BULL. AUSTRAL. MATH. SOC. 34B15
VOL. 54 (1996) [397–410]

BOUNDARY VALUE PROBLEMS FOR SYSTEMS OF DIFFERENTIAL EQUATIONS

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We give sufficient conditions for systems of the form $y' = f(x,y)$, x in $[0,1]$ and $y'' = f(x,y,y')$, x in $[0,1]$ to have solutions y with (x,y) in $\Omega \subseteq [0,1] \times \mathbf{R}^n$. We use degree theory and allow the shape of Ω to depend on x.

1. INTRODUCTION

We prove there are solutions, nonnegative in some cases, for various two point boundary value problems for systems of differential equations. There is an extensive literature on these problems as can be seen from the cited papers and their references.

In Section 2 we consider the problem

$$y' = f(x,y), \quad x \in [0,1] \tag{1.1}$$

$$y(0) = y(1) \tag{1.2}$$

where $f\colon [0,1] \times \mathbf{R}^n \to \mathbf{R}^n$ is continuous and $f(0,\cdot) = f(1,\cdot)$. A solution y is a continuously differentiable $\mathbf{R}^n$-valued function which satisfies (1.1) everywhere in $[0,1]$ and the boundary conditions (1.2). We use shooting arguments combined with Brouwer degree theory (see [**1**] and [**10**]) in contrast to the coincidence degree used in [**11**] and the Schauder degree used in [**2**]. We improve on some results of Bebernes [**2**] and of Gufstafson and Schmitt [**5**]. In Corollary 2.8 we apply our results to show that the condition $f(x,y) \geqslant -\alpha y$ for some $\alpha > 0$ and all $y \geqslant 0$ in Santanilla [**11**, Theorem 3.2] can be relaxed (see [**11**] for further discussion and references).

In Section 3 we consider

$$y'' = f(x,y,y') \quad x \in [0,1] \tag{1.3}$$

Received 11th December, 1995
I would like to thank Professor E.N. Dancer for some helpful conversations, and Professor N.S. Trudinger and the Centre for Mathematics and Applications for their support during the preparation of this manuscript.

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https://doi.org/10.1017/S000497270002181X Published online by Cambridge University Press

with the periodic, Picard and Neumann boundary conditions

$$y(0) = y(1), \quad y'(0) = y'(1), \tag{1.4}$$

$$y(0) = A, \quad y(1) = B, \text{ and} \tag{1.5}$$

$$y'(0) = A, \quad y'(1) = B, \tag{1.6}$$

respectively. A solution y is a twice continuously differentiable, $\mathbf{R}^n$-valued function satisfying (1.3) everywhere and the appropriate boundary conditions. Knobloch [7] was the first to consider problem (1.3) and (1.4) in the present context. Our results use Schauder degree theory and extend those of Bebernes and Schmitt [1], Habets and Schmitt [4], Gaines and Mawhin [3], Knobloch [7], Knobloch and Schmitt [8] and Lan [9].

The following notation will be useful.

For a bounded open subset T of $[0,1] \times \mathbf{R}^n$, let ∂T denote the boundary of T, let $\overline{T}$ denote its closure and for $x \in [0,1]$ let $T(x)$ denote its x–cross section and $\partial T(x)$ denote the boundary of $T(x)$. Thus $T(x) = \{y \in \mathbf{R}^n: (x,y) \in T\}$. By the boundary of T we mean the curved boundary so that we exclude the sets $\{0\} \times T(0)$ and $\{1\} \times T(1)$. As usual $\mathbf{R}^+ = \{x \in \mathbf{R}: x \geqslant 0\}$, I is the identity map on $\mathbf{R}^n$, $y = (y^1, \cdots, y^n) \in \mathbf{R}^n$, $y > 0$ means $y^i > 0$ for all $1 \leqslant i \leqslant n$ while $y \geqslant 0$ means $y^i \geqslant 0$ for all $1 \leqslant i \leqslant n$, and $B_r = \{y \in \mathbf{R}^n: |y| < r\}$. Also when A and B are subsets of $\mathbf{R}^n$ we denote by $C^m(A,B)$ the functions from A to B with continuous m–th order partial derivatives endowed with the usual maximum norm. If A is a bounded open subset of $\mathbf{R}^n$, $p \in \mathbf{R}^n$, $f \in C(A;\mathbf{R}^n)$ and $p \notin f(\partial A)$ we denote the Brouwer degree of f on A at p by $d(f,A,p)$.

We shall always use Ω to denote a bounded open subset of $[0,1] \times \mathbf{R}^n$. If $0 \in \Omega(x)$ for all $x \in [0,1]$, then we define $L: [0,1] \times \mathbf{R}^n \to \mathbf{R}$ by

$$L(x,y) = \begin{cases} 1, & \text{for } (x,y) \in \overline{\Omega} \\ \inf\{k > 0: (x, y/k) \in \Omega\}, & \text{otherwise.} \end{cases} \tag{1.7}$$

Thus $L \geqslant 1$ and $(x, y/L(x,y))$ belongs to the boundary of Ω for all (x,y) not in Ω. When y is a function of x uniquely determined from the context we shall abbreviate $L(x,y(x))$ to $L(x)$.

2. The first order problem

When considering problem (1.1) and (1.2) we assume that $\Omega(0) = \Omega(1)$.

Definition 2.1: We shall call Ω a star bounding set for (1.1) if it satisfies the following conditions. The cross–sections $\Omega(x)$ are star shaped with respect to the origin

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for all x in $[0,1]$. The mapping L defined by (1.7) is uniformly Lipschitz continuous on $[0,1] \times \mathbf{R}^n$. For each (t,u) in the boundary of Ω there exists $V(x,y;t,u) = V(x,y)$ with the following properties:

(1) V is continuously differentiable;
(2) there is a neighbourhood N of (t,u) in $[0,1] \times \mathbf{R}^n$ such that $\Omega \cap N \subseteq \{(x,y) \in N\colon V(x,y) < 0\}$;
(3) $V(t,u) = 0$;
(4) $V_x(t,u) + V_y(t,u)f(t,u) \geqslant 0$.

REMARK 2.2. If Ω is a star bounding set for (1.1), then for all $t, \theta \in [0,1]$ and $u \in \partial\Omega(t)$, $\theta u \in \overline{\Omega(t)}$ and thus $V_y(t,u)u \geqslant 0$. Moreover, since L is uniformly Lipschitz continuous, $V_y(t,u)u > 0$.

We call unit vector $n(x,y) = (n_1(x,y), n_2(x,y)) \in \mathbf{R} \times \mathbf{R}^n$ an outer normal to Ω if $(x,y) \in \partial\Omega$, and

$$\Omega \subset \{(t,u) \in [0,1] \times \mathbf{R}^n\colon n(x,y)^t(t-x, u-y) < 0\}.$$

REMARK 2.3. If Ω is a convex subset of $[0,1] \times \mathbf{R}^n$ with a uniformly Lipschitz boundary and to each (x,y) of the boundary there is an outer normal $n(x,y)$ such that

$$n(x,y)^t(1, f(x,y)) \geqslant 0,$$

then Ω is a star bounding set for (1.1). To see this, for $(t,u) \in \partial\Omega$ set $N = [0,1] \times \mathbf{R}^n$ and set $V(x,y;t,u) = n(t,u)(t-x, u-y)$. Since the boundary is uniformly Lipschitz, L is uniformly Lipschitz continuous. The boundary of Ω will be uniformly Lipschitz continuous if, for example, Ω contains $[0,1] \times 0$ and is the restriction to $[0,1] \times \mathbf{R}^n$ of an open convex subset of $[a,b] \times \mathbf{R}^n$ for some $a < 0 < 1 < b$.

In view of Remark 2.3, the following theorem is a generalisation of [**3**, Corollary V.31].

THEOREM 2.4. *Let $f\colon [0,1] \times \mathbf{R}^n \to \mathbf{R}^n$ be continuous and let Ω be a star bounding set for (1.1). Then there is a solution y of (1.1) and (1.2) with $(x,y) \in \overline{\Omega}$, for all $x \in [0,1]$.*

After the proof of Theorem 2.4 we indicate how the star shaped condition is a special case of a more general geometric condition.

If we weaken the geometric condition but strengthen the smoothness of the set Ω, and add a Brouwer degree condition related to the right hand side f of (1.1) and shape of the domain, we obtain binding sets. In Theorem 2.10 we prove an existence result for solutions of problem (1.1) and (1.2) via a slightly different technique.

DEFINITION 2.5: We call Ω a binding set for (1.1) if it satisfies the strict egress condition: for each $(x,y) \in \partial\Omega$, $f(x,y) \neq 0$ and there exist ε, $\eta(x,y) > 0$ such that

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$(x+\delta, y+\delta v) \in \Omega$ for all $-\eta < \delta < 0$ and $(x+\delta, y+\delta v) \notin \overline{\Omega}$ for all $0 < \delta < \eta$, whenever $x+\delta \in [0,1]$ and $v \in \mathbf{R}^n$ satisfies $|v - f(x,y)| < \varepsilon |f(x,y)|$. We call Ω a binding set for (1.1) and (1.2) if it is a binding set for (1.1) and there exists $K\colon [0,1] \times \mathbf{R}^n \to \mathbf{R}^n$ satisfying

(1) K is continuous,
(2) K is differentiable with respect to x uniformly in u, at $x = 0$,
(3) $K(x,\cdot)\colon \Omega(0) \to \Omega(x)$, and
(4) $K(0,u) = u = K(1,u)$ for all $u \in \Omega(0)$.

REMARK 2.6. It is easy to see that Ω is a binding set for (1.1) if for each $(t,u) \in \partial\Omega$ there exists $V(x,y;t,u) = V(x,y)$ with the following properties:

(1) V is continuously differentiable;
(2) there is a neighbourhood N of (t,u) in $[0,1] \times \mathbf{R}^n$ such that $\partial\Omega \cap N = \{(x,y) \in N\colon V(x,y) = 0\}$;
(3) $\Omega \subseteq \{(x,y) \in [0,1] \times \mathbf{R}^n\colon V(x,y) < 0\}$;
(4) $V_x(t,u) + V_y(t,u) f(t,u) > 0$.

If, in addition, $\Omega(0) \subseteq \Omega(x)$ for all x, then Ω is a binding set for (1.1) and (1.2) with $K(x,u) = u$ for all u in $\mathbf{R}^n$.

For the rest of this section g will denote a bounded, Lipschitz continuous function defined on $[0,1] \times \mathbf{R}^n$, and $y(t,u)$ will denote the solution at $x = t$ of the initial value problem

$$y' = g(x,y), \quad \text{for all } x \in [0,1] \tag{2.1}$$

$$y(0) = u. \tag{2.2}$$

Thus y and y' are defined and Lipschitz continuous on $[0,1] \times \mathbf{R}^n$.

PROOF (THEOREM 2.4): Let Ω be a star bounding set for (1.1). We assume first that f is continuously differentiable and that $h(t,u;f) = V_x(t,u) + V_y(t,u) f(t,u) > 0$, for all $(t,u) \in \partial\Omega$, where V is given in Definition 2.1. Let L be defined by (1.7) and $g\colon [0,1] \times \mathbf{R}^n \to \mathbf{R}^n$ be defined by

$$g(x,y) = L(x,y) f(x, y/L(x,y)), \quad \text{for all } (x,y) \in [0,1] \times \mathbf{R}^n.$$

As $g(x,y) = f(x,y)$ for all $(x,y) \in \overline{\Omega}$ it suffices to show problem (2.1) and (2.2) has a solution y with $(x, y(x)) \in \overline{\Omega}$ for all $x \in [0,1]$. As $L \geqslant 1$ is uniformly Lipschitz continuous, g, restricted to $\overline{\Omega}$, is uniformly Lipschitz continuous. As $(x, y/L(x,y)) \in \partial\Omega$ for all $(x,y) \notin \Omega$ and $\overline{\Omega}$ is compact it follows that g is uniformly Lipschitz continuous and $|g(x,y)| \leqslant c(|y| + 1)$ on $[0,1] \times \mathbf{R}^n$ for some constant $c > 0$. Thus the solution $y(t,u)$ of (2.1) and (2.2) exists and is unique and, moreover, $y'(t,u)$ is

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uniformly Lipschitz continuous on compact subsets of $[0,1] \times \mathbf{R}^n$. We define K on $[0,1] \times \partial\Omega(0)$ with $K(t,\cdot): \partial\Omega(0) \to \overline{\Omega}(t)$ as follows. For $(t,u) \in [0,1] \times \partial\Omega(0)$ let $m = s(t,u)$ be the unique solution of $mu \in \partial\Omega(t)$. As L is Lipschitz continuous $|s(t,u) - s(x,u)| \leqslant c\,|t-x|$, for some constant c independent of u. Since $[0,1] \times \{0\} \in \Omega$ and Ω is open there is $\varepsilon > 0$ such that $s > \varepsilon$. Moreover since $\partial\Omega(0) = \partial\Omega(1)$, $s(0,u) = 1 = s(1,u)$ for all $u \in \partial\Omega(0)$. Thus there exists $k \in C^2([0,1];[0,1])$ such that $k(0) = 1 = k(1)$, and $0 \leqslant k(t) \leqslant s(t,u)$ for all $(t,u) \in [0,1] \times \partial\Omega(0)$. We set $K(t,u) = k(t)u$ for all $(t,u) \in [0,1] \times \partial\Omega(0)$. Define $H: [-1,1] \times \mathbf{R}^n \to \mathbf{R}^n$ by

$$(2.4)\quad H(t,u) = \begin{cases} y(t,u) - K(t,u), & \text{for all } (t,u) \in (0,1] \times \partial\Omega(0) \\ (1+t)(g(0,u) - K_x(0,u)) - tu, & \text{for all } (t,u) \in [-1,0] \times \partial\Omega(0). \end{cases}$$

We show that our problem has a solution if $H(1,u) = 0$ for some $u \in \overline{\Omega}(0)$. We show $H(t,u) \neq 0$ for all $(t,u) \in [-1,1] \times \partial\Omega(0)$ and $d(H(t,\cdot),\Omega(0),0) = d(I,\Omega(0),0)$, for all $t \in [-1,1]$. Then $d(I,\Omega(0),0) = 1$ as $0 \in \Omega(0)$ and, by Brouwer degree theory, $H(1,u) = 0$ has a solution. Suppose that $H(t,u) = 0$ for some $(t,u) \in (0,1] \times \partial\Omega(0)$. Then there is $(t,u) \in (0,1] \times \partial\Omega(0)$ and a solution $y(x,u)$ of (2.1) and (2.2) with $y(t,u) = K(t,u)$. Set $v(x) = V(x,y(x,u);0,u)$. Then $v(0) = 0$ and

$$(2.5)\qquad v'(0) = h(0,u;g) = V_x(0,u) + V_y(0,u)g(0,u) > 0, \quad \text{for all } u \in \partial\Omega(0)$$

since $y(0,u) = u \in \partial\Omega(0)$ and $f(0,u) = g(0,u)$. There is $\delta > 0$ such that $(x,y(x,u)) \notin \overline{\Omega}$ for $0 < x < \delta$ since $\Omega \cap N \subseteq \{(x,y) \in N: V(x,y;0,u) < 0\}$, for some neighbourhood N of $(0,u)$. As $(t,K(t,u)) \in \overline{\Omega}$, by continuity there exists $t_0 > 0$ such that $(t_0,y(t_0,u)) \in \partial\Omega$ and $(x,y(x,u)) \notin \overline{\Omega}$ for $0 < x < t_0$. Since y is continuous and $L(x,y)$ is Lipschitz, $L(x,y(x,u))$ is a continuous function of x on $[0,t_0]$, and hence attains its maximum value l at $t_1 \in (0,t_0)$, say. Set $z(x,u) = y(x,u)/l$. Thus for all $0 < x < t_0$, $(x,z(x,u)) \in \overline{\Omega}$ and $(t_1,z(t_1,u)) \in \partial\Omega$, $z'(t_1,u) = g(t_1,y(t_1,u))/l = f(t_1,z(t_1,u))$ and $n(t_1,z(t_1,u))$ is an outer normal to Ω at $(t_1,z(t_1,u))$. Set $v(x) = V(x,z(x,u);t_1,u)$. Then $v(0) = 0$ and

$$v'(t_1) = h(t_1,u;f) = V_x(t_1,z(t_1,u)) + V_y(t_1,z(t_1,u))f(t_1,z(t_1,u)) > 0.$$

Again it follows that $(x,z(x,u)) \notin \overline{\Omega}$ for $t_1 < x < \eta$ for some $\eta > t_1$, a contradiction. Thus $H(t,u) \neq 0$ for all $(t,u) \in (0,1] \times \partial\Omega(0)$. If $H(1,u) = 0$ for some $u \in \overline{\Omega}(0)$, we see from this argument that $(x,y(x,u)) \in \overline{\Omega}$ for all $x \in [0,1]$, and hence y is the required solution. Set $w(x) = V(x,K(x,u);0,u)$. Since $w(0) = 0$, $(x,K(x,u)) \in \overline{\Omega}$ for all $x \in [0,1]$ and $\Omega \cap N \subseteq \{(x,y) \in N: V(x,y;0,u) < 0\}$, for some neighbourhood N of $(0,u)$,

$$(2.6)\qquad w'(0) = V_x(0,u) + V_y(0,u)K_x(0,u) \leqslant 0, \quad \text{for all } u \in \partial\Omega(0).$$

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By (2.5) and (2.6)

$$V_y(0,u)(g(0,u) - K_x(0,u)) > 0, \quad \text{for all } u \in \partial\Omega(0).$$

From this and Remark 2.2, $H(t,u) \neq 0$, for all $(t,u) \in [-1,0] \times \partial\Omega(0)$. Since H is continuous on $[-1,1] \times \overline{\Omega}$ except possibly from the right at $t = 0$, it suffices to connect the degrees of $H(0,\cdot)$ and $H(t,\cdot)$, $t \in (0,1]$. If $y_i(x,u)$ is a component of $y(x,u)$ then $y_i'(x,u)$ is Lipschitz continuous, so

$$y_i(x,u) = y_i(0,u) + xy_i'(0,u) + x\varepsilon(x,u),$$

where $|\varepsilon(x,u)| < k|x|$ for some constant k independent of $u \in \overline{\Omega}(0)$. Thus $|H(t,u)/t - H(0,u)| = |\varepsilon(t,u)| < kt$ for all $t \in (0,1]$. Thus if $t > 0$ is small enough,

$$d(H(0,\cdot),\Omega,0) = d(H(t,\cdot),\Omega,0).$$

If $h(t,u;f) = 0$ for some $(t,u) \in \partial\Omega$ or f is not continuously differentiable, for $m > 0$, by Remark 2.2 there is a constant $\delta \in (0,1/m)$ such that $h(x,y;f + y/m) > 2\delta > 0$ for all $(x,y) \in \partial\Omega$. By approximating f on $\overline{\Omega}$ to within δ by a smooth function f_δ, and replacing f by $f_m(x,y) = f_\delta(x,y) + y/m$ in the above argument, we obtain a sequence of solutions y_m of

$$y' = f_m(x,y) \text{ for all } x \in [0,1]$$

satisfying (1.2). A subsequence of these solutions converges to the required solution. $\square$

REMARK 2.7: On examining the proof, we see that the star shaped condition is used to scale the domain, to define K and compute degree. Thus this condition can be replaced by the following. As before let Ω be a bounded open subset of $[0,1] \times \mathbf{R}^n$ with uniformly Lipschitz continuous boundary and let $[0,1] \times \{0\} \subseteq \Omega$. Assume there is a map $\Psi \in C^{1,1}(\mathbf{R}^+ \times [0,1] \times \mathbf{R}^n; \mathbf{R}^n)$ such that

(1) $\Psi(0,x,y) = 0$, for all $(x,y) \in \overline{\Omega}$,
(2) $\Psi(1,x,y) = y$, for all $(x,y) \in \overline{\Omega}$,
(3) $\Psi\big(l,x,\overline{\Omega}(x)\big) \subset\subseteq \Psi(m,x,\Omega(x))$, for all $0 \leqslant x \leqslant 1$ and $0 \leqslant l < m$,
(4) $\bigcup_{n \in \mathbf{N}} \Psi(n,x,\Omega(x)) = \mathbf{R}^n$,
(5) $\Psi(l,x,\cdot)\colon \overline{\Omega}(x) \to \mathbf{R}^n$ is a diffeomorphism, and
(6) $\Psi_l(0,0,y) = y$ for all $y \in \partial\Omega(0)$.

Define $k \in C^2([0,1];[0,1])$ such that $k(0) = 1 = k(1)$, and $K(x,u) = \Psi(k(x),x,u) \in \overline{\Omega}(x)$ for all $x \in [0,1]$ where $u \in \partial\Omega(0)$. For $y \notin \overline{\Omega}(x)$, $x \in [0,1]$ set $L(x,y) = \sup\{l\colon y \notin \Psi\big(l,x,\overline{\Omega}(x)\big)\}$ and $g(x,y) = \Psi_x(L(x,y),x,z) + \Psi_y(L(x,y),x,z)f(x,z)$,

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and set $L(x,y) = 1$ and $g(x,y) = f(x,y)$ for $(x,y) \in \overline{\Omega}$ where $z \in \partial\Omega(x)$ is defined by $\Psi(L(x,y),x,z) = y$. From the assumptions L, Ψ_x and Ψ_y are uniformly Lipschitz continuous and satisfy $|\Psi_x(L(x,y),x,z)|$, $|\Psi_y(L(x,y),x,z)| \leqslant c(1+|y|)$, for some constant c, all $(x,z) \in \overline{\Omega}$ and all $(x,y) \in [0,1] \times \mathbf{R}^n$. The proof of Theorem 2.4 can be suitably modified to these hypotheses. In particular, we replace tu in (2.4) by $t\Psi_l(1,0,u)$, and y/m in the definition of f_m by $\Psi_l(1,x,y)/m$. Also we note that $V_y(x,u)\Psi_l(1,x,u) > 0$ for all $u \in \partial\Omega(x)$, and $\Psi_l(0,0,u) \neq 0$ for all $u \in \partial\Omega(0)$, by (3) and (5) of the definition of Ψ. In place of $z(x,u) = y(x,u)/l$ we let $z(x,u)$ be the solution of $\Psi(l,x,z(x,u)) = y(x,u)$. Moreover $d(\Psi_l(1,0,\cdot),\Omega(0),0) = 1$, using (6), since $H(t,u) = \Psi_l(t,0,u)$ is a homotopy. The star shaped region corresponds to the special case $\Psi(l,x,y) = ly$.

As an application of Theorem 2.4 we have Corollary 2.8, the generalisation of Santanilla [**11**, Theorem 3.2] mentioned in the Introduction.

COROLLARY 2.8. *If there exists $R > 0$ such that for all $y \geqslant 0$, $y^t f(x,y) \geqslant 0$ when $|y| = R$ and $f_i(x,y) \leqslant 0$ when $y_i = 0$ then problem (1.1) and (1.2) has a nonnegative solution.*

PROOF: Let $\Omega = [0,1] \times (B_R \cap \{y \in \mathbf{R}^n : y > 0\})$. By translating the origin, we may assume that $0 \in \Omega(x)$ for all $x \in [0,1]$. By Theorem 2.4 there is a solution y of (1) and (2) with $(x,y) \in \overline{\Omega}$ for all $x \in [0,1]$. By translating back to the original origin, we see that y is the required solution. ∎

REMARK 2.9. As E.N. Dancer (private communication) observed, the additional condition $f(x,y) \geqslant -\alpha y$ required in Santanilla is automatically satisfied if f is continuously differentiable. Thus the assumption can be removed for continuous f simply by approximating by continuously differentiable f and using compactness to select a subsequence which converges to a solution.

We now state and prove our existence result using binding functions.

THEOREM 2.10. *Let $f : [0,1] \times \mathbf{R}^n \to \mathbf{R}^n$ be continuous and let Ω be a binding set for (1.1) and (1.2). If*

$$d(J,\Omega(0),0) \neq 0 \tag{2.7}$$

where $J(y) = f(0,y) - K_x(0,y)$, for all y in $\mathbf{R}^n$, then there is a solution y of (1.1) and (1.2) with (x,y) in $\overline{\Omega}$ for all x in $[0,1]$.

PROOF: Assume first that f is continuously differentiable and $\Omega \subseteq [0,1] \times \overline{B}_R$ for some R. Define $g : [0,1] \times \mathbf{R}^n \to \mathbf{R}^n$ as follows:

$$g(x,y) = \begin{cases} f(x,y), & \text{for all } y \in \overline{B}_R \\ f(x, yR/|y|), & \text{otherwise.} \end{cases} \tag{2.8}$$

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Define $H\colon [0,1] \times \mathbf{R}^n \to \mathbf{R}^n$ by $H(t,u) = y(t,u) - K(t,u)$ where y is the solution of (2.1) and (2.2). We show that problem (1.1) and (1.2) has a solution if $H(1,u) = 0$ for some $u \in \Omega(0)$. To show that $H(1,u) = 0$ has a solution it suffices to show that

$$d(H(t,\cdot),\Omega(0),0) = d(J,\Omega(0),0) \neq 0 \tag{2.9}$$

for all $t \in (0,1]$. To connect the degrees of $H(t,\cdot)$ and J let $y^i(x,u)$ denote a component of the solution y, and K^i and H^i the corresponding components of K and H, respectively. Thus

$$H^i(x,u) = x\Big(y^{i'}(0,u) - K^{i'}(0,u)\Big) + \varepsilon(x,u)x \tag{2.10}$$

where $|\varepsilon(x,u)|$ converges to 0 uniformly in $u \in \partial\Omega(0)$ as x converges to 0^+. Thus $|H(t,u)/t - J(u)| < |\varepsilon(x,u)|$ for $t \in (0,1]$ and $u \in \partial\Omega(0)$. Thus if $t > 0$ is small enough, then (2.9) follows. Thus it suffices to show that $H(t,u) \neq 0$ for all $t \in (0,1]$ and $u \in \partial\Omega(0)$. Suppose that this is not the case. Then there are $u \in \partial\Omega(0)$ and $t \in (0,1]$ such that $y(t,u) = K(t,u) \in \overline{\Omega}(t)$. By the definition of binding sets, solutions $y(t,u) \in \partial\Omega(t)$ satisfy that $(t,y(t,u))$ strictly egresses from $\overline{\Omega}$. As $u \in \partial\Omega(0)$ it follows from (2.2) and continuity, that there is $t_0 > 0$ such that $(x,y(x,u)) \notin \overline{\Omega}$ for $0 < x < t_0$ and $(t_0,y(t_0,u)) \in \partial\Omega$. But $(t_0,y(t_0,u))$ is a strict egress point for (2.1) and $\partial\Omega$, a contradiction. The result follows.

If f is not smooth then $|f|$ attains a positive minimum $\overline{f}$ on $\partial\Omega$. Approximate f uniformly on $\overline{B}_R$ by a sequence of smooth functions, solve the approximating problems, and obtain the desired solution as the limit of a convergent subsequence of the solutions of the approximating problems. ∎

REMARK 2.11. In view of the proofs of Theorem 2.10 and Theorem 2.1 of [1] we have the following result.

Assume that there exists K satisfying (1) to (4) of Definition 2.5. Let $f\colon [0,1] \times \mathbf{R}^n \to \mathbf{R}^n$ be continuous and assume that all solutions of (1.1) and (2.2) exist on $[0,1]$. Let (2.7) hold and assume that $y(t,u) - K(t,u) \neq 0$ for all $(t,u) \in (0,1] \times \partial\Omega(0)$, where $y(t,u)$ is a solution at $x = t$ of (1.1) and (2.2). Then there is a solution y of (1.1) and (1.2) with (x,y) in $\overline{\Omega}$ for all x in $[0,1]$. This is very close to Theorem 2.1 of Bebernes and Schmitt [1]. They require $K(t,\cdot)$ to be one to one on $\overline{\Omega(0)}$, but do not require the uniformity in (2) of Definition 2.5. Apart from these differences we see that the difference between Theorem 2.1 of [1] and Theorem 2.10 is that the egress conditions of Definition 2.5 are used in Theorem 2.10 to guarantee that $y(t,u) - K(t,u) \neq 0$, whereas it is assumed in Theorem 2.1 of [1].

The difference between Theorems 2.4 and 2.10 is that because of the geometry of bounding sets a natural homotopy exists guaranteeing $d(H(1,\cdot),\Omega(0),0) \neq 0$, whereas this follows from assumption (2.7) in Theorem 2.10.

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The following example illustrates the use of Theorem 2.10. As Ω contains a hole, Ω is not a star bounding set and Theorem 2.4 does not apply. Moreover, to find a binding set for (1.1) and (1.2) we allow Ω to depend on x.

EXAMPLE 2.12. Consider

$$y' = A(x)y + \left(y^t B(x)y\right)y + \left(y^t C(x)y\right)^2 y^t D(x) \quad x \in [0,1] \tag{2.11}$$

where A, B, C and D are a non-singular $n \times n$ matrices which are smooth and of period one in x. Assume moreover that B is negative definite, C and D are positive definite, n is odd, and the fundamental matrix $Z(x)$ for

$$y' = A(x)y$$

is also of period one. Then there is a nontrivial periodic solution of (2.11) of period one. To see this let $\Delta = \{(x,y)\colon x \in [0,1],\ y = Z(x)u,\ u \in B_\varepsilon\}$ and $\Omega = [0,1] \times B_R \backslash \overline{\Delta}$. Thus $\partial\Omega = \Gamma_1 \bigcup \Gamma_2$ where $\Gamma_1 = \{(x,y)\colon x \in [0,1],\ y = Z(x)u,\ |u| = \varepsilon\} = \partial\Delta$ and $\Gamma_2 = \{(x,y)\colon x \in [0,1],\ |y| = R\} = [0,1] \times \partial B_R$. We choose $\varepsilon > 0$ sufficiently small that

$$\left(y^t B(x)y\right)y^t m(x,y) > \left(y^t C(x)y\right)^2 y^t D(x) m(x,y) \tag{2.12}$$

on Γ_1 where $m(x,y)$ is the interior unit normal to $\Delta(x) = \{y\colon y = Z(x)u,\ u \in B_\varepsilon\}$. We choose R sufficiently large that

$$\left(y^t B(x)y\right)y^t y < \left(y^t C(x)y\right)^2 y^t D(x)y \tag{2.13}$$

on Γ_2 and $R > \sup\{|y| : y \in \Gamma_1\}$. Clearly $m(x,y)^t A(x)y = 0$, so the strict egress condition is satisfied on Γ_1, by (2.12), and on Γ_2, by (2.13). Thus Ω is a binding set for (1.1) and (1.2). On $\Gamma_1(0)$, $f(0,y)m(0,y) > 0$ so $f(0,y)$ is homotopic to $m(0,y)$ on $\Delta(0)$. As $\Delta(0)$ is convex and 0 is an interior point, $m(0,y)$ is homotopic to $-I$ on $\Delta(0)$. As n is odd, $d(f(0,\cdot),\Delta(0),0) = -1$. By (2.13), $f(0,\cdot)$ is homotopic to I on B_R. By the excision property of degree, $d(f(0,\cdot),\Omega(0),0) = 2$. By Theorem 2.10, problem (1.1) and (1.2) has a nontrivial solution.

If we set $n = 3$ and $A_{12} = \pi$, $A_{21} = -\pi$, $A_{33} = \cos \pi x$, $A_{ij} = 0$ otherwise, it is difficult to find a domain Ω whose shape is independent of x and boundary consists entirely of 1 nonrecurrent points – in the terminology of Lloyd [**10**]. In particular, it is not clear how to apply Lloyd [**10**, Theorem 9.22].

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3. The second order problem

We consider problem (1.3) together with one of the boundary conditions (1.4) to (1.6).

Definition 3.1: We call Ω an admissible bounding set for problems (1.3) with boundary conditions (1.4), (1.5) or (1.6) if it is a bounded open subset of $[0,1] \times \mathbf{R}^n$ with the following properties:

(1) $[0,1] \times \{0\} \subset \Omega$;
(2) $\Omega(x)$ is star shaped with respect to the origin, for all $x \in [0,1]$;
(3) the mapping $L(x,y)$ defined by (1.7) is Lipschitz continuous on $[0,1] \times \mathbf{R}^n$;
(4) for each $(t,u) \in \partial\Omega$ with $t \in (0,1)$ there is a neighbourhood N and a twice continuously differentiable function $r: N \to \mathbf{R}$ such that
  (a) $\Omega \cap N \subset \{(x,y) \in N : r(x,y) < 0\}$,
  (b) $r(t,u) = 0$, and
  (c) for $p \in \mathbf{R}^n$ such that $r'(t,u,p) = r_x(t,u) + r_y^t(t,u)p = 0$,

$$r''_f(t,u,p) = r_{xx}(t,u) + 2r_{xy}^t(t,u)p + p^t r_{yy}(t,u)p + r_y^t(t,u)f(t,u,p) \geqslant 0;$$

(5) there is a constant $\delta > 0$ and a continuous vector field $n: [0,1] \times \mathbf{R}^n \to \mathbf{R}^n$ such that

$$n(t,u)^t r_y(t,u) \geqslant |r_y(t,u)|\,\delta > 0 \text{ and } n(t,u)^t y \geqslant |y|\,\delta > 0$$

for all $(t,u) \in \partial\Omega$ with $t \in (0,1)$, respectively $[0,1]$.

Remark 3.2. It is easy to see that (5) of Definition 3.1 is satisfied if $\partial\Omega$ is smooth and L is uniformly Lipschitz continuous. In place of (1) of Definition 3.1 we can allow

$$\{(x,\phi(x)) : x \in [0,1]\} \subset \Omega$$

for some $\phi \in C^2[0,1]$, with the appropriate changes in the other assumptions. In view of Remark 2.7, the star shaped condition can be generalised. One of the features of our work is that we do not require the usual assumption that r_{yy} is positive semi-definite. A second feature is the simpler modification of f we use in the proofs as compared, for example, with Lan [**9**].

Definition 3.3: We call f admissible for (1.3) if it satisfies

(1) $f \in C([0,1] \times \mathbf{R}^n \times \mathbf{R}^n; \mathbf{R}^n)$,
(2) $|f| \leqslant \phi(|p|)$, where $\int^\infty s/\phi(s)\,ds = \infty$ and
(3) $|f| \leqslant 2C\left(y^t f + |p|^2\right) + K$, where C, K are non-negative constants.

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Let $G\colon [0,1]\times[0,1] \to \mathbf{R}$ be the Green's function for (1.3) together with the homogeneous boundary conditions $A = 0 = B$ in (1.5). Thus

$$G(x,t) = \begin{cases} x(1-t), & \text{for } 0 \leqslant x \leqslant t \leqslant 1 \\ (1-x)t, & \text{for } 0 \leqslant t \leqslant x \leqslant 1. \end{cases}$$

Let $w(x,A,B) = A(1-x) + Bx$.

Theorem 3.4. *Let f be admissible and let Ω be an admissible bounding set for problem (1.3) and (1.4). Let f and Ω satisfy:*

(1) $f(0,y,y') = f(1,y,y')$ *for all* $(y,y') \in \mathbf{R}^n \times \mathbf{R}^n$;

(2) $\Omega(0) = \Omega(1)$;

(3) *each point* $(t,u) \in \partial\Omega$ *with* $t \in \{0,1\}$ *has a neighbourhood* N *and a continuously differentiable function* $r\colon N \to \mathbf{R}$ *such that* $\Omega \cap N \subset \{(x,y) \in N\colon r(x,y) < 0\}$, $r(t,u) = 0$ *and* $r'(0,u,p) \geqslant r'(1,u,p)$ *for all* $(0,u) \in \partial\Omega(0)$ *and* $p \in \mathbf{R}^n$.

Then problem (1.3) and (1.4) has a solution y with $(x,y) \in \overline{\Omega}$, for all $x \in [0,1]$.

Proof: We assume first that $r''_f > 0$ when $r = 0 = r'$. We use Schauder degree theory and need the following family of functions to construct a homotopy. Choose $R > 0$ and $\varepsilon \in (0,1)$ such that $B_{2\varepsilon} \subseteq \Omega(x) \subseteq B_R$ for all $x \in [0,1]$. Let $h \in C(\mathbf{R}^n;[0,1])$ satisfy

$$h(y) = \begin{cases} 0, & \text{if } |y| < \varepsilon \\ 1, & \text{if } |y| > 2\varepsilon, \end{cases}$$

(3.1) $$f_\lambda(x,y,p) = \lambda f(x,y,p) + (1-\lambda)h(y)\,|f|\,n(x,y)/\delta \quad \text{and}$$

(3.2) $$g_\lambda(x,y,p) = L(x,y)f_\lambda(x,y/L(x,y),p/L(x,y))$$

where L is given in (1.7). Thus g_λ is continuous, $f_\lambda = g_\lambda$ in $\Omega \times \mathbf{R}^n$ and it suffices to find a solution y of (3.2) and (1.4) with $\lambda = 1$ and $(x,y) \in \overline{\Omega}$. Let $\mathbf{\Delta} = \{y \in C^1([0,1];\mathbf{R}^n)\colon |y| < R \text{ and } |y'| < M\}$, where M is chosen below. Let $\mathbf{\Sigma} = \mathbf{\Delta} \times \Omega(0)$. For $y \in C^1([0,1];\mathbf{R}^n)$, we set $T(g_\lambda(y))(x) = -\int_0^1 G(x,t)g_\lambda(t,y(t),y'(t))\,dt$. If $(y,A) \in \overline{\mathbf{\Sigma}}$ is a solution of

(3.3.) $$(y(x) - T(g_1(y)) - w(x,A,A), y'(0) - y'(1)) = 0,$$

we show that $(x,y) \in \overline{\Omega}$. Thus, by the definition of T and g_λ, problem (1.3) and (1.4) has a solution y with $(x,y) \in \overline{\Omega}$ if and only if (y,A) is a solution of (3.3) in $\overline{\mathbf{\Sigma}}$. We choose M as follows. By continuity, $L \leqslant l$ on $[0,1] \times \overline{B}_R$, for some $l \geqslant 1$. Set

$$c = \sup\{|n(x,y)|/\delta\colon (x,y) \in \Omega\}.$$

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Thus $|g_\lambda| \leqslant c\phi(|p|)$ and $|g_\lambda| \leqslant 2(C+c/\varepsilon)\left(y^t f + |p|^2\right) + Kl^2$, for all $(x,y) \in \Omega$. By Hartman [**6**, Lemma 5.2, p.429] there is M such that solutions y of

$$y'' = g_\lambda(x,y,y')$$

with $(x,y) \in \overline{\Omega}$ satisfy $|y'| < M$. To show that (3.3) has a solution we use Schauder degree theory. Define $H_i \colon [0,1] \times \overline{\Sigma} \to C^1([0,1];\mathbf{R}^n) \times \mathbf{R}^n$ for $i = 1,2,3$, by

$$\begin{aligned}
&H_1(\lambda,(y,A))(x) = (y(x) - T(g_1(y))(x) - w(x,A,A),\ \lambda(y'(0) - y'(1)) + (1-\lambda)A),\\
&H_2(\lambda,(y,A))(x) = (y(x) - T(g_\lambda(y))(x) - w(x,A,A), A) \quad \text{and}\\
&H_3(\lambda,(y,A))(x) = (y(x) - \lambda(T(g_0(y))(x) - w(x,A,A)), A).
\end{aligned}$$

We show that either there is a solution to our problem or the above functions H_i define homotopies. Suppose $H_1(\lambda,(y,A)) = 0$ has a solution $(y,A) \in \partial\Sigma$. By the choice of M, $|y'| < M$. Suppose $y \in \partial\Delta$. By the choice of R there is $t \in (0,1)$ such that $|y(t)| = R$ and $L(t) > 1$. As $L(0) = 1 = L(1)$ and L is continuous, there is $t_0 \in (0,1)$ such that L has a maximum $l_0 \leqslant l$ say, at t_0. Let $z(x) = y(x)/l_0$. Thus $(x,z) \in \overline{\Omega}$ for all $x \in [0,1]$ and $(t_0, y(t_0)) \in \partial\Omega$. Thus $r(t_0,z) = 0$, $r'(t_0,z,z') = 0$ and $r''_f(t_0,z,z') > 0$, a contradiction, and $y \notin \partial\Delta$. Moreover, by this argument $(x,y) \in \overline{\Omega}$ for all $x \in [0,1]$. Suppose $A \in \partial\Omega(0)$. If $\lambda = 1$, then (y,A) is a solution to our problem, as required. If $0 \leqslant \lambda < 1$, then $0 \leqslant r'(1,y,y') \leqslant r'(0,y,y') \leqslant 0$ so $r_y(0,A)\lambda(y'(0) - y'(1)) + (1-\lambda)A = (1-\lambda)r_y(0,A)A > 0$, a contradiction. Thus $H_1(\lambda,(y,A)) \neq 0$ for any $(y,A) \in \partial\Sigma$.

Suppose $H_2(\lambda,(y,A)) = 0$ has a solution $(y,A) \in \partial\Sigma$. Suppose $y \in \partial\Delta$. Since $|y'| < M$, $|y(t)| = R$ for some $t \in (0,1)$. As above L has a maximum $l_0 > 1$ at $t_0 \in (0,1)$, and by setting $z(x) = y(x)/l_0$ we again get a contradiction, $r(t_0,z) = 0$, $r'(t_0,z,z') = 0$ and $r''_{g_\lambda}(t_0,z,z') > 0$, since $r_y(0,z)(n(t_0,z)h(z)\,|f(t_0,z,z')|/\delta - f(t_0,z,z')) \geqslant 0$, by condition (5) of the definition of admissibility of Ω. Now $0 \in \Omega(0)$ so $A \neq 0$ on $\partial\Omega(0)$. Thus $H_2 \neq 0$ on $\partial\Sigma$. Suppose $H_3(\lambda,(y,A)) = 0$ has a solution $(y,A) \in \partial\Sigma$. By the above arguments it suffices to show that there is no $t \in (0,1)$ such that $|y(t)| = R$. If such a t exists then $\lambda > 0$, $y(t)^t y'(t) = 0$ and $y'(t)^2 + y''(t)^t y(t) \geqslant \lambda\,|f|\,n(t,y)^t y(t) > 0$. This is a contradiction, so $H_3 \neq 0$ on $\partial\Sigma$. By the homotopy invariance of Schauder degree

$$d(H_i(\lambda,\cdot),\Sigma,0) = \text{ constant}$$

for all $\lambda \in [0,1]$ and $i = 1,2,3$. In particular,

$$d(H_1(1,\cdot),\Sigma,0) = d(H_2(\lambda,\cdot),\Sigma,0) = d(H_3(0,\cdot),\Sigma,0) = 1$$

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Thus there is a solution in Σ of $H_1(1,(y,A)) = 0$, and by the above arguments y is the required solution of (1.3) and (1.4). If $r''_f \geqslant 0$ when $r = 0 = r'$, consider the sequence of problems where f is replaced by $f + y/n$. By the above argument there exists a sequence of solutions y_n of these which will have a subsequence, convergent to the required solution. ∎

REMARK 3.5. Condition (2) of Theorem 3.4 is condition (e) of the definition of V of Bebernes [**4**, p.124]. An admissible bounding set Ω satisfies (2) of Theorem 3.4 only if $r_y(0,u) = r_y(1,u)$ and $r_x(0,u) \geqslant r_x(1,u)$. If $\Omega(0)$ has a unique tangent plane at each of its boundary points then the condition $r_y(0,u) = r_y(1,u)$ follows automatically, by suitably scaling r. Let $n = 1$ and suppose there exist lower and upper solutions α and β, respectively, satisfying $\alpha(0) = \alpha(1)$, $\beta(0) = \beta(1)$ and $\alpha(x) < \beta(x)$ for all x in $[0,1]$. If we set $\Omega = \{(x,y)\colon \alpha(x) < y < \beta(x),\ x \in [0,1]\}$ then this condition becomes $\alpha'(0) \geqslant \alpha'(1)$ and $\beta'(0) \leqslant \beta'(1)$. It is not difficult to construct an example to show that some additional assumptions of this kind are necessary to guarantee existence.

THEOREM 3.6. *Let f be admissible and Ω an admissible bounding set for problem (1.3) and (1.5). If $A \in \Omega(0)$ and $B \in \Omega(1)$, then problem (1.3) and (1.5) has a solution y with $(x,y) \in \overline{\Omega}$.*

The proof uses the extension ideas introduced in Theorem 3.4 together with the usual ideas. Schauder degree theory is used in the function space Δ and the proof is simpler than that of Theorem 3.4. The proof is omitted.

Our extension argument is simpler than that of Lan [**9**] even in the case that $\Omega(x)$ is convex and allows us to relax the usual positive definite restriction required on r_{yy} (see, for example, [**3**]) required for the homotopy argument.

REMARK 3.7. This result extends some results of Gaines and Mawhin [**3**] and Lan [**9**].

THEOREM 3.8. *Let f be admissible and Ω an admissible bounding set for problem (1.3) and (1.6). Assume that each point $(t,u) \in \partial\Omega$ with $t \in \{0,1\}$ has a neighbourhood N and a continuously differentiable function $r\colon N \to \mathbf{R}$ such that $\Omega \cap N \subset \{(x,y) \in N\colon r(x,y) < 0\}$, $r(t,u) = 0$,*

$$r'(0,u,A) \geqslant 0 \quad \text{for all } (0,u) \in \partial\Omega(0) \text{ and} \tag{3.4}$$

$$r'(0,u,B) \leqslant 0 \quad \text{for all } (0,u) \in \partial\Omega(1). \tag{3.5}$$

Then problem (1.3) and (1.6) has a solution y with $(x,y) \in \overline{\Omega}$.

PROOF: Again by approximating we may assume that $r''_f > 0$ when $r = 0 = r'$ and use Schauder degree theory. Let g_λ, T and Δ be as in the proof of Theorem 3.4. Let $\Sigma = \Delta \times \Omega(0) \times \Omega(1)$. Essentially we 'shoot' with the boundary values C and D of solutions y of (1.3). In particular we show that if $(y,C,D) \in \overline{\Sigma}$ is a solution of

$$(y(x) - T(g_1(y))(x) - w(x,C,D), y'(0) - A, y'(1) - B) = 0, \tag{3.6}$$

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then $(x,y) \in \overline{\Omega}$ and thus $y(x)$ is a solution of problem (1.3) and (1.6). We choose l, M and c as before. To show that (3.6) has the required solution we use Schauder degree theory. Define $H_i\colon [0,1] \times \overline{\Sigma} \to C^1([0,1];\mathbf{R}^n) \times \mathbf{R}^{2n}$, for $i = 1,2,3$ by

$$\begin{aligned} H_1(\lambda,(y,C,D)) = (y - T(g_1(y)) - w(\cdot,C,D), \lambda(y'(0) - A) - (1-\lambda)C\ ,\\ \lambda(y'(1) - B) + (1-\lambda)D)\,, \end{aligned} \tag{3.7}$$

$$H_2(\lambda,(y,C,D)) = (y - T(g_\lambda(y)) - w(\cdot,C,D), -C, D) \quad \text{and} \tag{3.8}$$

$$H_3(\lambda,(y,C,D)) = (y - \lambda T(g_0(y)) - w(\cdot,C,D), -C, D). \tag{3.9}$$

We show that either there is a solution of our problem or the above functions H_i define homotopies. Suppose $H_1(y,C,D) = 0$ for some $(y,C,D) \in \partial\Sigma$. As in the proof of Theorem 3.4, $(x,y) \in \overline{\Omega}$ for all $x \in [0,1]$. If $\lambda = 1$, then y is a solution to our problem, as required, so $\lambda \in [0,1)$. Suppose $C \in \partial\Omega(0)$. Now $y(0) = C$, $y'(0) = A + (1-\lambda)C/\lambda$ while $r'(0, C, A + (1-\lambda)C/\lambda) > 0$, by (3.4) since $r_y(0,C)C > 0$. Thus $(x,y) \notin \overline{\Omega}$ for some $x > 0$, a contradiction. The assumption $D \in \partial\Omega(1)$ leads to a similar contradiction using (3.5).

The rest of the proof follows that of Theorem 3.4. ∎

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