A CHARACTERIZATION OF A CLASS OF BARRELLED SEQUENCE SPACES

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1. Introduction. In a recent paper [4] Bennett and Kalton characterized dense, barrelled subspaces of an arbitrary FK space, E. In this note, it is shown that if E is assumed to be an AK space, then the characterization assumes a simpler and more explicit form.

2. Definition and preliminaries. ω denotes the vector space of sequences of complex numbers. A subspace E of ω is a K space if it is endowed with a locally convex topology τ such that the linear functionals

\[ x \to x_j \quad (j = 0, 1, 2, \ldots) \]

are continuous. In addition, if τ is complete and metrizable, then (E, τ) is an FK space.

If \( x = \{x_k\} \), let \( P_n x = \{x_0, x_1, \ldots, x_n, 0, \ldots\} \). If a K space (E, τ) has the property that \( P_n x \to x \) in τ for each \( x \in E \), then (E, τ) is called an AK space.

If E is an FK–AK space then the dual of E may be identified with

\[ E^\beta = \left\{ y \in \omega : \sum_{j=0}^{\infty} x_j y_j \text{ converges } \forall x \in E \right\}. \]

If F is a subspace of \( E^\beta \) containing the space φ of sequences with only finitely many non-zero terms then E, F form a separated pair under the bilinear form

\[ (x, y) = \sum_{j=0}^{\infty} x_j y_j. \]

\( \sigma(E, F), \tau(E, F) \) and \( \beta(E, F) \) denote the weak, Mackey and strong topologies, respectively, on E by F (see, e.g., [7]).

If \( A = (a_{nk}) \) is an infinite matrix of complex numbers, the sequence \( Ax = \{(Ax)_n\} \) is defined by

\[ (Ax)_n = \sum_{k=0}^{\infty} a_{nk} x_k \quad (n = 0, 1, 2, \ldots). \]

\( E_A = \{x : Ax \in E\} \), where E is a given sequence space. \( A' \) denotes the transpose of A.

The following theorem is established in [8].

**Theorem 2.1.** Let E and F be sequence spaces, each containing φ, such that \( (E^\beta, \sigma(E^\beta, E)) \) and \( (F, \sigma(F, F^\beta)) \) are sequentially complete. If \( A = (a_{nk}) \) is an infinite


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matrix, then the following are equivalent:

(i) \( F_A \) contains \( E \);
(ii) \( E_\beta^A \) contains \( E^\beta \);
(iii) \( F_{A^*} \) contains \( (E^\beta)^\beta \).

Proof. (i) \( \Rightarrow \) (ii). Let \( \{t_k\} \in F^\beta \) and \( \{x_k\} \in E \). Define the matrix \( B = (b_{nk}) \) by

\[
b_{nk} = \begin{cases} 
    t_k & (0 \leq k \leq n), \\
    0 & (k > n).
\end{cases}
\]

Then

\[
\sum_{n=0}^{\infty} t_n \sum_{k=0}^{\infty} a_{nk} x_k = \lim_{j \to \infty} \sum_{n=0}^{j} t_n \sum_{k=0}^{\infty} a_{nk} x_k = \lim_{j \to \infty} \sum_{k=0}^{\infty} x_k \sum_{n=0}^{j} a_{nk} = \lim_{j \to \infty} [(BA)x]_j.
\]

The hypotheses on \( E \) insure that

\[
\lim_{j \to \infty} [(BA)x]_j = \sum_{k=0}^{\infty} x_k \lim_{j \to \infty} [(BA)e^k]_j = \sum_{k=0}^{\infty} x_k \sum_{n=0}^{\infty} t_n a_{nk},
\]

where \( e^k \) denotes the sequence with a one in the \( k \)th coordinate and zeros elsewhere.

Since \( \{t_k\} \in F^\beta \), \( \{x_k\} \in E \) are arbitrary, it follows that \( A^* \) maps \( F^\beta \) to \( E^\beta \).

(ii) \( \Rightarrow \) (iii) follows from (i) \( \Rightarrow \) (ii) and the fact that \( F = (F^\beta)^\beta \) if \( (F, \sigma(F, F^\beta)) \) is sequentially complete [10, p. 974].

(iii) \( \Rightarrow \) (i) is trivial.

3. A class of barrelled spaces.

**Theorem 3.1.** Let \( E \) be an FK–AK space and \( E_0 \) a subspace of \( E \) containing \( \phi \). \( E_0 \) is barrelled in \( E \) if and only if

(i) \( E_0^\beta = E^\beta \), and
(ii) \( (E^\beta, \sigma(E^\beta, E_0)) \) is sequentially complete.

**Proof.** (Necessity) Let \( \{t_k\} \in E_0^\beta \), and define \( A = (a_{nk}) \) by

\[
a_{nk} = \begin{cases} 
    t_k & (0 \leq k \leq n), \\
    0 & (k > n).
\end{cases}
\]
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If \( c \) denotes the space of convergent sequences, then \( c_A \) includes \( E_0 \). Since \( c_A \) is an \( FK \) space [9, ch. 12], it follows from [4, Theorem 1] that \( c_A \) includes \( E \). Thus, for any \( x \in E, \sum_{k=0}^{\infty} t_k x_k \) converges. Consequently \( E^b \) includes \( E_0^b \). Since the reverse inclusion is satisfied, we have \( E_0^b = E^b \).

Let \( \{a^{(n)}\} \) be a sequence in \( E^b \) that is \( \sigma(E^b, E_0) \) Cauchy. If \( A = (a_{nk}) \) is defined by \( a_{nk} = a^{(n)}_k \), then \( c_A \) includes \( E_0 \). Consequently, \( c_A \) includes \( E \), [4, Theorem 1]. Condition (ii) now follows from the fact that \( E^b \) is \( \sigma(E^b, E) \) sequentially complete.

(Sufficiency). Let \( \{a^{(n)}\} \) be a sequence in \( E^b \) that is \( \sigma(E^b, E_0) \) bounded. Let \( m \) denote the space of bounded sequences, and define \( A = (a_{nk}) \) by \( a_{nk} = a^{(n)}_k \). Then \( m_A \) includes \( E_0 \). Conditions (i) and (ii) and Theorem 2.1 imply that \( m_A \) includes \( E \) since \( (m, \sigma(m, \ell)) \) is sequentially complete (\( \ell \) = space of absolutely convergent series). Thus, \( \sigma(E^b, E_0) \) and \( \sigma(E^b, E) \) define the same bounded sequences and, hence, the same bounded sets. Thus, the topology \( \beta(E_0, E^b) \) is the restriction of \( \beta(E, E^b) = \tau(E, E^b) = FK \) topology of \( E \) to \( E_0 \). It follows that \( E_0 \) is barrelled in \( E \).

REMARKS. If \( E_0 \) is monotone (i.e., the coordinatewise product \( xy \in E_0 \) if \( x \in E_0 \) and \( y \) is a sequence of zeros and ones) then condition (ii) of Theorem 3.1 can be omitted [3, p. 55].

Let \( \{r_n\} \) denote a non-decreasing unbounded sequence of positive integers with \( r_0 = 1 \) and \( r_n = o(n) \). For each \( x \in \omega \) and each \( n = 0, 1, 2, \ldots, \) let \( c_n(x) \) denote the number of non-zero elements in \( \{x_0, x_1, \ldots, x_n\} \). If \( E \) is a sequence space, a scarce copy of \( E \) is the linear span of

\[ \{x \in E : c_n(x) \leq r_n, n = 0, 1, 2, \ldots \}. \]

As corollaries to Theorem 3.1, we obtain Theorems 7, 8 and 10 of [2]. In each case the spaces are monotone and the verification of condition (i) of Theorem 3.1 is straightforward.

\( \omega \) has the topology of coordinatewise convergence, and, for \( p > 0, \ell^p = \left\{ x : \sum_{t=0}^{\infty} |x_t|^p < \infty \right\}. \)

**Corollary 3.2.** Every scarce copy of \( \omega \) is barrelled.

**Corollary 3.3.** Every scarce copy of \( \bigcap_{p > 0} \ell^p \) is barrelled as a subspace of \( \ell \).

**Corollary 3.4.** Let \( E \) be a monotone \( FK-AK \) space. The union of all the scarce copies of \( E \) is a barrelled subspace of \( E \).

It is noted that Corollary 3.4 strengthens Theorem 10 of [2], which is stated for solid spaces.

Another consequence of Theorem 3.1 is the following result.
COROLLARY 3.5. Let \( E \) be an FK–AK space and \( E_0 \) a subspace of \( E \) containing \( \phi \). The following are equivalent:

(i) \( E_0 \) is barrelled;

(ii) If \( G \) is a separable FK space containing \( E_0 \), then \( G \) contains \( E \).

Proof. (i) \( \Rightarrow \) (ii). This is a consequence of [4, Theorem 1]. (ii) \( \Rightarrow \) (i). Let \( \{t_k\} \in E_0^b \), and define \( A = (a_{nk}) \) by

\[
a_{nk} = \begin{cases} 
  t_k & (0 \leq k \leq n), \\
  0 & (k > n).
\end{cases}
\]

Then \( c_A \) includes \( E_0 \). Since \( c_A \) is a separable FK space [1, p. 199], \( c_A \) includes \( E \). Thus, \( \{t_k\} \in E^b \), and condition (i) of Theorem 3.1 is satisfied.

Let \( \{a^{(n)}\} \) be a sequence in \( E^b \) that is \( \sigma(E^b, E_0) \) Cauchy. If \( A = (a_{nk}) \) is the matrix defined by \( a_{nk} = a_{k}^{(n)} \), then \( c_A \) includes \( E_0 \). It follows that \( c_A \) includes \( E \). Since \( E^b \) is \( \sigma(E^b, E) \) sequentially complete, condition (ii) of Theorem 3.1 is satisfied. Thus, \( E_0 \) is barrelled.

Remark. For FK–AK spaces, (ii) \( \Rightarrow \) (i) of Corollary 3.5 improves (ii) \( \Rightarrow \) (i) of [4, Theorem 1].

In Theorem 3.1, if it is not assumed that \( E \) is an AK space, then (i) and (ii) are not sufficient to insure that \( E_0 \) is barrelled in \( E \).

Let \( E \) be \( ac_0 \), the space of sequences that are almost convergent to 0, (see [6]). For \( x \in ac_0 \), let

\[
\|x\| = \sup_n |x_n|.
\]

Let \( E_0 = bs + c_0 \), where

\[
c_0 = \left\{ x \in \omega : \lim_{n \to \infty} x_n = 0 \right\},
\]

\[
bs = \left\{ x \in \omega : \sup_n \left| \sum_{j=0}^{n} x_j \right| < \infty \right\}.
\]

Then \( E_0^b = E^b = \ell \), and \( E_0 \) is dense in \( E \) [5, p. 29]. Furthermore, \( \ell \) is \( \sigma(\ell, E_0) \) sequentially complete. However, \( E_0 \) is a normed FK space when topologized by

\[
\|x\| = \inf \left\{ \sup_n |y_n| + \sup_n \left| \sum_{j=0}^{n} z_j \right| : x = y + z, y \in c_0, z \in bs \right\}.
\]

It follows from [4, Theorem 1] that \( E_0 \) is not barrelled in \( E \).

REFERENCES


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