREM 2. *Let $\mathfrak{A} = A \cup B|H$ be an amalgam of residually finite groups*

A and B. Suppose that H has countable index in both A and B and A and B have filters

$$A_0 = A \geqq A_1 \geqq \cdots \geqq A_i \geqq \cdots,$$
$$B_0 = B \geqq B_1 \geqq \cdots \geqq B_j \geqq \cdots$$

such that

(i) $$\{A_i\} \cap H = \{B_j\} \cap H$$

and

(ii) $$\bigcap_i A_i H = H = \bigcap_j B_j H.$$

Then some permutational product on $\mathfrak{A}$ *is residually finite.*

COROLLARY 1. *Suppose* $\mathfrak{A} = A \cup B|H$ *is an amalgam of countable residually finite groups with filters* $\mathscr{A} = \{A_i | i \in I\}$ *and* $\mathscr{B} = \{B_j | j \in J\}$, *respectively, such that*

(i) $$\{A_i\} \cap H = \{B_j\} \cap H$$

and

(ii) $$\bigcap_i A_i H = H = \bigcap_j B_j H$$

Then some permutational product on $\mathfrak{A}$ *is residually finite.*

1. Preliminaries

Let $\mathfrak{A} = A \cup B|H$ be an arbitrary amalgam. Choose a transversal S of H in A and a transversal T of H in B.

Let the set $D = S \times T \times H$ and for $a \in A$, $b \in B$ define permutations on D as follows: let $d = (s, t, h) \in D$, $s \in S$, $t \in T$, and $h \in H$. Then put

$$(d)\rho(a) = (s', t', h'),$$

where $s'h' = sha(s' \in S, h' \in H)$, and $t' = t$. Similarly define

$$(d)\rho(b) = (s'', t'', h''),$$

where $t''h'' = thb(t'' \in T, h'' \in H)$, and $s'' = s$. If $h \in H$, the above permutations $\rho(h)$ are easily seen to be the same by either definition. We may consider $\rho(a)$ and $\rho(b)$ as right multiplications by a and b, so that if functions are composed from left to right, $\rho(A) \cong A$ and $\rho(B) \cong B$, where

$$\rho(A) = \{\rho(a) | a \in A\}.$$

In the group of all permutations of D, $\rho(A) \cap \rho(B) = \rho(H)$ (Neumann, [4]), so that the amalgam $A \cup B|H$ can be embedded in the subgroup $P(\mathfrak{A}; S, T)$ generated by $\rho(A)$ and $\rho(B)$ in the group of permutations of D. The group $P(\mathfrak{A}; S, T)$ is the *permutational product* on $A \cup B|H$ *depending on S and T*.

Now suppose $A \cup B|H = \mathfrak{A}$ is an amalgam, U and V are normal subgroups of A and B respectively, and that $U \cap H = V \cap H$. Then we may form a factor amalgam

$$\mathfrak{F} = \mathfrak{F}(\mathfrak{A}; U, V) = A/U \cup B/V|HU/U \tag{1}$$

by identifying HU/U and HV/V according to their natural isomorphisms with $H/H \cap U = H/H \cap V$.

THEOREM. *Let $\mathfrak{A} = A \cup B|H$ be an amalgam and $\mathfrak{F} = \mathfrak{F}(\mathfrak{A}; U, V)$ a factor amalgam. Suppose S and T are transversals of H in A and B, respectively, which map onto transversals S', T' of HU/U in A/U and B/V, respectively. Then there is a homomorphism f from $P(\mathfrak{A}; S, T)$ onto $P(\mathfrak{F}; S', T')$, where, if $x \in P(\mathfrak{A}; S, T)$ and*

$$x : (s, t, h) \to (s^*, t^*, h^*)$$

with $s, s^* \in S$, $t, t^* \in T$ and $h, h^* \in H$, then

$$xf : (sU, tV, hU) \to (s^*U, t^*V, h^*U).$$

PROOF. This is a simple consequence of the definition of a permutational product. (cf. Gregorac [3]).

We shall call the map f above the *natural homomorphism* from $P(\mathfrak{A}; S, T)$ onto $P(\mathfrak{F}; S', T')$. Mappings between permutational products will here be assumed to be natural homomorphisms as above, unless otherwise stated.

2. Proofs

PROOF OF THEOREM 1.

Suppose $A \cup B|H = \mathfrak{A}$ is given together with filters $\{A_i\}$ and $\{B_j\}$ and transversals S and T satisfying the hypotheses of Theorem 1. Let $1 \neq x \in P(\mathfrak{A}; S, T)$. Then there exists an element (s, t, h) of $S \times T \times H$ such that

$$x : (s, t, h) \to (s^*, t^*, h^*)$$

and either $s \neq s^*$, $t \neq t^*$ or $h \neq h^*$. Without loss of generality it may be assumed that $h \neq h^*$. Choose an element A_k of the given filter of A so that $h(h^*)^{-1} \notin A_k$ and choose B_k so that $A_k \cap H = B_k \cap H$. Then let f be the natural homomorphism from

$$P(\mathfrak{A}; S, T) \quad \text{onto} \quad P(\mathfrak{F}; S', T')$$

where

$$\mathfrak{F} = A/A_k \cup B/B_k|HA_k/A_k;$$

here S' and T' are the images of S and T under the canonical homomorphisms

$A \to A/A_k$ and $B \to B/B_k$ and are again transversals. Now

$$xf : (sA_k, tB_k, hA_k) \to (s^* A_k, t^* B_k, h^* A_k)$$

and $hA_k \neq h^* A_k$ since $h(h^*)^{-1} \notin A_k$. Therefore $xf \neq 1$. Finally, note that $P(\mathfrak{F}; S', T')$ is finite since it is a permutation group on a finite set, completing the proof.

PROOF OF THEOREM 2.

I thank Dr L. G. Kovács for suggesting the following proof of Theorem 2 answering a question I asked him.

It must be verified that transversals of H can be chosen as required in Theorem 1. Only the cases where at least one of A or B is infinite need be considered, so assume A is infinite.

Suppose $A_0 = A \geqq A_1 \geqq A_2 \geqq \cdots \geqq A_j \geqq \cdots$ is given filter of A. Let $S = \{h_0, s_1, s_2, \cdots\}$ be any transversal of H in A, where $h_0 \in H$. A transversal S^* of H in A shall be constructed such that

$$(*) \qquad \begin{cases} \text{if} & u, v \in S^* \quad \text{and} \quad uA_iH = vA_iH, \\ \text{then} & uA_i = vA_i, \quad i = 0, 1, 2, \cdots. \end{cases}$$

This can then be taken as the required transversal of H in A. A similar construction will yield a transversal T^* of H in B, when B is infinte. (If B is finite, choose the filter of B to be $\{\{1\}\}$.)

Let $S_0^* = \{h_0\}$ and suppose $S_i^* = \{h_0, s_1^*, \cdots, s_i^*\}$ has been constructed such that

(i′) $s_k^* H = s_k H$, if $s_k^* \in S_i^*$
(ii′) $\cap_j S_i^* A_j H = S_i^* H$ and
(iii′) property $(*)$ holds when $u, v \in S_i^*$.

Consider $s_{i+1} \in S$. Note that

$$s_{i+1} \notin \bigcap_j S_i^* A_j H = S_i^* H$$

by (i′), so there exists an integer j such that

(iv′) $s_{i+1} \in S_i^* A_j H$ and $s_{i+1} \notin S_i^* A_{j+1} H$.

Thus $s_{i+1} = s^* a_j h$ for some $s^* \in S_i^*$, $a_j \in A_j$ and $h \in H$. Fix a choice of s^*, a_j and h; let $s_{i+1}^* = s_{i+1} h^{-1} = s^* a_j$. It remains to be verified that

$$S_{i+1}^* = S_i^* \cup \{s_{i+1}^*\}$$

has properties (i′), (ii′) and (iii′).

Property (i′) is clear, since $s_{i+1}^* = s_{i+1} h^{-1}$. Let $g \in \cap_j S_{i+1}^* A_j H$. Then $g = s_j^* a_j h_j$, with $s_j^* \in S_{i+1}^*$ and $a_j h_j \in A_j H$ for infinitely many j; the set S_{i+1}^* is finite, so there are infinitely many representations of g such that s_j^* is fixed, say $s_j^* = \sigma$. Then $g = \sigma a_j h_j$ for infinitely many j, so

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$$\sigma^{-1}g \in \bigcap_j A_j H = H$$

since the given filter is a descending chain. Thus $g = \sigma h$, that is, property (ii′) holds.

Finally, it must be shown that if

$$s^*_{i+1}A_tH = s^*_nA_tH,\ s^*_n \in S^*_i, \quad \text{then} \quad s^*_{i+1}A_t = s^*_nA_t.$$

This is clear by (iii′) if $t \leqq j$, since $s^*_{i+1} = s^*a_j$ and $A_t \geqq A_j$; if $t > j$, then $s^*_{i+1} \notin s^*_nA_tH$ by the choice of j (see iv′)). The proof follows by induction on i.

PROOF OF COROLLARY 1.

Suppose an amalgam and filters are given as in the hypotheses of Corollary 1. It will be shown that filters satisfying Theorem 2 exist in A and B. Let $a_1, a_2, \cdots$ be the elements in A but not H. Since $\cap A_iH = H$, there exists a countable subset of $\mathscr{A}$, say $A_{i_1}, A_{i_2}, \cdots$, such that $a_j \notin A_{i_j}H$, $j = 1, 2, \cdots$, that is, $\cap_t A_{i_t}H = H$. Similarly choose $B_{i_1}, B_{i_2}, \cdots$, from B such that $\cap_t B_{i_t}H = H$. Since A and B are countable, the sets

$$\{A_{i_t} | t = 1, 2, \cdots\}, \qquad \{B_{i_t} | t = 1, 2, \cdots\},$$

can be extended to countable *filters* satisfying (ii) by adjoining some elements of $\mathscr{A}$ and $\mathscr{B}$, if necessary. The resulting countable filters, again denoted $\{A_{i_t}\}$ and $\{B_{i_t}\}$, may be further extended to countable filters satisfying (i) and (ii) by adding (when necessary) for each distinct $A_{i_t} \cap H$ an element B'_{i_t} of $\mathscr{B}$ such that $B'_{i_t} \cap H = A_{i_t} \cap H$ and for each distinct $B_{i_t} \cap H$ adding A'_{i_t} from $\mathscr{A}$ such that $A'_{i_t} \cap H = B_{i_t} \cap H$. Thus we may assume that the original filters $\mathscr{A}$ and $\mathscr{B}$ are countable and satisfy (i) and (ii). Next list in any order the distinct intersections $A_i \cap H$. If this list is finite repeat the last term so that the list $A_1 \cap H, A_2 \cap H, \cdots$, is infinite. List all elements of $\mathscr{A}$ and $\mathscr{B}$ having the intersection $A_k \cap H$ with H; by repeating terms if necessary, this also may be assumed to be an infinite set say

$$A_{i_{k1}}, A_{i_{k2}}, \cdots, B_{i_{k1}}, B_{i_{k2}}, \cdots.$$

Do this for all $k = 1, 2, \cdots$ thus listing all elements of $\mathscr{A}$ and $\mathscr{B}$. Define new filters $\{A^*_i\}$ and $\{B^*_i\}$ using a diagonalization process as follows.

Let $A^*_1 = A_{i_{11}}$, and $B^*_1 = B_{i_{11}}$, and define

$$A^*_n = (\bigcap_{t=1}^{n} A_{i_{1t}}) \cap (\bigcap_{t=1}^{n-1} A_{i_{2t}}) \cap \cdots \cap A_{i_{n1}},$$

and

$$B^*_n = (\bigcap_{t=1}^{n} B_{i_{1t}}) \cap (\bigcap_{t=1}^{n-1} B_{i_{2t}}) \cap \cdots \cap B_{i_{n1}},$$

for $n = 2, 3, \cdots$.

Then $\{A_n^*\}$ and $\{B_n^*\}$ are filters which are descending chains and $A_i^* \cap H = B_i^* \cap H$ for all $i = 1, 2, \cdots$. Let $x \in \cap_n A_n^* H$. Then $x \in A_n^* H$ for all n. Let $A_j H$ be given. Then $A_j = A_{i_{rs}}$ for some i, r and s and for some n_0, $A_{n_0}^* \leqq A_{i_{rs}}$ by construction. Since $x \in A_j H$, for all j, it follows that $\cap_n A_n^* H = H$ completing the proof.

It is known that if A and B have suitable varietal properties and the generalized free product on $A \cup B|H$ is residually finite, then filters satisfying (i) and (ii) in Theorem 1 must exist in A and B (Gregorac [2]). Thus in this case if the generalized free product on $A \cup B|H$ is countable and residually finite, then so also is some permutational product on $A \cup B|H$. When does the converse hold? In particular, if some permutational product on $A \cup B|H$ is residually finite what conditions can be imposed on A and B so that property (ii) of Theorem 1 must hold?

(*Added 24 July, 1968*). In reference to a question the referee asked, note that the countability imposed in Corollary 1 is not necessary. Simple uncountable examples which are residually finite may be constructed by taking the amalgamated subgroup to be an abelian direct factor of both constituents; in this case the permutational products will be the generalized direct product on the amalgam and the amalgamated subgroup will be a direct factor.

Finally, the following question was suggested by the referee and others, but we have not been able to answer it. If $\mathfrak{A} = A \cup B|H$ is an amalgam of residually finite groups and some permutational product on $\mathfrak{A}$ is residually finite, is every permutational product on $\mathfrak{A}$ residually finite?

(*Added in proof 16 July, 1969*). Dr R. B. J. T. Allenby has settled this last question negatively; some permutational products on a given amalgam can be residually finite, when others are not.

References

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