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VARIANTS OF THE HÖLDER INEQUALITY AND ITS INVERSES

BY
CHUNG-LIE WANG[(1)]

ABSTRACT. This paper presents variants of the Hölder inequality for integrals of functions (as well as for sums of real numbers) and its inverses. In these contexts, all possible transliterations and some extensions to more than two functions are also mentioned.

1. **Introduction.** Let $L_p = L_p(S, \Sigma, \mu)$, $-\infty < p < \infty$, be the space of all pth power non-negative integrable functions over a given finite measure space (S, Σ, μ) (where S may be considered as a bounded subset of real numbers). For f in L_p, we write $\|f\|_p = \{\int_S f^p\, d\mu\}^{1/p}$.

It is known that, for f_1, f_2 in L_2, the Schwarz inequality

$$\|f_1 f_2\|_1 \le \|f_1\|_2 \, \|f_2\|_2$$

has an inverse of the form

$$\|f_1\|_2 \, \|f_2\|_2 \le C_2 \, \|f_1 f_2\|_1$$

where C_2 is a positive constant depending on functions f_1 and f_2. For example, it was shown in [4, 5] that if f_1 and f_2 are such that

$$0 < m_i \le f_i(x) \le M_i < \infty \tag{1.1}$$

on S, where $m_i = \inf f_i(x)$, $M_i = \sup f_i(x)$, $i = 1, 2$, then

$$C_2 = (M_1 M_2 + m_1 m_2)/2(m_1 m_2 M_1 M_2)^{1/2}. \tag{1.2}$$

Similarly, the Hölder inequality (see [1, 11])

$$\|f_1 f_2\|_1 \le \|f_1\|_p \, \|f_2\|_q \tag{1.3}$$

for

$$1/p + 1/q = 1, \; p, q > 1 \tag{1.4}$$

has an inverse of the form (see [7])

$$\|f_1\|_p \, \|f_2\|_q \le C_{pq} \, \|f_1 f_2\|_1 \tag{1.5}$$

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if f_1 and f_2 are under suitable hypotheses. It was shown in [7] that for functions satisfying (1.1)

$$(1.6)\quad C_{pq}=\frac{M_1^pM_2^q-m_1^pm_2^q}{\{pm_2M_2(M_1M_2^{q-1}-m_1m_2^{q-1})\}^{1/p}\{qm_1M_1(M_2M_1^{p-1}-m_2m_1^{p-1})\}^{1/q}}$$

In the following we first extend the inequalities (1.3) and (1.5) to the case

$$(1.7)\qquad (1/p)+(1/q)=1/r,\ p,q,r>0$$

from the case (1.4). In these contexts, all possible transliterations and some extensions to more than two functions are also mentioned.

2. **Basic results.** We now give variants of (1.3) and (1.5) respectively.

THEOREM 1. *If f_1 is in L_p and f_2 is in L_q with* (1.7), *then f_1f_2 is in L_r and*

$$(2.1)\qquad \|f_1f_2\|_r\le\|f_1\|_p\,\|f_2\|_q.$$

THEOREM 2. *Under the assumptions of Theorem* 1, *with* (1.1),

$$(2.2)\qquad \|f_1\|_p\,\|f_2\|_q\le C_{pq}^r\,\|f_1f_2\|_r,$$

where

$$(2.3)\qquad C_{pq}^r=\frac{\{r(M_1^{p/2}M_2^{q/2}+m_1^{p/2}m_2^{q/2})M(p,q,r)\}^{1/r}}{\{p(m_2M_2)^{q/2}\}^{1/p}\{q(m_1M_1)^{p/2}\}^{1/q}}$$

with

$$M(p,q,r)=\max\{s^{(p/2)-r}t^{(q/2)-r}\mid s=M_1,m_1,\ t=M_2,m_2\}.$$

NOTE 1. A proof of Theorem 1 is indicated in [7] on page 527 involving the Riesz convexity theorem, but with the restriction $p,q,r\ge1$. However, it will be shown below that (2.1) is just a rewriting of the standard Hölder inequality.

Proof of Theorem 1. On the set $\{x\in S\mid f_2^r(x)\le f_1^{p-r}(x)\}$ we have $f_1^r(x)f_2^r(x)\le f_1^p(x)$, but on its complement we have $f_1^r(x)f_2^r(x)<f_2^q(x)$. Hence f_1f_2 is in L_r. Indeed (2.1) is just a rewriting of the standard Hölder inequality:

$$(2.4)\qquad [1/(p/r)]+[1/(p/r)]=1,\qquad f_1^r\in L_{p/r},\qquad f_2^r\in L_{q/r},$$

so

$$\int f_1^rf_2^r\le\left[\int(f_1^r)^{p/r}\right]^{r/p}\left[\int(f_2^r)^{q/r}\right]^{r/q}$$

which implies (2.1).

Proof of Theorem 2. From (1.1) it follows that $0\le M_1^{p/2}f_2^{q/2}-m_2^{q/2}f_1^{p/2}$ and $0\le M_2^{q/2}f_1^{p/2}-m_1^{p/2}f_2^{q/2}$ in S. Hence

$$(2.5)\qquad 0\le(M_1^{p/2}f_2^{q/2}-m_2^{q/2}f_1^{p/2})(M_2^{q/2}f_1^{p/2}-m_1^{p/2}f_2^{q/2})$$

in S. Integrating (2.5) over S, we obtain

$$(2.6)\quad (m_2M_2)^{q/2}\int_S f_1^p+(m_1M_1)^{p/2}\int_S f_2^q \le (M_1^{p/2}M_2^{q/2}+m_1^{p/2}m_2^{q/2})\int_S f_1^{p/2}f_2^{q/2}$$

Noting that $f_1^{p/2}f_2^{q/2}\le M(p,q,r)f_1^rf_2^r$ in S, (2.6) yields

$$(2.7)\quad \begin{aligned}(r/p)[(p/r)(m_2M_2)^{q/2}]\int_S f_1^p+(r/q)[(q/r)(m_1M_1)^{p/2}]\int_S f_2^q\\ \le (M_1^{p/2}M_2^{q/2}+m_1^{p/2}m_2^{q/2})M(p,q,r)\int_S f_1^rf_2^r.\end{aligned}$$

After applying the geometric-arithmetic inequality to the left-hand side of (2.7), a straight forward transposition yields (2.2).

NOTE 2. It is evident that the equality holds in (2.6) (or (2.7)) if and only if the equality holds in (2.5). Adopting the same argument used in [5] for the case $p=q=2$ by Diaz-Metcalf, it can be easily shown that the equality holds in (2.6) (or (2.7)) if and only if either $f_1=m_1$ and $f_2=M_2$ or $f_1=M_1$ and $f_2=m_2$ (see page 283 of [5] for detail).

NOTE 3. Indeed (2.2) can be obtained by transliteration from (1.5). Noting (2.4) we have

$$\left[\int (f_1^r)^{p/r}\right]^{r/p}\left[\int (f_2^r)^{q/r}\right]^{r/q}\le C_{p/r,q/r}\int f_1^rf_2^r$$

which implies (2.2), where (by also raising m_1, m_2, M_1, M_2, to the rth power)

$$(2.8)\quad \begin{aligned}C_{pq}^r&=(C_{p/r,q/r})^{1/r}\\ &=\frac{\{r(M_1^pM_2^q-m_1^pm_2^q)\}^{1/r}}{\{pm_2^rM_2^r(M_1^rM_2^{q-r}-m_1^rm_2^{q-r})\}^{1/p}\{qm_1^rM_1^r(M_2^rM_1^{p-r}-m_2^rm_1^{p-r})\}^{1/q}}\end{aligned}$$

It is well known that the spaces L_p ([0, 1], dx), $0<p<1$, where dx is Lebesque measure on [0, 1], are neither normed nor seminormed linear spaces. However they are F-spaces (or linear topological spaces). It should also be noted that it might sometimes be desirable, in discussing L_p spaces, to proceed without the standard $(1/p)+(1/q)=1$ normalization. In fact, we may consider a pth power integrable function for any real p (by relaxing the restriction $p\ge 1$) if we only concern integrability of functions (see also pp. 51, 171, 534–535 of [7] for more examples). Moreover, in connection with (2.1) and (2.2) above, two simple examples are in order:

EXAMPLE 1. Setting $p=q=1$, $r=1/2$, (2.1) and (2.2) imply

$$(2.9)\quad (C_{1,1}^{1/2})^{-1/2}\sqrt{\|f_1\|_1\,\|f_2\|_1}\le \|\sqrt{f_1f_2}\|_1\le\sqrt{\|f_1\|_1\,\|f_2\|_1}$$

where

$$C_{1,1}^{1/2}=(M_1^{1/2}M_2^{1/2}+m_1^{1/2}m_2^{1/2})^2/4(m_1m_2M_1M_2)^{1/2} \tag{2.3}$$

Note that (2.8) yields the same value of $C_{1,1}^{1/2}$ as computed by (2.3).

In particular, consider $f_1(x)=1/(1-x^2)$ and $f_2(x)=1/(1-k^2x^2)$, $k^2, x^2<1$. Then

$$\|\sqrt{f_1 f_2}\|_1=u(s)=\int_0^s \frac{dx}{\sqrt{(1-x^2)(1-k^2x^2)}}, \qquad 0\le s<1.$$

In view of (2.9) the elliptic integral $u(s)$ is bounded.

EXAMPLE 2. Rewrite (2.1) in terms of random variables X, Y and expectations over a probability space $(\Omega, \mathcal{F}, \mathcal{P})$ (see [3]):

$$E_r(XY)\le E_p(X)E_q(Y), \tag{2.10}$$

where

$$E_p(X)=\{E(|X|^p)\}^{1/p}, \quad \text{etc.}$$

and p, q, r satisfy (1.7), then the usual Lyapunov inequality

$$E_r(X)\le E_p(X), \tag{2.11}$$

under the restriction $1\le r\le p<\infty$, can be easily relaxed to the case $0\le r\le p\le \infty$ (by setting $Y\equiv 1$ in (2.10)). In particular,

$$E_0(X)\le E_r(X)\le E_\infty(X) \tag{2.12}$$

where

$$E_0(X)=\lim_{r\to 0} E_r(X), \qquad E_\infty(X)=\lim_{r\to\infty} E_r(X).$$

Finally, it might be interesting to consider a special case in connection with (2.11), (2.12). To this end, setting $\Omega=[0,1]$, $d\mathcal{P}=\mathcal{P}(dx)$, $X=x$ ($\mathcal{F}$ is naturally the Borel algebra on $[0,1]$) we have

$$E_r(x)=(1+r)^{-1/r}$$

for $0\le r\le\infty$. It is obvious that $E_r(x)$ is a strictly monotone increasing function of r for $0\le r\le\infty$ and $e^{-1}\le E_r(x)\le 1$. (Note that in general if $0\le r\le p\le\infty L_p\subset L_r$ for $\mu(S)<\infty$.)

3. **Transliteration.** As far as Theorems 1 and 2 are concerned, they are simply transliterations from the standard normalized case (1.4) to the case (1.7) which is believed to be new.

It should also be noted, there is only one Hölder inequality, for example, if $p>0$, $q<0$, $r>0$ and $(1/p)+(1/q)=1/r$, then

$$(1/r)+(1/-q)=1/p, \qquad f_2^{-1}\in L_{-q} \tag{3.1}$$

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If $f_1f_2 \in L_r$, then

$$(3.2)\qquad \|f_1\|_p = \left\{\int [(f_1f_2)f_2^{-1}]^p\right\}^{1/p} \le \|f_1f_2\|_r \|f_2\|_q^{-1}, \|f_1\|_p \|f_2\|_q \le \|f_1f_2\|_r.$$

This generalized reverse Hölder inequality (3.2) holds also, trivially, if $f_1f_2 \notin L_r$, so it holds in general.

We now transliterate inverses of the generalized Hölder inequality into inverses of the generalized reverse Hölder inequality. To this end, noting (2.2) and (3.1)

$$(3.3)\quad C^p_{r,-q}\|f_1\|_p = C^p_{r,-q}\left\{\int [(f_1f_2)f_2^{-1}]^p\right\}^{1/p} \ge \|f_1f_2\|_r \|f_2\|_q^{-1}, \|f_1f_2\|_r \le C^p_{r,-q}\|f_1\|_p \|f_2\|_q.$$

With f_1, f_2 restricted by (1.1), we have

$$(3.4)\quad 0 < m_1m_2 \le f_1(x)f_2(x) \le M_1M_2 < \infty \quad\text{and}\quad 0 < M_2^{-1} \le f_2^{-1}(x) \le m_2^{-1} < \infty$$

From (3.4) follows

$$C^p_{r,-q} = \begin{cases} (2.8) & \dfrac{\{p[(M_1M_2)^r m_2^q - (m_1m_2)^r M_2^q]\}^{1/p}\{q[(m_1m_2)^r M_1^p - (M_1M_2)^r m_1^p]\}^{1/q}}{\{r(M_1^p m_2^q - m_1^p M_2^q)\}^{1/r}} \\ (2.3) & \dfrac{\{p[(M_1M_2)^{r/2} m_2^{q/2} + (m_1m_2)^{r/2} M_2^{q/2}]M(r,-q,p)\}^{1/p}}{\{r(M_2m_2)^{q/2}\}^{1/r}\{-q(m_1m_2M_1M_2)^{r/2}\}^{1/-q}}, \end{cases}$$

with

$$M(r,-q,p) = \max\{s^{(r/2)-p}t^{(q/2)+p} \mid s = M_1M_2, m_1m_2, t = M_2, m_2\}.$$

There remain two (p, q, r) cases, (i) $p > 0$, $q < 0$, $r < 0$ and (ii) $p < 0$, $q < 0$, $r < 0$ for which generalized Hölder inequalities are now stated as follows:

(i) If $p > 0$, $q < 0$, $r < 0$ and $(1/p) + (1/q) = 1/r$, then $(1/p) + (1/-r) = (1/-q)$, $f_2^{-1} \in L_{-q}$. If $f_1f_2 \in L_r$, then

$$(3.5)\qquad \|f_2\|_q^{-1} = \left\{\int [f_1(f_1f_2)^{-1}]^{-q}\right\}^{1/-q} \le \|f_1\|^p \|f_1f_2\|_r^{-1},\quad \|f_1f_2\|_r \le \|f_1\|_p \|f_2\|_q.$$

The inequality (3.5) does not hold if $f_1f_2 \notin L_r$. Its corresponding inverse is

$$(3.6)\qquad \|f_1\|_p \|f_2\|_q \le C^{-q}_{p,-r}\|f_1f_2\|_r,$$

(ii) If $p > 0$, $q < 0$, $r < 0$ and $(1/p) + (1/q) = 1/r$ then $(1/-p) + (1/-q) = 1/-r$ and $(f_1f_2)^{-1} \in L_{-r}$ (if $f_1f_2 \in L_r$):

$$(3.7)\qquad \|f_1f_2\|_r^{-1} \le \|f_1\|_p^{-1} \|f_2\|_q^{-1},\qquad \|f_1\|_p \|f_2\|_q \le \|f_1f_2\|_r.$$

The inequality (3.7) holds also, trivially, if $f_1f_2 \notin L_r$, so it holds in general. its corresponding inverse is

$$(3.8)\qquad \|f_1f_2\|_r \le C^{-r}_{-p,-q}\|f_1\|_p \|f_2\|_q$$

NOTE 4. The calculations of the constants $C_{p,-r}^{-q}$ and $C_{-p,-q}^{-r}$ in (3.6) and (3.8) are similar to those of $C_{r,-q}^{p}$ given above with f_1 and f_2 restricted by (1.1), but we leave out the details.

Finally, there are extensions in all contexts to more than two functions, which are also believed to be new. To demonstrate how these can be done, three immediate extensions from Theorem 1 are listed here as corollaries with simple proofs omitted. for the sake of simplicity, we shall write $\sum$ and $\prod$ for $\sum_{j=1}^{n}$ and $\prod_{j=1}^{n}$ respectively unless confusion will occur.

COROLLARY 1. *Given positive numbers p_j and f_j in L_{p_j}, $j=1,\ldots,n$, then $\prod f_j$ is in L_r and $\|\prod f_j\|_r \le \prod \|f_j\|\, p_j$, where $1/r=\sum 1/p_j$.*

COROLLARY 2. *Given positive numbers $\alpha_1,\ldots,\alpha_n$ such that $\sum \alpha_j = 1$ and $f_1,\ldots,f_n$ in L_p, $p>0$, then $\prod f_j^{\alpha_j}$ is in L_p and $\|\prod f_j^{\alpha_j}\|_p \le \prod \|f_j\|_p^{\alpha_j}$.*

COROLLARY 3. *Given positive numbers $\alpha_1,\ldots,\alpha_n$, $\beta_1,\ldots,\beta_m$ such that $\sum_{j=1}^{n}\alpha_j=\sum_{k=1}^{m}\beta_k=1$ and $f_1,\ldots,f_n$ is in L_p and $g_1,\ldots,g_m$ is in L_q with* (1.7), *then $\prod f_j^{\alpha_j}\prod g_k^{\beta_k}$ is in L_r and $\|\prod f_j^{\alpha_j}\prod g_k^{\beta_k}\|_r \le \prod \|f_j\|_p^{\alpha_j}\prod \|g_k\|_q^{\beta_k}$.*

As above, under suitable hypotheses, we shall be able to find the inverses of inequalities given in Corollaries 1, 2 and 3. However, due to their similarity, we only give an inverse of the inequality in Corollary 1; namely,

$$\prod \|f_j\|_{p_j} \le C_{p_1\cdots p_n}^{r} \|\prod f_j\|_r. \tag{3.9}$$

In order to find a positive constant $C_{p_1\cdots p_n}^{r}$ so that (3.9) is true, we assume that

$$0<m_j=\inf f_j(x)\le f_j(x)\le \sup f_j(x)=M_j<\infty, \qquad 1\le j\le n. \tag{3.10}$$

It follows from (3.10) that

$$f_j^{p_j}\le K(p_j,r)\prod f_i^r, \qquad 1\le j\le n \tag{3.11}$$

where

$$K(p_j,r)=M_j^{p_j-r}/\prod_{1\le k\le n}^{j} m_k^r(p_j>r).$$

(Here $\prod_{1\le k\le n}^{j}$ indicates that k assumes values 1 through n except j). Integrating (3.11) over S and summing from 1 to n, we have

$$\sum \int_S f_j^{p_j} \le \left(\sum K(p_j,r)\right)\int_S \prod f_j^r$$

or

$$\sum (r/p_j)\left(p_j/r\int_S f_j^{p_j}\right)\le \left(\sum K(p_j,r)\right)\int_S \prod f_j^r. \tag{3.12}$$

Now noting that $\sum (r/p_j)=1$ and applying the geometric-arithmetic inequality to the left-hand side of (3.12), we obtain

$$\prod\left(p_j/r\int_S f_j^{p_j}\right)^{r/p_j} \leq \left(\sum K(p_j, r)\right)\int_S \prod f_j^r. \tag{3.13}$$

Comparing (3.13) with (3.9), we find

$$C^r_{p_1\cdots p_n} = \left(r\sum K(p_j, r)\right)^{1/r} / \prod p_j^{1/p_j}. \tag{3.14}$$

NOTE 5. It follows from (3.14) that

$$C^r_{pq} = \frac{\{(M_1^{p-r}/m_2^r + M_2^{q-r}/m_1^r)\}^{1/r}}{p^{1/p}q^{1/q}}, \tag{3.15}$$

where p, q, r satisfy (1.7). C^r_{pq} given in (3.15) is generally different from that given in (2.3) and (2.8) which are extensions of the results of Diaz–Metcalf [4, 5] and Diaz–Goldman–Metcalf [7] respectively. However, $C^1_{2,2} = C_{2,2} = C_2$ in any case.

4. **Concluding remarks.** In case $S = \{1, \ldots, n\}$ for some positive integer $n \geq 1$, μ is chosen to be the counting measure on S (see [1]). Now every f in L_p is a sequence $\{a_j\}$ of non-negative real numbers $a_1, \ldots, a_n$. We still use

$$\|f\|_p = \left\{\sum_{j=1}^{n} a_j^p\right\}^{1/p}.$$

Consequently, the above results for functions can be directly translated into the corresponding results for finite sequences of real numbers by replacing integrals with sums under suitable interpretation. Thus, if $p = q = 2$, our results are identical to those results in [4, 5]. In other words, Theorem 2 is a generalization of the results of Diaz–Metcalf [4,5] and a fortiari a generalization of the results of Greub-Rheinboldt [9], Kantorovich [10], Pólya–Szegö [12], and Schweitzer [13].

4.2 Although only non-negative functions (over a bounded subset of real numbers) and non-negative real numbers are considered here in the development of our results, complex functions (over a subset, of finite measure, of n-dimensional real Euclidean space, etc.) and complex numbers are also admissible (see [4, 5, 6]) with a slight modification of notation. However those will not be written here explicitly.

4.3 Since variants of the Hölder inequality and its inverses contain their normalized cases as special cases, it is expected that the results here will have a wide range of applications (e.g. see [14]).

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References

1. R. G. Bartle, *The elements of integration*, John-Wiley and Sons, New York, 1966.

2. E. F. Beckenbach and R. Bellman, *Inequalities*, Springer–Verlag, Berlin, 1961.

3. K. L. Chung, *A course in probability theory*. Harcourt, Brace & World, Inc., New York, 1968.

4. J. B. Diaz and F. T. Metcalf. *Complementary inequalities* I: *Inequalities complementary to Cauchy's inequality for sums of real numbers*, J. Math. Anal. and Appl. **9** (1964), 59–74.

5. J. B. Diaz and F. T. Metcalf, *Complementary inequalities II: Inequalities complementary to the Buniakowsky–Schwarz inequality for integrals*, J. Math. Anal. and Appl. **9** (1964), 278–293.

6. J. B. Diaz and F. T. Metcalf, *Stronger forms of a class of inequalities of G. Pólya–G. Szegö and Kantorovich*, Bull. Amer. Math. Soc. **69** (1963), 415–418.

7. J. B. Diaz, A. J. Goldman, and F. T. Metcalf, *Equivalence of certain inequalities complementing those of Cauchy–Schwarz and Hölder*, J. Res. NBS 68B (Math. and Math. Phys.) No. **2** (1964), 147–149.

8. N. Dunford and J. T. Schwartz, *Linear operators Part I: General Theory*, Interscience, New York, 1958.

9. W. Greub and W. Rheinboldt, *On a generalization of an inequality of L. V. Kantorovich*, Proc. America Math. Soc. **10** (1959), 407–415.

10. L. V. Kantorovich, *Functional analysis and applied mathematics, Uspehi Mat. Nauk* **3** (1948), 89–185 (also translated from the Russian by C. D. Benster, Nat. Bur. Standards Report No. **1509,** 1952).

11. M. E. Munroe, *Introduction to measure and integration*, Addison-Wesley, Reading, Mass., 1953.

12. G. Pólya and G. Szegö, *Aufgaben und Lehrsätze aus der Analysis*, Vol. **I.** Berlin, 1925.

13. P. Schweitzer, *Egy egyenlotlenség ax aritmetikai középértékröl* (*An inequality concerning the arithmetic mean*), Math. es phys. lopok **23** (1914), 257–261.

14. Chung-Lie Wang, *An extension of a Bellman inequality*, *Utilitas mathematica*, **8** (1975), 251–256.

Dept. of Mathematics,
University of Regina,
Regina, Canada,
S4S 0A2

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