A CHARACTERIZATION OF A CLASS OF BARRELLED SEQUENCE SPACES

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1. Introduction. In a recent paper [4] Bennett and Kalton characterized dense, barrelled subspaces of an arbitrary $FK$ space, $E$. In this note, it is shown that if $E$ is assumed to be an $AK$ space, then the characterization assumes a simpler and more explicit form.

2. Definition and preliminaries. $\omega$ denotes the vector space of sequences of complex numbers. A subspace $E$ of $\omega$ is a $K$ space if it is endowed with a locally convex topology $\tau$ such that the linear functionals $x \mapsto x_j$ ($j = 0, 1, 2, \ldots$) are continuous. In addition, if $\tau$ is complete and metrizable, then $(E, \tau)$ is an $FK$ space.

If $x = \{x_k\}$, let $P_n x = \{x_0, x_1, \ldots, x_n, 0, \ldots\}$. If a $K$ space $(E, \tau)$ has the property that $P_n x \to x$ in $\tau$ for each $x \in E$, then $(E, \tau)$ is called an $AK$ space.

If $E$ is an $FK$–$AK$ space then the dual of $E$ may be identified with

$$E^\beta = \left\{ y \in \omega : \sum_{j=0}^{\infty} x_j y_j \text{ converges } \forall x \in E \right\}.$$}

If $F$ is a subspace of $E^\beta$ containing the space $\phi$ of sequences with only finitely many non-zero terms then $E, F$ form a separated pair under the bilinear form

$$\langle x, y \rangle = \sum_{j=0}^{\infty} x_j y_j.$$

$\sigma(E, F), \tau(E, F)$ and $\beta(E, F)$ denote the weak, Mackey and strong topologies, respectively, on $E$ by $F$ (see, e.g., [7]).

If $A = (a_{nk})$ is an infinite matrix of complex numbers, the sequence $Ax = \{(Ax)_n\}$ is defined by

$$(Ax)_n = \sum_{k=0}^{\infty} a_{nk} x_k \quad (n = 0, 1, 2, \ldots).$$

$E_A = \{x : Ax \in E\}$, where $E$ is a given sequence space. $A'$ denotes the transpose of $A$.

The following theorem is established in [8].

**Theorem 2.1.** Let $E$ and $F$ be sequence spaces, each containing $\phi$, such that $(E^\beta, \sigma(E^\beta, E))$ and $(F, \sigma(F, F^\beta))$ are sequentially complete. If $A = (a_{nk})$ is an infinite

matrix, then the following are equivalent:

(i) \( F_A \) contains \( E \);

(ii) \( E_A^B \) contains \( F^B \);

(iii) \( F_A^B \) contains \( (E^B)^B \).

Proof. (i) \( \Rightarrow \) (ii). Let \( \{t_k\} \in F^B \) and \( \{x_k\} \in E \). Define the matrix \( B = (b_{nk}) \) by

\[
b_{nk} = \begin{cases} t_k & (0 \leq k \leq n), \\ 0 & (k > n). \end{cases}
\]

Then

\[
\sum_{n=0}^{\infty} t_n \sum_{k=0}^{\infty} a_{nk} x_k = \lim_{j \to \infty} \sum_{n=0}^{j} t_n \sum_{k=0}^{\infty} a_{nk} x_k = \lim_{j \to \infty} \sum_{k=0}^{\infty} x_k \sum_{n=0}^{j} t_n a_{nk} = \lim_{j \to \infty} [(BA)x]_j.
\]

The hypotheses on \( E \) insure that

\[
\lim_{j \to \infty} [(BA)x]_j = \sum_{k=0}^{\infty} x_k \lim_{j \to \infty} [(BA)e^k]_j = \sum_{k=0}^{\infty} x_k \sum_{n=0}^{\infty} t_n a_{nk},
\]

where \( e^k \) denotes the sequence with a one in the \( k \)th coordinate and zeros elsewhere.

Since \( \{t_k\} \in F^B \), \( \{x_k\} \in E \) are arbitrary, it follows that \( A' \) maps \( F^B \) to \( E^B \).

(ii) \( \Rightarrow \) (iii) follows from (i) \( \Rightarrow \) (ii) and the fact that \( F = (F^B)^B \) if \( (F, \sigma(F, F^B)) \) is sequentially complete [10, p. 974].

(iii) \( \Rightarrow \) (i) is trivial.

3. A class of barrelled spaces.

**Theorem 3.1.** Let \( E \) be an FK–AK space and \( E_0 \) a subspace of \( E \) containing \( \phi \). \( E_0 \) is barrelled in \( E \) if and only if

(i) \( E_0^B = E^B \), and

(ii) \( (E^B, \sigma(E^B, E_0)) \) is sequentially complete.

**Proof.** (Necessity) Let \( \{t_k\} \in E_0^B \), and define \( A = (a_{nk}) \) by

\[
a_{nk} = \begin{cases} t_k & (0 \leq k \leq n), \\ 0 & (k > n). \end{cases}
\]
If \( c \) denotes the space of convergent sequences, then \( c_A \) includes \( E_0 \). Since \( c_A \) is an FK space \([9, \text{ch. 12}]\), it follows from \([4, \text{Theorem 1}]\) that \( c_A \) includes \( E \). Thus, for any \( x \in E \), \( \sum_{k=0}^{\infty} t_k x_k \) converges. Consequently \( E^\beta \) includes \( E_0^\beta \). Since the reverse inclusion is satisfied, we have \( E_0^\beta = E^\beta \).

Let \( \{a^{(n)}\} \) be a sequence in \( E^\beta \) that is \( \sigma(E^\beta, E_0) \) Cauchy. If \( A = (a_{nk}) \) is defined by \( a_{nk} = a_k^{(n)} \), then \( c_A \) includes \( E_0 \). Consequently, \( c_A \) includes \( E \), \([4, \text{Theorem 1}]\). Condition (ii) now follows from the fact that \( E^\beta \) is \( \sigma(E^\beta, E) \) sequentially complete.

(Sufficiency). Let \( \{a^{(n)}\} \) be a sequence in \( E^\beta \) that is \( \sigma(E^\beta, E_0) \) bounded. Let \( m \) denote the space of bounded sequences, and define \( A = (a_{nk}) \) by \( a_{nk} = a_k^{(n)} \). Then \( m_A \) includes \( E_0 \). Conditions (i) and (ii) and Theorem 2.1 imply that \( m_A \) includes \( E \) since \( (m, \sigma(m, \ell)) \) is sequentially complete (\( \ell = \) space of absolutely convergent series). Thus, \( \sigma(E^\beta, E_0) \) and \( \sigma(E^\beta, E) \) define the same bounded sequences and, hence, the same bounded sets. Thus, the topology \( \beta(E_0, E^\beta) \) is the restriction of \( \beta(E, E^\beta) = \tau(E, E^\beta) = FK \) topology of \( E \) to \( E_0 \). It follows that \( E_0 \) is barrelled in \( E \).

**Remarks.** If \( E_0 \) is monotone (i.e., the coordinatewise product \( xy \in E_0 \) if \( x \in E_0 \) and \( y \) is a sequence of zeros and ones) then condition (ii) of Theorem 3.1 can be omitted \([3, \text{p. 55}]\).

Let \( \{r_n\} \) denote a non-decreasing unbounded sequence of positive integers with \( r_0 = 1 \) and \( r_n = o(n) \). For each \( x \in \omega \) and each \( n = 0, 1, 2, \ldots \), let \( c_n(x) \) denote the number of non-zero elements in \( \{x_0, x_1, \ldots, x_n\} \). If \( E \) is a sequence space, a scarce copy of \( E \) is the linear span of

\[
\{x \in E : c_n(x) \leq r_n, n = 0, 1, 2, \ldots\}.
\]

As corollaries to Theorem 3.1, we obtain Theorems 7, 8 and 10 of \([2]\). In each case the spaces are monotone and the verification of condition (i) of Theorem 3.1 is straightforward.

\( \omega \) has the topology of coordinatewise convergence, and, for \( p > 0 \), \( \ell^p = \left\{ \begin{array}{l} x : \sum_{j=0}^{\infty} |x_j|^p < \infty \end{array} \right\} \).

**Corollary 3.2.** Every scarce copy of \( \omega \) is barrelled.

**Corollary 3.3.** Every scarce copy of \( \bigcap_{p>0} \ell^p \) is barrelled as a subspace of \( \ell \).

**Corollary 3.4.** Let \( E \) be a monotone FK–AK space. The union of all the scarce copies of \( E \) is a barrelled subspace of \( E \).

It is noted that Corollary 3.4 strengthens Theorem 10 of \([2]\), which is stated for solid spaces.

Another consequence of Theorem 3.1 is the following result.
**COROLLARY 3.5.** Let $E$ be an FK–AK space and $E_0$ a subspace of $E$ containing $\phi$. The following are equivalent:

(i) $E_0$ is barrelled;

(ii) If $G$ is a separable FK space containing $E_0$, then $G$ contains $E$.

**Proof.** (i) $\Rightarrow$ (ii). This is a consequence of [4, Theorem 1]. (ii) $\Rightarrow$ (i). Let $\{t_k\} \in E_0^\beta$, and define $A = (a_{nk})$ by

$$a_{nk} = \begin{cases} t_k & (0 \leq k \leq n), \\ 0 & (k > n). \end{cases}$$

Then $c_A$ includes $E_0$. Since $c_A$ is a separable FK space [1, p. 199], $c_A$ includes $E$. Thus, $\{t_k\} \in E_0^\beta$, and condition (i) of Theorem 3.1 is satisfied.

Let $\{a^{(n)}\}$ be a sequence in $E_0^\beta$ that is $\sigma(E_0^\beta, E_0)$ Cauchy. If $A = (a_{nk})$ is the matrix defined by $a_{nk} = a^{(k)}_{nk}$, then $c_A$ includes $E_0$. It follows that $c_A$ includes $E$. Since $E_0^\beta$ is $\sigma(E_0^\beta, E)$ sequentially complete, condition (ii) of Theorem 3.1 is satisfied. Thus, $E_0$ is barrelled.

**REMARK.** For FK–AK spaces, (ii) $\Rightarrow$ (i) of Corollary 3.5 improves (ii) $\Rightarrow$ (i) of [4, Theorem 1].

In Theorem 3.1, if it is not assumed that $E$ is an AK space, then (i) and (ii) are not sufficient to insure that $E_0$ is barrelled in $E$.

Let $E$ be $ac_0$, the space of sequences that are almost convergent to 0, (see [6]). For $x \in ac_0$, let

$$\|x\| = \sup_n |x_n|.$$

Let $E_0 = bs + c_0$, where

$$c_0 = \left\{ x \in \omega : \lim_{n \to \infty} x_n = 0 \right\},$$

$$bs = \left\{ x \in \omega : \sup_n \left| \sum_{j=0}^{n} x_j \right| < \infty \right\}.$$ 

Then $E_0^\beta = E^\beta = \ell$, and $E_0$ is dense in $E$ [5, p. 29]. Furthermore, $\ell$ is $\sigma(\ell, E_0)$ sequentially complete. However, $E_0$ is a normed FK space when topologized by

$$\|x\| = \inf \left\{ \sup_n |y_n| + \sup_n \left| \sum_{j=0}^{n} z_j \right| : x = y + z, y \in c_0, z \in bs \right\}.$$

It follows from [4, Theorem 1] that $E_0$ is not barrelled in $E$.

**REFERENCES**