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RESEARCH ARTICLE

On homomorphisms into Weyl modules corresponding to partitions with two parts

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Abstract
Let K be an infinite field of characteristic $p > 0$ and let λ, μ be partitions, where μ has two parts. We find sufficient arithmetic conditions on p, λ, μ for the existence of a nonzero homomorphism $\Delta(\lambda) \to \Delta(\mu)$ of Weyl modules for the general linear group $GL_n(K)$. Also, for each p we find sufficient conditions so that the corresponding homomorphism spaces have dimension at least 2.

1. Introduction

In the representation theory of the general linear group $GL_n(K)$, where K is an infinite field of characteristic $p > 0$, the Weyl modules $\Delta(\lambda)$ are of central importance. These are parameterized by partitions λ with at most n parts. Over a field of characteristic zero, the modules $\Delta(\lambda)$ are irreducible. However over fields of positive characteristics, this is no longer true and determining their structure is a major problem. In particular, very little is known about homomorphisms between them.

For $GL_3(K)$, all homomorphisms between Weyl modules have been classified when $p > 2$ by Cox and Parker [5]. Some of the few general results are the nonvanishing theorems of Carter and Payne [4] and Koppinen [11], and the row or column removal theorems of Fayers and Lyle [14] and Kulkarni [12].

In [17], we examined homomorphisms into hook Weyl modules and obtained a classification result. This has been obtained also by Loubert [13] for $p > 2$. In the present paper, we consider homomorphisms $\Delta(\lambda) \to \Delta(\mu)$, where μ has two parts. The main result, Theorem 3.1, provides sufficient arithmetic conditions on λ, μ, and p so that $\mathrm{Hom}_S(\Delta(\lambda), \Delta(\mu)) \neq 0$, where S is the Schur algebra for $GL_n(K)$ of appropriate degree. An explicit map is provided that corresponds to the sum of all standard tableaux of shape μ and weight λ. The main tool of the proof is the description of Weyl modules by generators and relations of Akin et al. [2].

The first examples of pairs of Weyl modules with homomorphism spaces of dimension greater than 1 were obtained by Dodge [6]. Shortly after, more were found by Lyle [14]. In Corollary 6.2, we find sufficient conditions on λ, μ and p so that $\dim \mathrm{Hom}_S(\Delta(\lambda), \Delta(\mu)) > 1$ and thus we have new examples of homomorphism spaces between Weyl modules of dimension greater than 1.

By a classical theorem of Carter and Lusztig [3], the results in Theorem 3.1 and Corollary 6.2 have analogues for Specht modules for the symmetric group when $p > 2$, see Remark 3.2 and the Remark after Corollary 6.2.

Section 2 is devoted to notation and preliminaries. In Section 3, we state the main result, and in Section 4, we consider the straightening law needed later. The proof of the main result is in Section 5. In Section 6, we consider homomorphism spaces of dimension greater than 1.

2. Preliminaries

2.1. *Notation*

Throughout this paper, K will be an infinite field of characteristic $p > 0$. We will be working with homogeneous polynomial representations of $GL_n(K)$ of degree r, or equivalently, with modules over the Schur algebra $S = S_K(n, r)$. A standard reference here is [8].

In what follows we fix notation and recall from Akin and Buchsbaum [1], and also Akin et al. [2] important facts.

Let $V = K^n$ be the natural $GL_n(K)$-module. The divided power algebra $DV = \sum_{i\geq 0} D_i V$ of V is defined as the graded dual of the Hopf algebra $S(V^*)$, where V^* is the linear dual of V and $S(V^*)$ is the symmetric algebra of V^*, see [2], I.4. For $v \in V$ and i, j nonnegative integers, we will use many times relations of the form

$$v^{(i)}v^{(j)} = \binom{i+j}{j} v^{(i+j)},$$

where $\binom{i+j}{j}$ is the indicated binomial coefficient.

By $\wedge(n, r)$, we denote the set of sequences $a = (a_1, \ldots, a_n)$ of nonnegative integers that sum to r and by $\wedge^+(n, r)$ we denote the subset of $\wedge(n, r)$ consisting of sequences $\lambda = (\lambda_1, \ldots, \lambda_n)$ such that $\lambda_1 \geq \lambda_2 \cdots \geq \lambda_n$. Elements of $\wedge^+(n, r)$ are referred to as partitions of r with at most n parts. The transpose partition $\lambda^t = (\lambda_1^t, ..., \lambda_q^t) \in \wedge^+(\lambda_1, r)$, $q = \lambda_1$, of a partition $\lambda = (\lambda_1, ..., \lambda_n) \in \wedge^+(n, r)$ is defined by $\lambda_j^t = \#\{i : \lambda_i \geq j\}$.

If $a = (a_1, \ldots, a_n) \in \wedge(n, r)$, we denote by $D(a)$ or $D(a_1, \ldots, a_n)$ the tensor product $D_{a_1}V \otimes \cdots \otimes D_{a_n}V$. All tensor products in this paper are over K.

The exterior algebra of V is denoted $\Lambda V = \sum_{i\geq 0} \Lambda^i V$. If $a = (a_1, \ldots, a_n) \in \wedge(n, r)$, we denote by $\Lambda(a)$ the tensor product $\Lambda^{a_1}V \otimes \cdots \otimes \Lambda^{a_n}V$.

For $\lambda \in \wedge^+(n, r)$, we denote by $\Delta(\lambda)$ the corresponding Weyl module for S. In [2], Definition II.1.4, the module $\Delta(\lambda)$ (denoted $K_\lambda F$ there), was defined as the image a particular map $d'_\lambda : D(\lambda) \to \Lambda(\lambda^t)$. For example, if $\lambda = (r)$, then $\Delta(\lambda) = D_r V$, and if $\lambda = (1^r)$, then $\Delta(\lambda) = \Lambda^r V$.

2.2. *Relations for Weyl modules.*

We recall from [2], Theorem II.3.16, the following description of $\Delta(\lambda)$ in terms of generators and relations.

Theorem 2.1 ([2]). *Let $\lambda = (\lambda_1, \ldots, \lambda_m) \in \wedge^+(n, r)$, where $\lambda_m > 0$. There is an exact sequence of S-modules*

$$\sum_{i=1}^{m-1}\sum_{t=1}^{\lambda_{i+1}} D(\lambda_1, \ldots, \lambda_i + t, \lambda_{i+1} - t, \ldots, \lambda_m) \xrightarrow{\square_\lambda} D(\lambda) \xrightarrow{d'_\lambda} \Delta(\lambda) \to 0,$$

where the restriction of $\square_\lambda$ to the summand $M_i(t) = D(\lambda_1, \ldots, \lambda_i + t, \lambda_{i+1} - t, \ldots, \lambda_m)$ is the composition

$$M_i(t) \xrightarrow{1\otimes\cdots\otimes\Delta\otimes\cdots 1} D(\lambda_1, \ldots, \lambda_i, t, \lambda_{i+1} - t, \ldots, \lambda_m) \xrightarrow{1\otimes\cdots\otimes\eta\otimes\cdots 1} D(\lambda),$$

where $\Delta : D(\lambda_i + t) \to D(\lambda_i, t)$ and $\eta : D(t, \lambda_{i+1} - t) \to D(\lambda_{i+1})$ are the indicated components of the comultiplication and multiplication, respectively, of the Hopf algebra DV and d'_λ is the map in [2], Def.II.13.

2.3. *Standard basis of* $\Delta(\mu)$

We will record here and in the next subsection two important facts from [2] and [1] specified to the case of partitions consisting of two parts.

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Let us fix the order $e_1 < e_2 < ... < e_n$ on the set $\{e_1, e_2, ..., e_n\}$ of the canonical basis elements of the natural module V of $GL_n(K)$. We will denote each element e_i by its subscript i. For a partition $\mu = (\mu_1, \mu_2) \in \wedge^+(n, r)$, a *tableau* of shape μ is a filling of the diagram of μ with entries from $\{1, ..., n\}$. Such a tableau is called *standard* if the entries are weakly increasing across the rows from left to right and strictly increasing in the columns from top to bottom. (The terminology used in [2] is 'co-standard').

The set of standard tableaux of shape μ will be denoted by $\mathrm{ST}(\mu)$. The *weight* of a tableau T is the tuple $\alpha = (\alpha_1, ..., \alpha_n)$, where α_i is the number of appearances of the entry i in T. The subset of $\mathrm{ST}(\mu)$ consisting of the (standard) tableaux of weight α will be denoted by $\mathrm{ST}_\alpha(\mu)$.

For example, the following tableau of shape $\mu = (6, 4)$.

$$T = \begin{array}{|c|c|c|c|c|c|} \hline 1 & 1 & 1 & 2 & 2 & 4 \\ \hline 2 & 2 & 3 & 4 & \multicolumn{2}{c}{} \\ \cline{1-4} \end{array}$$

is standard and has weight $\alpha = (3, 4, 1, 2)$.

We will use 'exponential' notation for standard tableaux. Thus for the above example, we write

$$T = \begin{matrix} 1^{(3)}2^{(2)}4 \\ 2^{(2)}34 \end{matrix}.$$

To each tableau T of shape $\mu = (\mu_1, \mu_2)$, we may associate an element

$$x_T = x_T(1) \otimes x_T(2) \in D(\mu_1, \mu_2),$$

where $x_T(i) = 1^{(a_{i1})} \cdots n^{(a_{in})}$ and a_{ij} is equal to the number of appearances of j in the i-th row of T. For example, the T depicted above yields $x_T = 1^{(3)}2^{(2)}4 \otimes 2^{(2)}34$. According to [2], Theorem II.2.16, we have the following.

Theorem 2.2 ([2]). *The set $\{d'_\mu(x_T) : T \in \mathrm{ST}(\mu)\}$ is a basis of the K-vector space $\Delta(\mu)$.*

If $x = 1^{(a_1)}2^{(a_2)} \cdots n^{(a_n)} \otimes 1^{(b_1)}2^{(b_2)} \cdots n^{(b_n)} \in D(\mu)$, we will denote the element $d'_\mu(x) \in \Delta(\mu)$ by

$$\begin{bmatrix} 1^{(a_1)}2^{(a_2)} \cdots n^{(a_n)} \\ 1^{(b_1)}2^{(a_2)} \cdots n^{(b_n)} \end{bmatrix}.$$

2.4. Weight subspaces of $\Delta(\mu)$

Suppose $n \geq r$. Let $\nu \in \wedge(n, r)$ and $\mu = (\mu_1, \mu_2) \in \wedge^+(2, r)$. According to [1], equation (11), a basis of the K-vector space $\mathrm{Hom}_S(D(\nu), \Delta(\mu))$ is in 1-1 correspondence with set $\mathrm{ST}_\nu(\mu)$ of standard tableaux of shape μ and weight ν.

For the computations to follow, we need to make the above correspondence explicit. Let $\nu = (\nu_1, ..., \nu_n) \in \wedge(n, r)$ and $T \in \mathrm{ST}_\nu(\mu)$. Let a_i (respectively, b_i) be the number of appearances of i in the first row (respectively, second row) of T. We note that $\nu_i = a_i + b_i$ for each i. In particular, we have $a_1 = \nu_1$ because of standardness of T. Define the map

$$\phi_T : D(\nu) \to \Delta(\mu),$$

$$x_1 \otimes x_2 \otimes \cdots \otimes x_n \mapsto \sum_{i_2,...,i_n} d'_\mu\left(x_1 x_{2i_2}(a_2) \cdots x_{ni_n}(a_n) \otimes x_{2i_2}(b_2)' \cdots x_{ni_n}(b_n)'\right),$$

where $\sum_{i_s} x_{si_s}(a_s) \otimes x_{si_s}(b_s)'$ is the image of x_s under the component

$$D(\nu_s) \to D(a_s, b_s),$$

of the diagonalization $\Delta : DV \to DV \otimes DV$ of the Hopf algebra DV for $s = 2, ..., n$. Thus we have that a basis of the K-vector space $\mathrm{Hom}_S(D(\nu), \Delta(\mu))$ is the set

$$\{\phi_T : T \in \mathrm{ST}_\nu(\mu)\}.$$

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In particular, suppose $\lambda = (\lambda_1, ..., \lambda_m) \in \wedge^+(n, r)$ is a partition and $\mu = (\mu_1, \mu_2) \in \wedge^+(2, r)$ satisfies $\mu_2 \leq \lambda_1$. This inequality means that each tableau of shape μ that has the form

$$\begin{array}{l} 1^{(\lambda_1)}2^{(a_2)} \cdots m^{(a_m)} \\ 2^{(b_2)} \cdots m^{(b_m)} \end{array}$$

is standard. Hence, we have the following result.

Lemma 2.3. *Suppose $n \geq r$. Let $\lambda, \mu \in \wedge^+(n, r)$, where $\lambda = (\lambda_1, ..., \lambda_m)$ and $\mu = (\mu_1, \mu_2)$. If $\mu_2 \leq \lambda_1$, than a basis of the K- vector space* $\mathrm{Hom}_S(D(\lambda), \Delta(\mu))$ *is given by the elements ϕ_T, where*

$$T = \begin{array}{l} 1^{(\lambda_1)}2^{(a_2)} \cdots m^{(a_m)} \\ 2^{(b_2)} \cdots m^{(b_m)} \end{array},$$

are such that

$$a_i, b_i \geq 0,\ a_i + b_i = \lambda_i, (i = 2, ..., m),$$
$$a_2 + \cdots + a_m = \mu_1 - \lambda_1,\ b_2 + \cdots + b_m = \mu_2.$$

Example. Suppose $\lambda = (\lambda_1, 3, 3)$ and $\mu = (\lambda_1 + 4, 2)$, where $\lambda_1 \geq 3$. Then $\{[T_1], [T_2], [T_3]\}$ is a basis of $\mathrm{Hom}_S(D(\lambda), \Delta(\mu))$, where

$$T_1 = \begin{array}{l} 1^{(\lambda_1)}2^{(3)}3 \\ 3^{(2)} \end{array}, T_2 = \begin{array}{l} 1^{(\lambda_1)}2^{(2)}3^{(2)} \\ 23 \end{array}, T_3 = \begin{array}{l} 1^{(\lambda_1)}23^{(3)} \\ 2^{(2)} \end{array}.$$

For $x = 1^{(\lambda_1)} \otimes 1^{(2)}2 \otimes 3^{(3)} \in D(\lambda)$ and $T = T_2$, we have

$$\phi_T(x) = \binom{\lambda_1 + 2}{2} \begin{bmatrix} 1^{(\lambda_1+2)}3^{(2)} \\ 23 \end{bmatrix} + \binom{\lambda_1 + 1}{1} \begin{bmatrix} 1^{(\lambda_1+1)}23^{(2)} \\ 13 \end{bmatrix},$$

where the binomial coefficients come from multiplication in the divided power algebra DV.

3. Main result

In order to state the main result of this paper, we use the following notation. If x, y are positive integers, let

$$R(x, y) = \gcd\left\{\binom{x}{1}, \binom{x+1}{2}, ..., \binom{x+y-1}{y}\right\}.$$

If x is a positive integer, let $R(x, 0) = 0$.

Theorem 3.1. *Let K be an infinite field of characteristic $p > 0$ and let $n \geq r$ be positive integers. Let $\lambda, \mu \in \wedge^+(n, r)$ be partitions such that $\lambda = (\lambda_1, ..., \lambda_m)$ and $\mu = (\mu_1, \mu_2)$, where $\lambda_m \neq 0$, $m \geq 2$ and $\mu_2 \leq \lambda_1 \leq \mu_1$. If p divides all of the following integers*

$$R(\lambda_1 - \mu_2 + 1, l), l = \min\{\lambda_2, \mu_1 - \lambda_1\}$$
$$R(\lambda_i + 1, \lambda_{i+1}), i = 2, ..., m - 1,$$

then the map

$$\psi = \sum_{T \in \mathrm{ST}_\lambda(\mu)} \phi_T$$

induces a nonzero homomorphism $\Delta(\lambda) \to \Delta(\mu)$.

Remark 3.2. *Consider the symmetric group $\mathfrak{S}_r$ on r symbols. For a partition λ of r, let* $\mathrm{Sp}(\lambda)$ *be the corresponding Specht module defined in Section 6.3 of [8]. From Theorem 3.7 of [3], we have*

$$\dim \mathrm{Hom}_S(\Delta(\lambda), \Delta(\mu)) \leq \dim \mathrm{Hom}_{\mathfrak{S}_r}(\mathrm{Sp}(\mu), \mathrm{Sp}(\lambda)),$$

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for all partitions λ, μ of r. (In fact we have equality if $p > 2$ according to loc. cit.) Hence, our Theorem 3.1 may be considered as a nonvanishing result for homomorphisms between Specht modules.

Remark 3.3. *Here we make some comments concerning the inequalities $n \geq r$, $m \geq 2$ and $\mu_2 \leq \lambda_1 \leq \mu_1$ in the statement of the above theorem.*

(1) The assumption $n \geq r$ is needed so that the Weyl modules $\Delta(\lambda)$, $\Delta(\mu)$ are nonzero. As is usual with such results, it turns out that this assumption may be relaxed to $n \geq m$, since m is the maximum number of parts of the partitions λ, μ. This follows from the proof of the theorem to be given in Section 5.

It is well known that if $\mathrm{Hom}_S(\Delta(\lambda), \Delta(\mu)) \neq 0$, then $\lambda \trianglelefteq \mu$ in the dominance ordering, meaning in particular that $\lambda_1 \leq \mu_1$.

If $m = 1$, then by the previous remark, $\mathrm{Hom}_S(\Delta(\lambda), \Delta(\mu)) = 0$, unless $\mu = \lambda$, in which case $\mathrm{Hom}_S(\Delta(\lambda), \Delta(\mu)) = K$ by [10], the analogue for Weyl modules of II.2.8 Proposition.

(2) In the above remarks, the corresponding inequalities were needed to avoid trivial situations. The nature of the assumption $\mu_2 \leq \lambda_1$ is different. There are cases where nonzero homomorphisms $\Delta(\lambda) \to \Delta(\mu)$ exist if $\mu_2 > \lambda_1$. For example, let $n = 3$, $p = 2$, $\lambda = (2, 2, 2)$ and $\mu = (3, 3)$. One may check that the map ϕ_T, where $T = \begin{smallmatrix} 1^{(2)}2 \\ 23^{(2)} \end{smallmatrix}$, induces a nonzero map $\Delta(\lambda) \to \Delta(\mu)$. It would be interesting to find general results.

The main point for us of the assumption $\mu_2 \leq \lambda_1$ is that every tableau T in Lemma 2.3 is standard.

(3) If $\lambda_1 = \mu_1$, then $R(\lambda_1 - \mu_2 + 1, l) = 0$ and the first divisibility condition of the theorem holds for all p. The remaining divisibility conditions are exactly those for which we have $\mathrm{Hom}_{S'}(\Delta(\lambda_2, ..., \lambda_m), \Delta(\mu_2)) \neq 0$, where $S' = S_K(n, r - \lambda_1)$. This follows, for example, from Theorem 3.1 of [17]. Hence, in this case, we have an instance of row removal which states that $\dim \mathrm{Hom}_S(\Delta(\lambda), \Delta(\mu)) = \dim \mathrm{Hom}_{S'}(\Delta(\lambda_2, ..., \lambda_m), \Delta(\mu_2))$. *See the paper by Fayers and Lyle [7], Theorem 2.2 (stated for Specht modules), or the paper by Kulkarni [12], Proposition 1.2.*

For further use, we note that the divisibility assumptions of Theorem 3.1 may be stated in a different way. For a positive integer y, let $l_p(y)$ be the least integer i such that $p^i > y$. From James [9], Corollary 22.5, we have the following result.

Lemma 3.4 ([9]). *Let $x \geq y$ be positive integers. Then p divides R(x, y) if and only if $p^{l_p(y)}$ divides x.*

4. Straightening

For the proof of Theorem 3.1, we will need the following identities involving binomial coefficients. Our convention is that $\binom{a}{b} = 0$ if $b > a$ or $b < 0$.

Lemma 4.1.

(1) Let $a, m_1, ..., m_s$ be nonnegative integers and $m = m_1 + \cdots + m_s$.

a. We have

$$\sum_{j_1+\cdots+j_s=a} \binom{m_1}{j_1} \cdots \binom{m_s}{j_s} = \binom{m}{a},$$

where the sum ranges over all nonnegative integers $j_1, ..., j_s$ such that $j_1 + \cdots + j_s = a$.

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b. *If $m > 0$, then*

$$\sum_{j_0+\cdots+j_s=m} (-1)^{j_0} \binom{m_1}{j_1} \cdots \binom{m_s}{j_s} = 0,$$

where the sum ranges over all nonnegative integers $j_0, ..., j_s$ such that $j_0 + \cdots + j_s = m$.

(2) *Let a, b, c be nonnegative integers such that $b \leq a$. Then*

$$\sum_{j=0}^{c} (-1)^{c-j} \binom{a+j}{j} \binom{b}{c-j} = \binom{a-b+c}{c} = \sum_{j=0}^{c} (-1)^{j} \binom{a+c-j}{c-j} \binom{b}{j}.$$

Proof.

(1) The identity in (a) is Vandermonde's identity. For (b), we have

$$\begin{aligned}
\sum_{j_0+\cdots+j_s=m} (-1)^{j_0} \binom{m_1}{j_1} \cdots \binom{m_s}{j_s} &= \sum_{j_0=0}^{m} \sum_{j_1+\cdots+j_s=m-j_0} (-1)^{j_0} \binom{m_1}{j_1} \cdots \binom{m_s}{j_s} \\
&= \sum_{j_0=0}^{m} (-1)^{j_0} \sum_{j_1+\cdots+j_s=m-j_0} \binom{m_1}{j_1} \cdots \binom{m_s}{j_s} \\
&= \sum_{j_0=0}^{m} (-1)^{j_0} \binom{m}{m-j_0} \\
&= 0.
\end{aligned}$$

(2) The second identity is Lemma 2.6 of [14] for $q = 1$. The first follows from the second with the substitution $j \mapsto c - j$. ☐

We will also need the following explicit form of the straightening law concerning violations of standardness in the first column.

Lemma 4.2. *Let $\mu = (\mu_1, \mu_2) \in \wedge^+(n, r)$, $(a_1, ..., a_n) \in \wedge(n, \mu_1)$ and $(b_1, ..., b_n) \in \wedge(n, \mu_2)$.*

(1) *If $a_1 + b_1 > \mu_1$, then* $\begin{bmatrix} 1^{(a_1)} \cdots n^{(a_n)} \\ 1^{(b_1)} \cdots n^{(b_n)} \end{bmatrix} = 0.$

(2) *If $a_1 + b_1 \leq \mu_1$, then in $\Delta(\mu)$ we have*

$$\begin{bmatrix} 1^{(a_1)} \cdots n^{(a_n)} \\ 1^{(b_1)} \cdots n^{(b_n)} \end{bmatrix} = (-1)^{b_1} \sum_{i_2,...,i_n} \binom{b_2+i_2}{b_2} \cdots \binom{b_n+i_n}{b_n} \begin{bmatrix} 1^{(a_1+b_1)} 2^{(a_2-i_2)} \cdots n^{(a_n-i_n)} \\ 2^{(b_2+i_2)} \cdots n^{(b_n+i_n)} \end{bmatrix},$$

where the sum ranges over all nonnegative integers $i_2, ..., i_n$ such that $i_2 + \cdots + i_n = b_1$ and $i_s \leq a_s$ for all $s = 2, ..., n$.

Proof.

(1) This is clear since there is no element in $\Delta(\mu)$ of weight $(\nu_1, ..., \nu_n)$ satisfying $\nu_1 > \mu_1$.

(2) We proceed by induction on b_1, the case $b_1 = 0$ being clear. Suppose $b_1 > 0$. Consider the element $x \in D(\mu_1 + b_1, \mu_2 - b_1)$, where

$$x = 1^{(a_1+b_1)} 2^{(a_2)} \cdots n^{(a_n)} \otimes 2^{(b_2)} \cdots n^{(b_n)},$$

and the map

$$\delta : D(\mu_1 + b_1, \mu_2 - b_1) \xrightarrow{\Delta\otimes 1} D(\mu_1, b_1, \mu_2 - b_1) \xrightarrow{1\otimes\eta} D(\mu_1, \mu_2).$$

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According to the analogue of Lemma II.2.9 of [2] for divided powers in place of exterior powers, we have $d'_\mu(\delta(x)) = 0$ in $\Delta(\mu_1, \mu_2)$. Thus

$$\begin{bmatrix} 1^{(a_1)} \cdots n^{(a_n)} \\ 1^{(b_1)} \cdots n^{(b_n)} \end{bmatrix} = -\sum_{j_1,\dots,j_n} \binom{b_2+j_2}{b_2} \cdots \binom{b_n+j_n}{b_n} \begin{bmatrix} 1^{(a_1+b_1-j_1)} 2^{(a_2-j_2)} \cdots n^{(a_n-j_n)} \\ 1^{(j_1)} 2^{(b_2+j_2)} \cdots n^{(b_n+j_n)} \end{bmatrix},$$

where the sum ranges over all nonnegative integers $j_1, \dots, j_n$ such that $j_1 + \cdots + j_n = b_1, j_1 < b_1$ and $j_s \le a_s$ for all $s = 2, \dots, n$. Let X be the right hand side of the above equality. By induction, we have

$$X = -\sum_{j_1,\dots,j_n} \binom{b_2+j_2}{b_2} \cdots \binom{b_n+j_n}{b_n} (-1)^{j_1} \sum_{k_2,\dots,k_n} \binom{b_2+j_2+k_2}{b_2+j_2} \cdots \binom{b_n+j_n+k_n}{b_n+j_n} \begin{bmatrix} 1^{(a_1+b_1)} 2^{(a_2-j_2-k_2)} \cdots n^{(a_n-j_n-k_n)} \\ 2^{(b_2+j_2+k_2)} \cdots n^{(b_n+j_n+k_n)} \end{bmatrix},$$

where the new sum ranges over all nonnegative integers $k_2, \dots, k_n$ such that $k_2 + \cdots + k_n = j_1$ and $k_s \le a_s - j_s$ for all $s = 2, \dots, n$. Using the identities

$$\binom{b_s+j_s}{b_s} \binom{b_s+j_s+k_s}{b_s+j_s} = \binom{b_s+j_s+k_s}{b_s} \binom{j_s+k_s}{j_s},$$

for $s = 2, \dots, n$, we obtain

$$X = -\sum_{j_1,\dots,j_n,k_2,\dots,k_n} (-1)^{j_1} \binom{b_2+j_2+k_2}{b_2} \cdots \binom{b_n+j_n+k_n}{b_n} \binom{j_2+k_2}{j_2} \cdots \binom{j_n+k_n}{j_n} \begin{bmatrix} 1^{(a_1+b_1)} 2^{(a_2-j_2-k_2)} \cdots n^{(a_n-j_n-k_n)} \\ 2^{(b_2+j_2+k_2)} \cdots n^{(b_n+j_n+k_n)} \end{bmatrix}.$$

The coefficient c of

$$\begin{bmatrix} 1^{(a_1+b_1)} 2^{(a_2-i_2)} \cdots n^{(a_n-i_n)} \\ 2^{(b_2+i_2)} \cdots n^{(b_n+i_n)} \end{bmatrix}$$

in the right hand side of the above equation is equal to

$$-\sum_{\substack{j_1,\dots,j_n,k_2,\dots,k_n \\ j_s+k_s=i_s}} (-1)^{j_1} \binom{b_2+j_2+k_2}{b_2} \cdots \binom{b_n+j_n+k_n}{b_n} \binom{j_2+k_2}{j_2} \cdots \binom{j_n+k_n}{j_n},$$

where the sum is restricted over those $j_1, \dots, j_n$ and $k_2, \dots, k_n$ that satisfy the additional conditions $j_s + k_s = i_s$ for all $s = 2, \dots, n$. Hence

$$\begin{aligned} c &= -\sum_{j_1,\dots,j_n} (-1)^{j_1} \binom{b_2+i_2}{b_2} \cdots \binom{b_n+i_n}{b_n} \binom{i_2}{j_2} \cdots \binom{i_n}{j_n} \\ &= -\binom{b_2+i_2}{b_2} \cdots \binom{b_n+i_n}{b_n} \sum_{j_1,\dots,j_n} (-1)^{j_1} \binom{i_2}{j_2} \cdots \binom{i_n}{j_n}. \end{aligned}$$

Remembering that in the last sum we have $j_1 < b_1$, Lemma 4.1 (1)(b) yields

$$\sum_{j_1,\dots,j_n} (-1)^{j_1} \binom{i_2}{j_2} \cdots \binom{i_n}{j_n} = 0 - (-1)^{b_1}.$$

Thus $c = (-1)^{b_1} \binom{b_2+i_2}{b_2} \cdots \binom{b_n+i_n}{b_n}$. □

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5. Proof of the main theorem

Consider the map $\psi \in \mathrm{Hom}_S(D(\lambda), \Delta(\mu))$ given by the sum

$$\psi = \sum_{T\in \mathrm{ST}_\lambda(\mu)} \phi_T,$$

in the statement of Theorem 3.1 We will show, according to Theorem 2.1, that $\psi(x) = 0$ for every $x \in Im(\square_\lambda)$. First we look at the relations corresponding to rows 1 and 2 of $\Delta(\lambda)$.

Relations from rows 1 and 2

Let $x = 1^{(\lambda_1)} \otimes 1^{(t)}2^{(\lambda_2-t)} \otimes 3^{(\lambda_3)} \cdots m^{(\lambda_m)} \in Im(\square_\lambda)$, where $t \le \lambda_2$, and let $T \in \mathrm{ST}_\lambda(\mu)$. Then T is of the form

$$T = \begin{matrix} 1^{(\lambda_1)}2^{(a_2)}\cdots m^{(a_m)} \\ 2^{(b_2)}\cdots m^{(b_m)} \end{matrix} \in \mathrm{ST}_\lambda(\mu),$$

where the a_i, b_i satisfy the conditions of Lemma 2.3. Using the definition of ϕ_T from 2.4, we have

$$\phi_T(x) = \sum_{i\le t}\binom{\lambda_1+i}{i}\begin{bmatrix} 1^{(\lambda_1+i)}2^{(a_2-i)}3^{(a_3)}\cdots m^{(a_m)} \\ 1^{(t-i)}2^{(\lambda_2-t-a_2+i)}3^{(b_3)}\cdots m^{(b_m)} \end{bmatrix}.$$

If $(\lambda_1 + i) + (t - i) \ge \mu_1$, then by the first part of Lemma 4.2 we obtain $\phi_T(x) = 0$. Hence we may assume that $t \le min\{\lambda_2, \mu_1 - \lambda_1\}$. Using the second part of Lemma 4.2, we have

$$\phi_T(x) = \sum_{i\le t}\binom{\lambda_1+i}{i}(-1)^{t-i}\sum_{k_2+\cdots+k_m=t-i}\binom{b_2-k_3-\cdots-k_m}{k_2}\binom{b_3+k_3}{k_3}\cdots\binom{b_m+k_m}{k_m}$$
$$\begin{bmatrix} 1^{(\lambda_1+t)}2^{(a_2+k_3+\cdots+k_m)}3^{(a_3-k_3)}\cdots m^{(a_m-k_m)} \\ 2^{(b_2-k_3-\cdots-k_m)}3^{(b_3+k_3)}\cdots m^{(b_m+k_m)} \end{bmatrix}.$$

Let $c \in K$ be the coefficient of $\begin{bmatrix} 1^{(\lambda_1+t)}2^{(a_2+k_3+\cdots+k_m)}3^{(a_3-k_3)}\cdots m^{(a_m-k_m)} \\ 2^{(b_2-k_3-\cdots-k_m)}3^{(b_3+k_3)}\cdots m^{(b_m+k_m)} \end{bmatrix}$ in the right hand side of the last equation and let $k = k_3 + \cdots + k_m$. Then

$$\begin{aligned} c &= \left(\sum_{i=0}^{t}\binom{\lambda_1+i}{i}(-1)^{t-i}\binom{b_2-k}{t-k-i}\right)\binom{b_3+k_3}{k_3}\cdots\binom{b_m+k_m}{k_m} \\ &= (-1)^k\left(\sum_{i=0}^{t-k}\binom{\lambda_1+i}{i}(-1)^{t-k-i}\binom{b_2-k}{t-k-i}\right)\binom{b_3+k_3}{k_3}\cdots\binom{b_m+k_m}{k_m} \\ &= (-1)^k\binom{\lambda_1-b_2+t}{t-k}\binom{b_3+k_3}{k_3}\cdots\binom{b_m+k_m}{k_m}, \end{aligned}$$

where in the third equality we used the first identity of Lemma 4.1 (2). Thus

$$\phi_T(x) = \sum_{k_3,\ldots,k_m}(-1)^k\binom{\lambda_1-b_2+t}{t-k}\binom{b_3+k_3}{k_3}\cdots\binom{b_m+k_m}{k_m}\begin{bmatrix} 1^{(\lambda_1+t)}2^{(a_2+k)}3^{(a_3-k_3)}\cdots m^{(a_m-k_m)} \\ 2^{(b_2-k)}3^{(b_3+k_3)}\cdots m^{(b_m+k_m)} \end{bmatrix},$$

where $k = k_3 + \cdots + k_m$ and the sum ranges over all nonnegative integers $k_3, \ldots, k_m$ such that $k \le b_2$ and $k_s \le a_s$ for all $s = 3, \ldots, m$.

By summing with respect to $T \in \mathrm{ST}_\lambda(\mu)$ and using Lemma 2.3, we obtain

$$\psi(x) = \sum_{b_2,\ldots,b_m}\sum_{k_3,\ldots,k_m}(-1)^k\binom{\lambda_1-b_2+t}{t-k}\binom{b_3+k_3}{k_3}\cdots\binom{b_m+k_m}{k_m} \tag{1}$$
$$\begin{bmatrix} 1^{(\lambda_1+t)}2^{(a_2+k)}3^{(a_3-k_3)}\cdots m^{(a_m-k_m)} \\ 2^{(b_2-k)}3^{(b_3+k_3)}\cdots m^{(b_m+k_m)} \end{bmatrix},$$

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where the new sum is over all nonnegative integers $b_2, ..., b_m$ such that $b_i \leq \lambda_i (i = 2, ..., m)$ and $b_2 + \cdots + b_m = \mu_2$.

Fix

$$[S] = \begin{bmatrix} 1^{(\lambda_1+t)} 2^{(a_2+k)} 3^{(a_3-k_3)} \cdots m^{(a_m-k_m)} \\ 2^{(b_2-k)} 3^{(b_3+k_3)} \cdots m^{(b_m+k_m)} \end{bmatrix} \in \Delta(\mu),$$

in the right hand side of (1) and let $q = \mu_2 - (b_3 + k_3) - \cdots - (b_m + k_m)$. Then $q = b_2 - k$. The coefficient of $[S]$ in (1) is equal to

$$\begin{aligned} &\sum_k (-1)^k \binom{\lambda_1 - q - k + t}{t - k} \sum_{k_3 + \cdots + k_m = k} \binom{b_3 + k_3}{k_3} \cdots \binom{b_m + k_m}{k_m} \\ &= \sum_k (-1)^k \binom{\lambda_1 - q - k + t}{t - k} \binom{\mu_2 - q}{k} \\ &= \binom{\lambda_1 - \mu_2 + t}{t} \\ &= 0, \end{aligned}$$

where in the first equality we used Lemma 4.1 (1)(a) and in the second equality we used the second identity of Lemma 4.1 (2).

Relations from rows i and $i+1$ $(i > 1)$.

This computation is similar to the previous one but simpler as there is no straightening. Let $y = 1^{(\lambda_1)} \otimes \cdots \otimes i^{(\lambda_i)} \otimes i^{(t)}(i+1)^{(\lambda_{i+1}-t)} \otimes \cdots \otimes m^{(\lambda_m)} \in Im(\square_\lambda)$, where $i > 1$ and $t \leq \lambda_{i+1}$. As before let

$$T = \begin{matrix} 1^{(a_1)} \cdots m^{(a_m)} \\ 2^{(b_2)} \cdots m^{(b_m)} \end{matrix} \in \mathrm{ST}_\lambda(\mu).$$

The definition of ϕ_T yields

$$\phi_T(y) = \sum_{j \leq t} \binom{a_i + j}{j} \binom{b_i + t - j}{t - j} \begin{bmatrix} 1^{(\lambda_1)} 2^{(a_2)} \cdots i^{(a_i+j)} (i+1)^{(a_{i+1}-j)} \cdots m^{(a_m)} \\ 2^{(b_2)} \cdots i^{(b_i+t-j)} (i+1)^{(b_{i+1}-t+j)} \cdots m^{(b_m)} \end{bmatrix}.$$

By summing with respect to $T \in \mathrm{ST}_\lambda(\mu)$ and using Lemma 2.3, we have

$$\psi(y) = \sum_{b_2,...,b_m} \sum_{j \leq t} \binom{\lambda_i - b_i + j}{j} \binom{b_i + t - j}{t - j} \tag{2}$$

$$\begin{bmatrix} 1^{(\lambda_1)} 2^{(\lambda_2-b_2)} \cdots i^{(\lambda_i-b_i+j)} (i+1)^{(\lambda_{i+1}-b_{i+1}-j)} \cdots m^{(\lambda_m-b_m)} \\ 2^{(b_2)} \cdots i^{(b_i+t-j)} (i+1)^{(b_{i+1}-t+j)} \cdots m^{(b_m)}, \end{bmatrix}$$

where the new sum ranges over all nonnegative integers $b_2, ..., b_m$ such that $b_i \leq \lambda_i$ $(i = 2, ..., m)$ and $b_2 + \cdots + b_m = \mu_2$.

Fix

$$[S] = \begin{bmatrix} 1^{(\lambda_1)} 2^{(\lambda_2-b_2)} \cdots i^{(\lambda_i-b_i+j)} (i+1)^{(\lambda_{i+1}-b_{i+1}-j)} \cdots m^{(\lambda_m-b_m)} \\ 2^{(b_2)} \cdots i^{(b_i+t-j)} (i+1)^{(b_{i+1}-t+j)} \cdots m^{(b_m)} \end{bmatrix} \in \Delta(\mu),$$

in the right hand side of (2) and let $q = b_i - j$. The coefficient of $[S]$ in (2) is equal to

$$\sum_{j \leq t} \binom{\lambda_j - q}{j} \binom{t + q}{t - j} = \binom{\lambda_i + t}{t} = 0,$$

where in the first equality we used Lemma 4.1 (1)(a).

We have shown thus far that the map $\psi = \sum_{T\in \mathrm{ST}(\lambda,\mu)} \phi_T$ induces a homomorphism of S-modules $\bar{\psi} : \Delta(\lambda) \to \Delta(\mu)$ and it remains to be shown that $\bar{\psi} \neq 0$. Let $z = 1^{(\lambda_1)} \otimes \cdots \otimes m^{(\lambda_m)} \in D(\lambda)$ and $T \in \mathrm{ST}_\lambda(\mu)$. Then from the definition of ϕ_T we have $\phi_T(x) = [T]$ and hence

$$\psi(x) = \sum_{T\in \mathrm{ST}_\lambda(\mu)} [T].$$

The right hand side is a sum of distinct basis elements in $\Delta(\mu)$ (each with coefficient 1) according to Theorem 2.2 and hence nonzero. The proof is complete.

Remark 5.1 *Lyle has shown in [15], Propositions 2.19 through 2.27 and subsection 3.3, that the homomorphism spaces between Specht modules corresponding to partitions $\lambda = (\lambda_1, ..., \lambda_n)$, $\mu = (\mu_1, \mu_2)$ of r with $\mu_2 \leq \lambda_1$, over the complex Hecke algebra $\mathcal{H} = \mathcal{H}_{\mathbb{C},q}(\mathfrak{S}_r)$ of the symmetric group $\mathfrak{S}_r$, where q is a complex root of unity, are at most 1 dimensional. Furthermore, she proves exactly when they are nonzero and provides a generator which turns out to correspond to the sum of all standard tableaux in $\mathrm{ST}_\lambda(\mu)$. (Note that our λ, μ are reversed). In the statement of Theorem 3.1, a similar map is considered and there are some technical similarities between the proof of our main result and [15]. However, we show in the next section, our modular homomorphism spaces may have dimension greater than 1.*

6. Homomorphism spaces of dimension greater than 1

As mentioned in Introduction, the first examples of Weyl modules $\Delta(\lambda), \Delta(\mu)$ such that $\dim \mathrm{Hom}_S(\Delta(\lambda), \Delta(\mu)) > 1$ were obtained by Dodge [6]. More examples were found by Lyle [14], in fact in the q-Schur algebra setting. The purpose of this section is to observe that the homomorphism spaces of Theorem 3.1 may have dimension > 1, see Corollary 6.2 and Example 6.4 below.

We recall the following special case of the classical nonvanishing result of Carter and Payne [4]. Here, boxes are raised between consecutive rows. See [16], 1.2 Lemma, for a proof of this particular case in our context.

Proposition 6.1. ([4]). *Let $n \geq r$. Let $\lambda, \mu \in \wedge^+(n, r)$ such that for some some $d > 0$ we have $\mu = (\lambda_1 + d, \lambda_2 - d, \lambda_3, ..., \lambda_m)$, where $\lambda = (\lambda_1, ..., \lambda_m)$. Suppose p divides $R(\lambda_1 - \lambda_2 + d + 1, d)$. Then the map*

$$\begin{aligned}\alpha : D(\lambda_1, \lambda_2, ..., \lambda_m) &\xrightarrow{1\otimes\Delta\otimes 1} D(\lambda_1, d, \lambda_2 - d, ..., \lambda_m)\\ &\xrightarrow{\eta\otimes 1} D(\lambda_1 + d, \lambda_2 - d, ..., \lambda_m),\end{aligned}$$

where $\Delta : D(\lambda_2) \to D(d, \lambda_2 - d)$ is the indicated diagonalization and $\eta : D(\lambda_1, d) \to D(\lambda_1 + d)$ and the indicated multiplication, induces a nonzero homomorphism $\Delta(\lambda) \to \Delta(\mu)$. The main result of this section is the following.

Corollary 6.2. *Let $n \geq r$. Let $\lambda, \mu \in \wedge^+(n, r)$ such that $\lambda = (\lambda_1, ..., \lambda_m)$, $\lambda_m \neq 0$, $m \geq 3$ and $\mu = (\mu_1, \mu_2)$. Define $d = \mu_1 - \lambda_1$ and assume $0 < d \leq \lambda_2 - \lambda_3$ and $\mu_2 \leq \lambda_1$. If p divides all of the following integers*

(1) $R(\lambda_1 - \mu_2 + 1, d)$,
(2) $R(\lambda_i + 1, \lambda_{i+1})$, $i = 2, ..., m - 1$,
(3) $R(\lambda_1 - \lambda_2 + d + 1, d)$,
(4) $R(\lambda_2 - d + 1, \lambda_3)$,

then the dimension of the K-vector space $\mathrm{Hom}_S(\Delta(\lambda), \Delta(\mu))$ is at least 2.

Proof. By the first two divisibility conditions, the map

$$\psi_1 = \sum_{T\in \mathrm{ST}_\lambda(\mu)} \phi_T : D(\lambda) \to D(\mu),$$

induces a nonzero homomorphism $\bar{\psi}_1 : \Delta(\lambda) \to \Delta(\mu)$ according to Theorem 3.1.

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Next consider the following maps

$$\alpha : D(\lambda_1, \lambda_2, ..., \lambda_m) \to D(\lambda_1 + d, \lambda_2 - d, ..., \lambda_m),$$

as in Proposition 6.1 and

$$\beta : D(\lambda_1 + d, \lambda_2 - d, ..., \lambda_m) \xrightarrow{1\otimes\eta'} D(\lambda_1 + d, \lambda_2 - d + \lambda_3 + \cdots + \lambda_m),$$

where $\eta' : D(\lambda_2 - d, ..., \lambda_m) \to D(\lambda_2 - d + \lambda_3 + \cdots + \lambda_m)$ is the indicated multiplication.

Under assumption (3), we have that α induces a nonzero map

$$\bar{\alpha} : \Delta(\lambda) \to D(\lambda_1 + d, \lambda_2 - d, ..., \lambda_m),$$

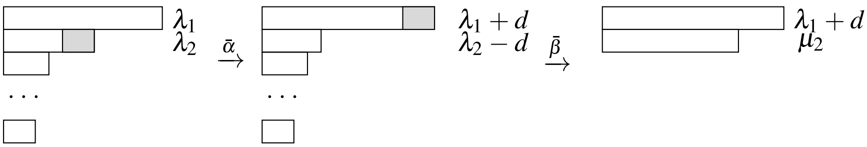
according to Proposition 6.1

Under assumptions (2) and (4), we have that β induces a nonzero map

$$\bar{\beta} : \Delta(\lambda_1 + d, \lambda_2 - d, ..., \lambda_m) \to \Delta(\lambda_1 + d, \lambda_2 - d + \lambda_3 + \cdots + \lambda_m),$$

according to Theorem 2.1

Consider the composition $\bar{\psi}_2 = \bar{\beta}\bar{\alpha} : \Delta(\lambda) \to \Delta(\mu)$ depicted below, where Weyl modules are indicated by the diagrams of the corresponding partitions.



It remains to be shown that the homomorphisms $\bar{\psi}_1, \bar{\psi}_2$ are linearly independent. Let $z = d'_\lambda(1^{(\lambda_1)} \otimes \cdots \otimes m^{(\lambda_m)}) \in \Delta(\lambda)$. From the definitions of the maps, we have

$$\bar{\psi}_1(z) = \sum_{T \in \mathrm{ST}_\lambda(\mu)} [T],$$

and

$$\bar{\psi}_2(z) = \begin{bmatrix} 1^{(\lambda_1)}2^{(d)} \\ 2^{(\lambda_2-d)} \cdots m^{(\lambda_m)} \end{bmatrix}.$$

It is clear that $\begin{matrix} 1^{(\lambda_1)}2^{(d)} \\ 2^{(\lambda_2-d)} \cdots m^{(\lambda_m)} \end{matrix} \in \mathrm{ST}_\lambda(\mu)$. Since $\lambda_3 > 0$, the set $\mathrm{ST}_\lambda(\mu)$ contains at least two elements. Hence, from the above equations and Theorem 2.2, it follows that the maps $\bar{\psi}_1, \bar{\psi}_2$ are linearly independent. $\square$

Remark *The assumptions of Corollary 6.2 imply that for the corresponding Specht modules we have* $\dim \mathrm{Hom}_{\mathfrak{S}_r}(\mathrm{Sp}(\mu), \mathrm{Sp}(\lambda)) \geq 2$. *See Remark 3.2.*

Example 6.3. *Let p be a prime and a an integer such that* $a \geq (p^2 + 1)(p - 1)$ *and*

$$a \equiv p - 2 \mod p^2.$$

Consider the following partitions

$$\lambda = (a, 2p - 1, (p - 1)^{p^2}),$$
$$\mu = (a + p, (p^2 + 1)(p - 1)),$$

where $p - 1$ *appears* p^2 *times as a row in* λ. *Using Lemma 3.4, it easily follows that the assumptions (1) - (4) of Corollary 6.2 are satisfied. For example, we have*

$$\lambda_1 - \mu_2 + 1 \equiv p - 2 - (p^2 + 1)(p - 1) + 1 \equiv 0 \mod p^2,$$

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and hence by Lemma 3.3, $d = p$ divides $R(\lambda_1 - \mu_2 + 1, d)$ which is assumption (1). Thus, $\dim \mathrm{Hom}_S(\Delta(\lambda), \Delta(\mu)) \geq 2.$[1]

For $p = 2$, the least a that satisfies the above requirements is $a = 8$ and thus we have the partitions $\lambda = (8, 3, 1, 1, 1, 1)$, $\mu = (10, 5)$. This pair appears in Example 4, Subsection 2.3, of Lyle's paper [15] which prompted us to consider Corollary 6.2 and in particular the composition $\bar{\psi}_2 = \bar{\beta}\bar{\alpha} : \Delta(\lambda) \to \Delta(\mu)$.

Conflict of interest. The authors declare they have no conflict of interest.

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[1] We note that for fixed p, it follows from the main result of [18] that the dimension of $\mathrm{Hom}_S(\Delta(\lambda), \Delta(\mu))$ does not depend on a. For $p = 3$, this means that $\dim \mathrm{Hom}_S(\Delta(\lambda), \Delta(\mu)) = 2$ for all a, see [18], Example 2.4.