A NOTE ON PRODUCTS OF NORMAL SUBGROUPS

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By a <u>group theoretic class</u> we mean a class of groups which contains the trivial group, denoted E, and any group isomorphic to a group in the class. Let $\mathfrak{X}$ be a group theoretic class. Following P. Hall [4, p. 533], we define $E\mathfrak{X}$, $C\mathfrak{X}$, $S\mathfrak{X}$, $Q\mathfrak{X}$, and $N_0\mathfrak{X}$ to be the (group theoretic) classes consisting of extensions of $\mathfrak{X}$ groups by $\mathfrak{X}$ groups, cartesian products of $\mathfrak{X}$ groups, subgroups of $\mathfrak{X}$ groups, homorphic images of $\mathfrak{X}$ groups and products of two normal $\mathfrak{X}$ subgroups of a group, respectively. If T is one of the above operations on classes of groups and $T\mathfrak{X} = \mathfrak{X}$, we say X is T <u>closed</u>.

In this note we exhibit some well known classes of groups that are not N_0 closed.

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The first result is an application of a lemma in [1, p. 145], which states that if K is a split extension of a group L by an abelian group A, then K is contained isomorphically in a group K* which is the product of two normal subgroups isomorphic to the restricted (standard) wreath product LwrA.

THEOREM 1. <u>The class of SI* groups is not N_0 closed</u>.

<u>Proof</u>. Following P. Hall [3, p. 350], let Λ be the ring of dyadic rationals $2^{-n}m$, where n and m run through the integers, V a vector space with basis v_λ, $\lambda \in \Lambda$, over the field of p elements, and $x_{\lambda\mu}$, with $\lambda < \mu$ a pair in Λ, the linear transformation defined by

$$x_{\lambda\mu} : v_\lambda \to v_\lambda + v_\mu, \; v_\rho \to v_\rho \; (\rho \neq \lambda);$$

let M be the group generated by all $x_{\lambda\mu}$. Let t and u be linear transformations of V defined by

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$$t : v_\lambda \to v_{2\lambda} \quad \text{and} \quad u : v_\lambda \to v_{\lambda+1},$$

and let H be the group generated by t and u. Let $G = \langle M, H \rangle$. Hall shows that $G = HM$, $H \cap M = 1$ and M is a minimal normal subgroup of G. He also shows that $t^{-1}ut = u^2$. Hence, $G = \langle t \rangle \langle u \rangle M$.

D.H. McLain shows [5, p. 641] that M is SI*. But G is not even SI (See [2, p. 351]). If the group $\langle u \rangle M$ is SI*, then G is a split extension of $\langle u \rangle M$ by the cyclic group $\langle t \rangle$. If $\langle u \rangle M$ is not SI*, then $\langle u \rangle M$ is a split extension of the SI* group M by the cyclic group $\langle u \rangle$. In either case, we obtain a non SI* group K which is a split extension of an SI* group L by a cyclic group A. By the lemma mentioned above it follows K is isomorphic to a subgroup of K*, a product of normal subgroups isomorphic to LwrA. It is easily verified that the restricted wreath product of two SI* groups is SI*. But the product of these two normal SI* subgroups is not SI*, since subgroups of SI* groups are SI*, and it follows that SI* is not N_0 closed.

Denote the class of all groups by $\mathfrak{O}$ and the class consisting of trivial groups by $\mathfrak{E}$. The next theorem shows that these are the only N_0 closed varieties of groups. (See [6] for notation.)

THEOREM 2. <u>The only</u> C, S, Q <u>and</u> N_0 <u>closed classes of groups are</u> $\mathfrak{E}$ <u>and</u> $\mathfrak{O}$.

<u>Proof</u>. Let the class $\mathfrak{X}$ be C, S, Q and N_0 closed. Then $\mathfrak{X}$ is a variety by Theorem 15.51 of [6, p. 17]. Therefore, if the group G is generated by its normal $\mathfrak{X}$ subgroups, then G is an $\mathfrak{X}$ group. For any law that is violated in G is violated in a subgroup which is a product of a finite number of normal subgroups, which must be an $\mathfrak{X}$ group since $\mathfrak{X}$ is N_0 closed. By a theorem of Wielandt [8, p. 246], any group generated by subnormal $\mathfrak{X}$ subgroups (A is subnormal in G if there is a finite normal series from A to G) is also an $\mathfrak{X}$ group.

Now suppose that $\mathfrak{X}$ is not $\mathfrak{E}$ or $\mathfrak{O}$. If $\mathfrak{X}$ has an exponent law, there is a least positive integer $m \neq 1$ such that $x^m = 1$ is a law of $\mathfrak{X}$. Let p^r be the largest power of a prime p dividing m, where $r > 0$. Let J_t denote a cyclic group of order t. Then $J_{p^r} \in \mathfrak{X}$ and $J_{p^{r+1}} \notin \mathfrak{X}$. The standard wreath product $G = J_{p^r} \mathrm{wr}\, J_{p^r}$ is nilpotent, so every cyclic subgroup of G is subnormal in G. But G is generated by two elements of order p^r, so $G \in \mathfrak{X}$. However, G has an element of order p^{r+1}, [7, p. 99], contradicting the fact that $J_{p^{r+1}} \notin \mathfrak{X}$. So $\mathfrak{X}$ has no nontrivial exponent laws.

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Therefore, if J denotes an infinite cyclic group, $J \in \mathfrak{X}$. Let F be a nonabelian free group, $\mathfrak{N}_n$ the variety of groups nilpotent of class at most n and $F(\mathfrak{N}_n)$ the free $\mathfrak{N}_n$ group defined by F. Then $F(\mathfrak{N}_n)$ is generated by subnormal infinite cyclic groups, so $F(\mathfrak{N}_n)$ is an $\mathfrak{X}$ group. Letting F^m denote the m th term in the descending central series of F, we have $\bigcap_{m=1}^{\infty} F^m = E$ by Magnus' Theorem. Hence F is contained isomorphically in the cartesian product $\prod_{m=1}^{\infty} F(\mathfrak{N}_m)$, which is an $\mathfrak{X}$ group. So F is an $\mathfrak{X}$ group and the only law of $\mathfrak{X}$ is the empty law; therefore, $\mathfrak{X} = \mathcal{O}$, a contradiction.

COROLLARY. The only E, C, S and Q closed group theoretic classes are $\mathcal{E}$ and $\mathcal{O}$.

Proof. For any class of groups, E and Q closure imply N_o closure.

Remark. None of the conditions of Theorem 2 or its corollary may be omitted. Most possibilities are easily ruled out. We leave it to the reader to verify that if $\mathfrak{X}$ is the class of groups for which every nontrivial subgroup has a proper commutator subgroup, then $\mathfrak{X}$ is S, C, N and E closed but not Q closed. Also, the class of groups for which every non-trivial factor group has nontrivial commutator subgroup is C, Q, N and E closed but not S closed. It may be possible to obtain other interesting combinations of Hall closure operations (See [4] for notation) which yield results similar to Theorem 2.

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