

**SOME CONVOLUTION IDENTITIES BASED UPON  
 RAMANUJAN'S BILATERAL SUM**

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Recently, Bhargava, Adiga and Somashekara made use of Ramanujan's  ${}_1\Psi_1$  summation formula to prove a convolution identity for certain coefficients generated by the quotient of two infinite products. As special cases of this identity, they deduced several results (connected especially with the generalised Frobenius partition functions) including, for example, the convolution identities given earlier by Kung-Wei Yang. In this sequel to the aforementioned works, we provide a complete answer to an interesting question raised by Bhargava, Adiga and Somashekara in connection with one class of their convolution identities.

For a real or complex number  $q$  ( $|q| < 1$ ), put

$$(1) \quad (\lambda; q)_\infty = \prod_{n=1}^{\infty} (1 - \lambda q^{n-1})$$

and let  $(\lambda; q)_\mu$  be defined by

$$(2) \quad (\lambda; q)_\mu = \frac{(\lambda; q)_\infty}{(\lambda q^\mu; q)_\infty}$$

for arbitrary parameters  $\lambda$  and  $\mu$ , so that

$$(3) \quad (\lambda; q)_n = \begin{cases} 1 & (n = 0) \\ (1 - \lambda)(1 - \lambda q) \cdots (1 - \lambda q^{n-1}) & (n \in \mathbb{N} = \{1, 2, 3, \dots\}). \end{cases}$$

One of the most celebrated results in the theory of basic bilateral hypergeometric series (see, for details, Slater [2, Chapter 7]) is Ramanujan's  ${}_1\Psi_1$  summation formula:

$$(4) \quad {}_1\Psi_1 \left[ \begin{matrix} a; \\ b; \end{matrix} q, z \right] = \sum_{n=-\infty}^{\infty} \frac{(a; q)_n}{(b; q)_n} z^n = \frac{(az; q)_\infty (q/az; q)_\infty (b/a; q)_\infty (q; q)_\infty}{(z; q)_\infty (b/az; q)_\infty (b; q)_\infty (q/a; q)_\infty} \quad (|b/a| < |z| < 1).$$

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Making use of (4), Bhargava, Adiga and Somashekara [1] proved an interesting generalisation of several results (connected especially with the generalised Frobenius partition functions) including, for example, the convolution identities of Yang [3] for the coefficients  $A_n(q)$  defined by

$$(5) \quad \prod_{n=1}^{\infty} (1 + xq^n + x^2q^{2n}) = \sum_{n=-\infty}^{\infty} A_n(q)x^n.$$

We find it to be convenient to recall here the main result of [1] as

**THEOREM 1.** [1, p.157, Theorem 2.1]. *If the coefficients  $B_n(\alpha, \theta, q)$  are generated by*

$$(6) \quad \frac{f(x; \theta, q)}{f(\alpha x; \theta, q)} = \sum_{n=-\infty}^{\infty} B_n(\alpha, \theta, q)x^n,$$

where, for convenience,

$$(7) \quad f(x; \theta, q) = \prod_{n=1}^{\infty} (1 + 2xq^n \cos \theta + x^2q^{2n}),$$

then

$$(8) \quad \begin{aligned} & \sum_{n=-\infty}^{\infty} q^{-n} B_{m+n}(\alpha, \theta, q) B_n(\beta, \theta, q) \\ &= \frac{(-\alpha q e^{i\theta})^m (1/\alpha; q)_m}{(\beta q; q)_m} \left\{ \frac{(\alpha q; q)_{\infty} (\beta q; q)_{\infty}}{(q; q)_{\infty} (\alpha \beta q; q)_{\infty}} \right\}^2 \\ & \quad \times {}_2\Psi_2 \left[ \begin{matrix} q^m/\alpha, 1/\beta; \\ \alpha q, \beta q^{m+1}; \end{matrix} \quad q, \alpha \beta q e^{2i\theta} \right] \end{aligned}$$

for every integer  $m$ .

Since (see Definitions (5) and (6))

$$(9) \quad A_n(q) = B_n(0, \pi/3, q),$$

the convolution identities of Yang [3] are easily recovered as special cases of Theorem 1 (for details, see [1, p.159]). Furthermore, in its special case when  $\theta = \pi/2$ , the bilateral  ${}_2\Psi_2$  series occurring on the right-hand side of (8) can be summed by means of the

known result [2, p.248, Equation (IV.13)]:

$$\begin{aligned}
 (10) \quad {}_2\Psi_2 \left[ \begin{matrix} b, & c; \\ & q, & -\frac{aq}{bc} \\ aq/b, & aq/c; \end{matrix} \right] &= \frac{(aq/bc; q)_\infty (aq^2/b^2; q^2)_\infty (aq^2/c^2; q^2)_\infty (q/a; q^2)_\infty}{(q/b; q)_\infty (q/c; q)_\infty (aq/b; q)_\infty} \\
 &\times \frac{(aq; q^2)_\infty (q^2; q^2)_\infty}{(aq/c; q)_\infty (-aq/bc; q)_\infty},
 \end{aligned}$$

in which the  ${}_2\Psi_2$  series is obviously *well-poised*. In this special case, Theorem 1 finally yields

**THEOREM 2.** [1, p.159, Theorem 3.1]. *If the coefficients  $C_n(\alpha, q)$  are generated by*

$$(11) \quad \frac{g(x; q)}{g(\alpha x; q)} = \sum_{n=-\infty}^{\infty} C_n(\alpha, q)x^n,$$

where, for convenience,

$$(12) \quad g(x; q) = \prod_{n=1}^{\infty} (1 + x^2 q^{2n}),$$

then

$$\begin{aligned}
 (13) \quad \sum_{n=-\infty}^{\infty} q^{-n} C_{m+n}(\alpha, q) C_n(\beta, q) &= \frac{(-i\alpha q)^m (1/\alpha; q)_m (a^2/q^{m-2}; q^2)_\infty (\alpha q; q)_\infty}{\{(q; q)_\infty\}^2 (\alpha\beta q; q)_\infty} \\
 &\times \frac{(\beta^2 q^{m+2}; q^2)_\infty (q^2; q^2)_\infty (q^{m+1}; q^2)_\infty (1/q^{m-1}; q^2)_\infty}{(\alpha/q^{m-1}; q)_\infty (-\alpha\beta q; q)_\infty}
 \end{aligned}$$

for every integer  $m$ .

While discussing the partition-theoretic interpretations of many of the convolution identities derivable as special cases of Theorem 1 and Theorem 2, especially their connections with the generalised Forbenius partition functions, Bhargava, Adiga and Somashekara [1, p.161] raised an interesting question concerning the derivability of the convolution identity (13) as a consequence of some  ${}_2\Phi_1$  summation formula instead of the  ${}_1\Psi_1$  and  ${}_2\Psi_2$  sums (4) and (10) which they used. A complete answer to their question is indeed provided by the following substantially modified (and simplified) version of Theorem 2:

**THEOREM 3.** *If the coefficients  $C_n(\alpha, q)$  are defined by (11), then*

$$(14) \quad \sum_{n=0}^{\infty} q^{-2n} C_{2m+2n}(\alpha, q) C_{2n}(\beta, q) = \frac{(-\alpha^2 q^2)^m (1/\alpha^2; q^2)_m (\alpha^2 q^2; q^2)_{\infty} (\beta^2 q^{2m+2}; q^2)_{\infty}}{(q^2; q^2)_{\infty} (\alpha^2 \beta^2 q^2; q^2)_{\infty}}$$

for every nonnegative integer  $m$ .

PROOF: First of all, upon comparing the definitions (1) and (12), we have

$$(15) \quad g(x; q) = (-x^2 q^2; q^2)_{\infty},$$

so that the definition (11) assumes the form:

$$(16) \quad \frac{(-x^2 q^2; q^2)_{\infty}}{(-\alpha^2 x^2 q^2; q^2)_{\infty}} = \sum_{n=-\infty}^{\infty} C_n(\alpha, q) x^n.$$

Next, by the  $q$ -binomial theorem [2, p.248, Equation (IV.11)]:

$$(17) \quad {}_1\Phi_0 \left[ \begin{matrix} a; \\ -; \end{matrix} q, z \right] = \sum_{n=0}^{\infty} \frac{(a; q)_n}{(q; q)_n} z^n = \frac{(az; q)_{\infty}}{(z; q)_{\infty}} \quad (\max\{|z|, |q|\} < 1),$$

which incidentally is an immediate consequence of Ramanujan’s bilateral sum (4) with  $b = q$ , we can rewrite the definition (16) in its equivalent form:

$$(18) \quad \sum_{n=0}^{\infty} \frac{(1/\alpha^2; q^2)_n}{(q^2; q^2)_n} (-\alpha^2 x^2 q^2)^n = \sum_{n=-\infty}^{\infty} C_n(\alpha, q) x^n,$$

which readily implies that

$$(19) \quad C_{2n}(\alpha, q) = \frac{(1/\alpha^2; q^2)_n}{(q^2; q^2)_n} (-\alpha^2 q^2)^n, \quad C_{2n+1}(\alpha, q) = 0 \quad (n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\})$$

and

$$(20) \quad C_{-n}(\alpha, q) = 0 \quad (n \in \mathbb{N}).$$

In view of the observations (19) and (20), the first member of the convolution identity (13) simplifies easily to

$$\begin{aligned} & \sum_{n=0}^{\infty} q^{-2n} C_{2m+2n}(\alpha, q) C_{2n}(\beta, q) \\ &= (-\alpha^2 q^2)^m \frac{(1/\alpha^2; q^2)_m}{(q^2; q^2)_m} \sum_{n=0}^{\infty} \frac{(q^{2m}/\alpha^2; q^2)_n (1/\beta^2; q^2)_n}{(q^{2m+2}; q^2)_n} \frac{(\alpha^2 \beta^2 q^2)^n}{(q^2; q^2)_n}, \end{aligned}$$

which leads us at once to the second member of the assertion (14) of Theorem 3 by means of the familiar  $q$ -Gauss summation theorem [2, p.247, Equation (IV.2)]:

$$(21) \quad {}_2\Phi_1 \left[ \begin{matrix} a, & b; \\ & q, & \frac{c}{ab} \end{matrix} \right] = \sum_{n=0}^{\infty} \frac{(a; q)_n (b; q)_n (c/ab)^n}{(c; q)_n (q; q)_n} \\ = \frac{(c/a; q)_{\infty} (c/b; q)_{\infty}}{(c; q)_{\infty} (c/ab; q)_{\infty}}.$$

This evidently completes the proof of Theorem 3, using only the well-known  ${}_2\Phi_1$  summation theorem (21), that is, *without* using Ramanujan's  ${}_1\Psi_1$  summation (4) and the well-poised  ${}_2\Psi_2$  sum (10). Indeed, in view of the definitions (1), (2) and (3), it is not difficult to verify that the second member of the assertion (13) of Theorem 2 (with  $m$  replaced by  $2m$ ) is precisely the same as the right-hand side of the assertion (14) of Theorem 3.  $\square$

#### REFERENCES

- [1] S. Bhargava, C. Adiga and D.D. Somashekara, 'Ramanujan's remarkable summation formula and an interesting convolution identity', *Bull. Austral. Math. Soc.* **47** (1993), 155–162.
- [2] L.J. Slater, *Generalized hypergeometric functions* (Cambridge University Press, Cambridge, London and New York, 1966).
- [3] K.-W. Yang, 'On the product  $\prod_{n \geq 1} (1 + q^n x + q^{2n} x^2)$ ', *J. Austral. Math. Soc. Ser. A* **48** (1990), 148–151.

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