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ON THE STRUCTURE OF FINITELY PRESENTED LATTICES

G. GRATZER, A. P. HUHN AND H. LAKSER

1. Introduction. A lattice L is finitely presented (or presentable) if and only if it can be described with finitely many generators and finitely many relations. Equivalently, L is the lattice freely generated by a finite partial lattice A, in notation, $L = F(A)$. (For more detail, see Section 1.5 of [**6**].)

It is an old "conjecture" of lattice theory that in a finitely presented (or presentable) lattice the elements behave "freely" once we get far enough from the generators.

In this paper we prove a structure theorem that could be said to verify this conjecture.

THEOREM 1. *Let L be a finitely presentable lattice. Then there exists a congruence relation θ such that L/θ is finite and each congruence class is embeddable in a free lattice.*

COROLLARY. *Every finitely presentable lattice L can be written in the form $L = C_1 \cup \ldots \cup C_n$, where the C_i are pairwise disjoint convex sublattices of L and each C_i is a sublattice of a free lattice.*

As an application we shall prove that if L is a finitely presentable lattice, then modularity implies finiteness.

2. Proof of theorem 1. Let X be a finite set and let A be a partial lattice defined on X. We outline the solution to the word problem in $F(A)$, due to R. A. Dean [**3**]; see also H. Lakser [**9**].

Denote by $P(X)$ the algebra of polynomial symbols in the two binary operation symbols $\vee$ and $\wedge$ generated by the set X. Then $F(A)$ is the image of $P(X)$ under a homomorphism ρ_A: $P(X) \to F(A)$ with $x\rho_A = x$ for $x \in X$. We describe the kernel of ρ_A.

For each $p \in P(X)$ we define an ideal p_A and a dual ideal p^A of A. For $p \in X$, $p_A = (p]$ (in A) and $p^A = [p)$,

$$(p \vee q)_A = p_A \vee q_A,$$

and

$$(p \wedge q)_A = p_A \wedge q_A,$$

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the $\vee$ and $\wedge$ on the right hand sides denoting the ideal join and meet in A.

Dually,

$$(p \wedge q)^A = p^A \vee q^A$$

and

$$(p \vee q)^A = p^A \wedge q^A.$$

We define a quasiorder $\leqq_A$ on $P(X)$; $p \leqq_A q$ if and only if it follows from one of the five rules below:

(W_C) $p^A \cap q_A \neq \emptyset$;
$({}_\vee W)$ $p = p_1 \vee p_2, p_1 \leqq_A q$ and $p_2 \leqq_A q$;
$({}_\wedge W)$ $p = p_1 \wedge p_2, p_1 \leqq_A q$ or $p_2 \leqq_A q$;
$(W_\vee)$ $q = q_1 \vee q_2, p \leqq_A q_1$ or $p \leqq_A q_2$;
$(W_\wedge)$ $q = q_1 \wedge q_2, p \leqq_A q_1$ and $p \leqq_A q_2$.

Note that, since A is finite we can compute in the lattice $I(A)$ of ideals of A and in the lattice $D(A)$ of dual ideals of A, and thus $p \leqq_A q$ is decidable.

Finally we define the relation $\sim_A$ on $P(X)$;

$$p \sim_A q \text{ if and only if } p \leqq_A q \text{ and } q \leqq_A p.$$

The solution to the word problem is:

PROPOSITION 1 (R. A. Dean [**3**]). (i) *If $p, q \in P(X)$, then $p\rho_A \leqq q\rho_A$ if and only if $p \leqq_A q$.*

(ii) *If $p, q \in P(X)$, then $p\rho_A = q\rho_A$ if and only if $p \sim_A q$, that is,* $\sim_A = \operatorname{Ker} \rho_A$.

(iii) *If $p \in P(X)$, then*

$$p_A = (p\rho_A] \cap X$$

and

$$p^A = [p\rho_A) \cap X.$$

A converse to Proposition 1(i) is provided by the following proposition.

PROPOSITION 2. *Let $p, q \in P(X)$ and let $p \leqq_A q$.*

(i) *If $p = p_1 \vee p_2$, then $p_1 \leqq_A q$ and $p_2 \leqq_A q$.*

(ii) *If $q = q_1 \wedge q_2$, then $p \leqq_A q_1$ and $p \leqq_A q_2$.*

(iii) *If $p = p_1 \wedge p_2, q = q_1 \vee q_2$, and $p^A \cap q_A = \emptyset$, then*

$$p^A \cap (q_1)_A = p^A \cap (q_2)_A = (p_1)^A \cap q_A = (p_2)^A \cap q_A = \emptyset$$

and either $p_i \leqq_A q$, $i = 1$ or 2, or $p \leqq_A q_i$, $i = 1$ or 2.

Proof. (i) and (ii) follow immediately from Proposition 1(i). As for (iii),

$$\begin{aligned}\emptyset &= p^A \cap q_A = p^A \cap ((q_1)_A \vee (q_2)_A) \supseteq p^A \cap ((q_1)_A \cup (q_2)_A) \\ &= (p^A \cap (q_1)_A) \cup (p^A \cap (q_2)_A),\end{aligned}$$

and similarly for $p_i{}^A \cap q_A$. Finally, since $p \leqq_A q$ does not follow from condition (W_C), it must follow from either condition $({}_\wedge W)$ or condition $(W_\vee)$, which is the final assertion of (iii).

For a pair (I, J) with $I \in I(A)$ and $J \in D(A)$, let

$$P_A(I, J) = \{p \in P(X) | p_A = I \text{ and } p^A = J\},$$

and let

$$C_A(I, J) = (P_A(I, J))\rho_A.$$

PROPOSITION 3. *If $P_A(I, J) \neq \emptyset$, then it is a subalgebra of $P(X)$ and $C_A(I, J)$ is a convex sublattice of $F(A)$. Moreover,*

$$C_A(I, J) = \{a \in F(A) | (a] \cap X = I \textit{ and } [a) \cap X = J\}.$$

Proof. If $p, q \in P_A(I, J)$, then e.g., $(p \vee q)_A = p_A \vee q_A = I \vee I = I$. Thus $P_A(I, J)$ is a subalgebra of $P(X)$ and so $C_A(I, J)$ is a sublattice of $F(A)$.

If $p, q \in P_A(I, J)$ and $p \leqq_A r \leqq_A q$ then, by 1(i) and 1(iii), $I = p_A \leqq r_A \leqq q_A = I$; thus $r_A = I$, and, similarly, $r^A = J$. Consequently, $C_A(I, J)$ is a convex sublattice of $F(A)$.

The last clause of the proposition follows from Proposition 1(iii), concluding the proof of the proposition.

The sublattices of $F(A)$ of the form $C_A(I, J)$ are finite in number, and it can be verified easily that the relation

$$a \equiv b(\theta) \text{ if and only if } (a] \cap X = (b] \cap X \quad \text{and} \quad [a) \cap X = [b) \cap X$$

is a congruence relation on $F(A)$. Thus to complete the proof of Theorem 1 we need only show that each sublattice $C_A(I, J)$ can be embedded in a free lattice. To do this, we first embed $C_A(I, J)$ into the completely free lattice generated by the underlying poset of A. Now this poset can be regarded as a partial lattice B where joins and meets are defined only for comparable elements and then the completely free lattice is $F(B)$. Without too much extra work we can establish a more general embedding theorem.

Let A and B be partial lattices defined on the finite set X. We say that B is *weaker than* A if they have the same underlying poset and any join or meet defined in B is also defined in A and has the same value. Note that under this condition, any ideal (dual ideal) of A is an ideal (dual ideal) of B.

THEOREM 2. *Let A and B be partial lattices on the finite set X. If B is weaker than A, then there is an order-embedding f^*: $F(A) \to F(B)$ with the property that if $I \in I(A)$, $J \in D(A)$, and $C_A(I, J) \neq \emptyset$, then f^* restricted to $C_A(I, J)$ is a lattice embedding.*

Proof. We define a mapping f: $P(X) \to P(X)$. Since $\wedge$ and $\vee$ in $P(X)$ are neither commutative nor associative we first introduce a technical convention: we assign a fixed linear order (completely unrelated to the partial order determined by A and B) to X. The action of f is defined by the following three conditions:

(f_0) If $p \in X$, then $pf = p$;

($f_\vee$) If $p = p_1 \vee p_2$ and $p_A = \{x_1, \ldots, x_n\}$ with $x_1, \ldots, x_n$ listed in the above linear order, then

$$pf = (\ldots((p_1f \vee p_2f) \vee x_1) \vee \ldots) \vee x_n;$$

($f_\wedge$) If $p = p_1 \wedge p_2$ and $p^A = \{y_1, \ldots, y_m\}$ with $y_1, \ldots, y_m$ in the linear order, then

$$pf = (\ldots((p_1f \wedge p_2f) \wedge y_1) \wedge \ldots) \wedge y_m.$$

Note first of all that, for each $p \in P(X)$, $p_B \subseteq p_A$ and $p^B \subseteq p^A$ (as subsets of X). These facts are immediate from the definitions since ideals (dual ideals) of A are ideals (dual ideals) of B; consequently, if I, J are ideals (dual ideals) of A then $I \vee J$ in B is a subset of $I \vee J$ in A.

We now verify some statements.

Statement (1). If $p \in P(X)$, then $p_A = (pf)_B$ and $p^A = (pf)^B$.

The proof is by induction on the rank of p. If $p \in X$, then $p_A = (pf)_A = (pf)_B$ by definition. If $p = p_1 \vee p_2$ and $p_A = \{x_1, \ldots, x_n\}$, then

$$\begin{aligned}(pf)_B &= (p_1f)_B \vee (p_2f)_B \vee (x_1] \vee \ldots \vee (x_n] \\ &= (p_1f)_B \vee (p_2f)_B \vee p_A \\ &= (p_1)_A \vee (p_2)_A \vee p_A \text{ (by induction)} \\ &= p_A\end{aligned}$$

since

$$(p_1)_A, (p_2)_A \subseteq p_A$$

where all ideal joins on the right are in $I(B)$. If $p = p_1 \wedge p_2$ and $p^A = \{x_1, \ldots, x_n\}$, then

$$\begin{aligned}(pf)_B &= (p_1f)_B \cap (p_2f)_B \cap (x_1] \cap \ldots \cap (x_n] \\ &= (p_1)_A \cap (p_2)_A \cap (x_1] \cap \ldots \cap (x_n] \text{ (by induction)} \\ &= p_A \cap (x_1] \cap \ldots \cap (x_n] \\ &= p_A\end{aligned}$$

since $p_A \subseteq (x_i]$ by Proposition 1(iii). The dual argument establishes $p^A = (pf)^B$.

Statement (2). Let $x_1, \ldots, x_m, y_1, \ldots, y_n \in X$, let $p_1, p_2, q_1, q_2 \in P(X)$, and let $p = (\ldots((p_1 \wedge p_2) \wedge x_1) \wedge \ldots) \wedge x_m$, $q = (\ldots((q_1 \vee q_2) \vee y_1) \vee \ldots) \vee y_n$. If $p \leqq_B q$ and $p^B \cap q_B = \emptyset$, then either $p_i \leqq_B q$ for $i = 1$ or 2 or $p \leqq_B q_i$ for $i = 1$ or 2.

The proof proceeds by induction on $m + n$. If $m + n = 0$, then the result follows by Proposition 2(iii). In general, by Proposition 2(iii) one of the following four conditions must hold

(i) $x_m \leqq_B q$;

(ii) $p \leqq_B y_n$;

(iii) $(\ldots((p_1 \wedge p_2) \wedge x_1) \wedge \ldots) \wedge x_{m-1} \leqq_B q$;

(iv) $p \leqq_B (\ldots((q_1 \vee q_2) \vee y_1) \vee \ldots) \vee y_{n-1}$.

(i) and (ii) contradict the hypothesis $p^B \cap q_B = \emptyset$. From (iii) we conclude by induction that either

$$p_i \leqq_B q, \quad i = 1 \text{ or } 2, \quad \text{or}$$

$$(\ldots((p_1 \wedge p_2) \wedge x_1) \wedge \ldots) \wedge x_{m-1} \leqq_B q_i, \qquad i = 1 \text{ or } 2.$$

Since $p \leqq_B (\ldots((p_1 \wedge p_2) \wedge x_1) \wedge \ldots) \wedge x_{m-1}$ we conclude, in the latter case, that $p \leqq_B q_i$, $i = 1$ or 2. If condition (iv) holds we proceed in a similar manner.

Statement (3). Let $p, q \in P(X)$. Then $p \leqq_A q$ if and only if $pf \leqq_B qf$.

First assume that $p \leqq_A q$. If $p \leqq_A q$ by condition (W_C), that is, if $p^A \cap q_A \neq \emptyset$, then, by Statement (1), $(pf)^B \cap (qf)_B \neq \emptyset$, that is, $pf \leqq_B qf$. Otherwise, we proceed by induction on the sum of the ranks of p and q. If $p \leqq_A q$ by condition $({}_\vee W)$, that is, if $p = p_1 \vee p_2$ and $p_1 \leqq_A q$, $p_2 \leqq_A q$, then $p_1 f \leqq_B qf$ and $p_2 f \leqq_B qf$ and, if $x \in p_A$, then $x \leqq_A p \leqq_A q$ and so $x = xf \leqq_B qf$. Thus $pf \leqq_B qf$ by successive applications of $({}_\vee W)$.

If $p \leqq_A q$ follows by $({}_\wedge W)$, that is, if $p = p_1 \wedge p_2$ and say, $p_1 \leqq_A q$, then $p_1 f \leqq_B qf$ and so $pf \leqq_B qf$ by successive applications of $({}_\wedge W)$.

If $p \leqq_A q$ follows by condition $(W_\wedge)$ $(W_\vee)$, the situation is the dual of the above. We thus conclude that $p \leqq_A q$ implies $pf \leqq_B qf$.

Now assume that $pf \leqq_B qf$. If $(pf)^B \cap (qf) \neq \emptyset$ then, by Statement (1), $p^A \cap q_A \neq \emptyset$ and so $p \leqq_A q$. We can thus assume that $(pf)^B \cap (qf)_B = \emptyset$. If $p = p_1 \vee p_2$, then $pf \leqq_B qf$ implies that $p_1 f \leqq_B qf$ and $p_2 f \leqq_B qf$.

By induction on the sum of the ranks of p and q we conclude that $p_1 \leqq_A q$ and $p_2 \leqq_A q$, that is, $p \leqq_A q$ by condition $({}_\vee W)$.

The dual situation obtains if $q = q_1 \wedge q_2$. We are thus left only with the case $p = p_1 \wedge p_2$ and $q = q_1 \vee q_2$. By $(f_\vee)$ and $(f_\wedge)$,

$$pf = (\ldots((p_1 f \wedge p_2 f) \wedge x_1) \wedge \ldots) \wedge x_m$$

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and

$$qf = (\ldots((q_1f \vee q_2f) \vee y_1) \vee \ldots) \vee y_n,$$

where $p^A = \{x_1, \ldots, x_m\}$ and $q_A = \{y_1, \ldots, y_n\}$. Since $(pf)^B \cap (qf)_B = \emptyset$, we apply Statement (2) and conclude that $p_if \leqq_B qf$, $i = 1$ or 2 or $pf \leqq_B q_if$, $i = 1$ or 2. Again, by induction on the sum of the ranks of p and q, we conclude that $p_i \leqq_A q$, $i = 1$ or 2, or $p \leqq_A q_i$, $i = 1$ or 2, that is, that $p \leqq_A q$. Thus Statement (3) has been proved.

From Statement (3) we get a one-to-one isotone map $f^*: F(A) \to F(B)$ by setting

$$p\rho_A f^* = pf\rho_B.$$

Statement (4). *If* $p, q \in P_A(I, J)$, then

$$(p \vee q)f \sim_B pf \vee qf$$

and

$$(p \wedge q)f \sim_B pf \wedge qf.$$

Since

$$(p \vee q)f = (\ldots((pf \vee qf) \vee x_1) \vee \ldots) \vee x_n,$$

where $I = \{x_1, \ldots, x_n\}$, we conclude immediately that

$$pf \vee qf \leqq_B (p \vee q)f.$$

If $x \in I = p_A$, then, by Statement (1), $x \in (pf)_B$, that is, $x \leqq_B pf$. Thus $(p \vee q)f \leqq_B pf \vee qf$, proving that $(p \vee q)f \sim_B pf \vee qf$. Dually,

$$(p \wedge q)f \sim_B pf \wedge qf,$$

concluding the proof of Statement (4).

From Statement (4), f^* restricted to $C_A(I, J)$ is a lattice embedding, concluding the proof of Theorem 2.

To prove Theorem 1, let B be the partial lattice structure defined on X with joins and meets defined only for comparable elements. Obviously, B is weaker than A. By Theorem 2, we conclude that each $C_A(I, J)$ is isomorphic to a sublattice of the lattice $F(B)$. By a result of [**2**], $F(B)$ is isomorphic to a sublattice of a free lattice, concluding the proof of Thoerem 1.

3. Some comments. There are two key ideas in this paper: the "canonical" decomposition of $F(A)$ (which appears to be new) and the (Scholl's) normal form pf for a polynomial p. The latter has some history behind it. In [**7**], a map pf is defined which gives the join support with lower cover. This is used to provide a short proof of Sorkin's theorem on isotone maps. A more refined version of this idea appears in [**1**] to show that a free product has an isotone embedding in a $CF(P)$; this they utilize to investigate chain conditions in free products. Finally, the "smooth representation" [**8**] is almost identical with pf; in [**8**] this is used to investigate common refinement properties of amalgamated free products.

A sublattice of a free lattice has many nice properties: $(SD_\wedge)$, $(SD_\vee)$, (W) (for notation and historic references, see [**6**]), every element is a join of join-irreducibles, every element is a meet of meet-irreducibles, every chain is countable, etc.

Thus Theorem 1 yields a very powerful decomposition. Here is one illustration.

Let L be a modular lattice having a decomposition $C_1 \cup \ldots \cup C_m$ into sublattices satisfying $(SD_\wedge)$. Then all C_i are distributive. This property of L (having a decomposition into the union of finitely many distributive sublattices) is preserved under the formation of sublattices and homomorphic images.

We use this to conclude a result of [**4**]: $F_M(4)$ (the free modular lattice on 4 generators) is not finitely presentable.

Indeed, if $F_M(4)$ is finitely presented, then it has a decomposition $C_1 \cup \ldots \cup C_n$ into distributive sublattices. But $F_M(4)$ has M_{3n} as a sublattice of a quotient (namely, the rational projective plane). Hence we obtain an M_3 in some C_i, contradicting that it is distributive.

In fact, we can prove more:

THEOREM 3. *Let L be a finitely presented lattice. If L is modular, then L is finite.*

Proof. We proceed as before, and decompose the lattice:

$$L = C_1 \cup \ldots \cup C_n,$$

where each C_i is a distributive sublattice of a free lattice. By a result of [**5**], each C_i has width $\leqq 3$, hence the width of $L \leqq 3n$, which is finite. By [**10**], a finitely generated modular lattice of finite width is finite, hence L is finite, concluding the proof of Theorem 3.

The structure theorem leads naturally to the following problem: which sublattices of a free lattice can occur in a representation of a finitely presented lattice?

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University of Manitoba,
Winnipeg, Manitoba

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