ON THE INTENSITY OF CROSSINGS BY A SHOT NOISE PROCESS

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Abstract

The crossing intensity of a level by a shot noise process with a monotone response is studied, and it is shown that the intensity can be naturally expressed in terms of a marginal probability.

Consider the shot noise process

\[ X(t) = \sum_{\tau \leq t} h(t - \tau), \quad t \in \mathbb{R}, \]

where the \( \tau \)'s are the points of a stationary Poisson process \( \eta \) on \( \mathbb{R} \) with mean rate \( \lambda > 0 \), and \( h \), the impulse response, is a non-negative function on \([0, \infty)\) such that

(i) \( h \) is non-increasing,
(ii) \( h \) is finite except possibly at 0,
(iii) \( \int_{u}^{\infty} h(x) \, dx < \infty \) for some large \( u \).

By Theorem 1 of Daley (1971), the conditions (ii) and (iii) ensure that \( X(t) < \infty \) almost surely for each \( t \).

Observe that the sample function of \( X \) increases only at the points of \( \eta \). Thus it is clear to define that \( X \) upcrosses the level \( u \) at \( t \), where \( u \geq 0 \), if \( X(t-) \leq u \) and \( X(t) > u \). For \( u \geq 0 \), write \( N_u \) for the point process (cf. Kallenberg (1976)) that consists of the points at which upcrossings of level \( u \) by \( X \) occur. Thus for each Borel set \( B \), \( N_u(B) \) denotes the number of upcrossings of \( u \) by \( X \) in \( B \). \( N_u \) is a stationary point process, which may be viewed as a thinned process of \( \eta \). The purpose of this communication is to derive the following result.

**Theorem 1.** For each \( u \geq 0 \),
\[ \mathbb{E}[N_u(B)] = \lambda m(B) P[u - h(0) < X(0) \leq u] \]
for each Borel set \( B \), where \( m \) is Lebesgue measure.

To prove Theorem 1, first enumerate the points of \( \eta \) in \((-\infty, 0)\) by letting \( \rho_i \) be the \( i \)th largest point of \( \eta \) to the left of 0 for \( i = 1, 2, 3, \ldots \). The \( \rho_i \) are almost surely well defined, and \( -\rho_1, \rho_1 - \rho_2, \rho_2 - \rho_3, \ldots \) are independent and identically distributed random variables. The following result is useful.

**Lemma 2.** For each \( i = 1, 2, \ldots \),
\[ P[X(\rho_i-) = \sum_{j \geq i+1} h(\rho_i - \rho_j)] = 1 \]
where \( X(\rho_i-) \) denotes the left-hand limit of \( X \) at \( \rho_i \). From this it follows immediately that \( X(\rho_i-) \) is independent of \( \rho_i \), and \( X(\rho_i-) \) has the same distribution as \( X(0) \).
Proof. Let $i \geq 1$ be fixed. Since $h$ is monotone, it is almost everywhere continuous. Using the continuity of $\rho_i - \rho_j$, $j \geq i + 1$, we obtain
\[
\lim_{\epsilon \to 0} h(\rho_i - \rho_j - \epsilon) = h(\rho_i - \rho_j) \quad \text{a.s. for } j \geq i + 1.
\]
Also by the monotonicity of $h$, $h(\rho_i - \rho_j - \epsilon) \leq h(\rho_{i+1} - \rho_j)$ for $0 < \epsilon < \rho_i - \rho_{i+1}$, $j \geq i + 2$, where \( \sum_{j=i+2}^{i+2} h(\rho_{i+1} - \rho_j) \) is almost surely finite since it has the same distribution as $X(0)$. Thus it follows from dominated convergence that almost surely
\[
\lim_{\epsilon \to 0} X(\rho_i - \epsilon) = \lim_{\epsilon \to 0} \sum_{j=i+1}^{i+1} h(\rho_i - \rho_j - \epsilon) = \sum_{j=i+1}^{i+1} h(\rho_i - \rho_j).
\]

Proof of Theorem 1. By stationarity, it apparently suffices to show that $N_u(B)$ equals $\lambda m(B)P[u - h(0) < X(0) \leq u]$ for each Borel set $B$ in $(-\infty, 0)$, where $m(B)$ denotes the Lebesgue measure of $B$. Since
\[
X(\rho_i) = h(0) + \sum_{j=i+1}^{i+1} h(\rho_i - \rho_j),
\]
Lemma 2 implies that almost surely
\[
N_u(B) = \sum_{i \geq 1} 1(u - h(0) < X(\rho_i) \leq u, \rho_i \in B),
\]
where $1(\cdot)$ is the indicator function. Applying the facts that $X(\rho_i)$ is independent of $\rho_i$ and $X(\rho_i)$ is equal in distribution to $X(0)$, we get
\[
\mathbb{E}N_u(B) = \sum_{i \geq 1} \mathbb{E}1(u - h(0) < X(\rho_i) \leq u) \mathbb{E}1(\rho_i \in B)
\]
\[
= P[u - h(0) < X(0) \leq u] \lambda m(B).
\]
We mention the following for completeness.
(a) By stationarity, the downcrossing intensity of a level by $X$ is also given by Theorem 1.
(b) We assumed, for simplicity of illustration, that the impulse response $h$ is deterministic. Lifting this restriction, it is readily seen that Theorem 1 continues to hold if the impulse responses brought about by the points of $\eta$ are independent of $\eta$, and are independent and identically distributed.
(c) For methods of obtaining the marginal distribution of $X$ see Gilbert and Pollak (1960).
(d) The crossing intensities of some other shot noise processes were studied by Rice (1944), and Bar–David and Nemirovsky (1972). A result in the latter paper can be reduced to one which is similar to Theorem 1. However, our assumptions on $h$ are considerably simpler.

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References
