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ON SOME SUM–TO–PRODUCT IDENTITIES

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To George Szekeres on his ninetieth birthday

We give new proofs of some sum–to–product identities due to Blecksmith, Brillhart and Gerst, as well as some other such identities found recently by us.

1. INTRODUCTION AND STATEMENT OF RESULTS

In the first two of a sequence of papers, Blecksmith, Brillhart and Gerst [**2, 3**] give five pairs of simple and beautiful sum–to–product identities, Theorems 2–6 below. We give another such in Theorem 1. We do several new things — we prove Theorem 1 (Section 2), we show that Theorem 2 follows from Theorem 1 (Section 3), we present new proofs of Theorems 3 and 4 (Section 4) and we show that Theorems 5 and 6 follow from Theorems 3 and 4 (Section 5).

In order to state our results, we must explain some by now fairly standard notation.

$$(a;q)_\infty = (1-a)(1-aq)(1-aq^2)\ \cdots\ ,$$
$$(a_1,a_2,\ \ldots\ a_n;q)_\infty = (a_1;q)_\infty(a_2;q)_\infty\ \cdots\ (a_n;q)_\infty,$$
$$\begin{pmatrix} a_1,a_2,\ \ldots\ ,a_m \\ b_1,b_2,\ \ldots\ ,b_n \end{pmatrix}_\infty = \frac{(a_1,a_2,\ \ldots\ ,a_m;q)_\infty}{(b_1,b_2,\ \ldots\ ,b_n;q)_\infty},$$
$$(q)_\infty = (q;q)_\infty.$$

We make use of Jacobi's triple product identity [**1,** (2.2.10)]

$$(aq,a^{-1},q;q)_\infty = \sum_{-\infty}^{\infty}(-1)^n a^n q^{(n^2+n)/2}$$

and the quintuple product identity [**4, 5**]

$$\begin{pmatrix} a^2q,a^{-2},q \\ aq,a^{-1} \end{pmatrix}_\infty = \sum_{-\infty}^{\infty}(-1)^n a^{3n} q^{(3n^2+n)/2} + \sum_{-\infty}^{\infty}(-1)^n a^{3n-1} q^{(3n^2-n)/2}.$$

The theorems referred to above are:

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THEOREM 1.

$$\sum_{-\infty}^{\infty} q^{n^2} + \sum_{-\infty}^{\infty} q^{2n^2} = 2\left(\begin{matrix} q^3, q^5, q^8 \\ q, q^4, q^7 \end{matrix}; q^8\right)_\infty,$$

$$\sum_{-\infty}^{\infty} q^{n^2} - \sum_{-\infty}^{\infty} q^{2n^2} = 2q\left(\begin{matrix} q, q^7, q^8 \\ q^3, q^4, q^5 \end{matrix}; q^8\right)_\infty.$$

THEOREM 2. [2, Theorem 1]

$$\sum_{-\infty}^{\infty} (-1)^n q^{2n^2} + \sum_{-\infty}^{\infty} (-1)^n q^{n^2} = 2(q)_\infty/(q^4, q^6, q^8, q^{10}, q^{22}, q^{24}, q^{26}, q^{28}; q^{32})_\infty,$$

$$\sum_{-\infty}^{\infty} (-1)^n q^{2n^2} - \sum_{-\infty}^{\infty} (-1)^n q^{n^2} = 2q(q)_\infty/(q^2, q^8, q^{12}, q^{14}, q^{18}, q^{20}, q^{24}, q^{30}; q^{32})_\infty.$$

THEOREM 3. [2, Theorem 3]

$$\sum_{-\infty}^{\infty} q^{n^2} + \sum_{-\infty}^{\infty} q^{3n^2} = 2\left(\begin{matrix} q^2, q^6, q^{10}, q^{12} \\ q, q^3, q^9, q^{11} \end{matrix}; q^{12}\right)_\infty,$$

$$\sum_{-\infty}^{\infty} q^{n^2} - \sum_{-\infty}^{\infty} q^{3n^2} = 2q\left(\begin{matrix} q^2, q^6, q^{10}, q^{12} \\ q^3, q^5, q^7, q^9 \end{matrix}; q^{12}\right)_\infty.$$

THEOREM 4. [3, Theorem 3]

$$\sum_{-\infty}^{\infty} q^{n^2} + \sum_{-\infty}^{\infty} q^{5n^2} = 2\left(\begin{matrix} q^2, q^8, q^{10}, q^{12}, q^{18}, q^{20} \\ q, q^4, q^9, q^{11}, q^{16}, q^{19} \end{matrix}; q^{20}\right)_\infty,$$

$$\sum_{-\infty}^{\infty} q^{n^2} - \sum_{-\infty}^{\infty} q^{5n^2} = 2q\left(\begin{matrix} q^4, q^6, q^{10}, q^{14}, q^{16}, q^{20} \\ q^3, q^7, q^8, q^{12}, q^{13}, q^{17} \end{matrix}; q^{20}\right)_\infty.$$

THEOREM 5. [3, Theorem 1]

$$\sum_{-\infty}^{\infty} q^{2n(n+1)} + \sum_{-\infty}^{\infty} q^{6n(n+1)+1} = 2\left(\begin{matrix} q^2, q^5, q^7, q^{12}, q^{17}, q^{19}, q^{22}, q^{24} \\ q, q^4, q^6, q^{11}, q^{13}, q^{18}, q^{20}, q^{23} \end{matrix}; q^{24}\right)_\infty,$$

$$\sum_{-\infty}^{\infty} q^{2n(n+1)} - \sum_{-\infty}^{\infty} q^{6n(n+1)+1} = 2\left(\begin{matrix} q, q^{10}, q^{11}, q^{12}, q^{13}, q^{14}, q^{23}, q^{24} \\ q^4, q^5, q^6, q^7, q^{17}, q^{18}, q^{19}, q^{20} \end{matrix}; q^{24}\right)_\infty.$$

and

THEOREM 6. [3, Theorem 4]

$$\sum_{-\infty}^{\infty} q^{n(n+1)} + \sum_{-\infty}^{\infty} q^{5n(n+1)+1} = 2\left(\begin{matrix} q^3, q^7, q^{10}, q^{13}, q^{17}, q^{20} \\ q, q^6, q^9, q^{11}, q^{14}, q^{19} \end{matrix}; q^{20}\right)_\infty,$$

$$\sum_{-\infty}^{\infty} q^{n(n+1)} - \sum_{-\infty}^{\infty} q^{5n(n+1)+1} = 2\left(\begin{matrix} q, q^9, q^{10}, q^{11}, q^{19}, q^{20} \\ q^2, q^3, q^7, q^{13}, q^{17}, q^{18} \end{matrix}; q^{20}\right)_\infty.$$

2. Proof of Theorem 1

We shall start by obtaining the 2-dissections

$$(1)\qquad \begin{aligned}(q^3,q^5,q^8;q^8)_\infty^2 &= \sum_{u,v=-\infty}^{\infty} q^{8u^2-2u+8v^2} - q^3 \sum_{u,v=-\infty}^{\infty} q^{8u^2+6u+8v^2+8v},\\ (q,q^7,q^8;q^8)_\infty^2 &= \sum_{u,v=-\infty}^{\infty} q^{8u^2-6u+8v^2} - q \sum_{u,v=-\infty}^{\infty} q^{8u^2+2u+8v^2+8v}.\end{aligned}$$

We have

$$\begin{aligned}(q^3,q^5,q^8;q^8)_\infty^2 &= \left\{\sum_{-\infty}^{\infty}(-1)^n q^{4n^2-n}\right\}^2\\ &= \sum_{m,n=-\infty}^{\infty} (-1)^{m+n} q^{4m^2-m+4n^2-n}\\ &= \sum_{m\equiv n \pmod 2} q^{4m^2-m+4n^2-n} - \sum_{m\not\equiv n \pmod 2} q^{4m^2-m+4n^2-n}\\ &= \sum_{u,v=-\infty}^{\infty} q^{4(u+v)^2-(u+v)+4(u-v)^2-4(u-v)}\\ &\qquad - \sum_{u,v=-\infty}^{\infty} q^{4(u+v+1)^2-4(u+v+1)+4(u-v)^2-(u-v)}\\ &= \sum_{u,v=-\infty}^{\infty} q^{8u^2-2u+8v^2} - q^3 \sum_{u,v=-\infty}^{\infty} q^{8u^2+6u+8v^2+8v}\end{aligned}$$

as claimed. The other identity is proved similarly.

It follows that

$$(2)\qquad \begin{aligned}(q^3,q^5,q^8;q^8)_\infty^2 + q(q,q^7,q^8;q^8)_\infty^2 &= \frac{(q^2)_\infty^4}{(q)_\infty (q^4)_\infty},\\ (q^3,q^5,q^8;q^8)_\infty^2 - q(q,q^7,q^8;q^8)_\infty^2 &= \frac{(q)_\infty (q^4)_\infty^6}{(q^2)_\infty^3 (q^8)_\infty^2}.\end{aligned}$$

For,

$$\begin{aligned}&(q^3,q^5,q^8;q^8)_\infty^2 + q(q,q^7,q^8;q^8)_\infty^2\\ &\quad= \sum_{u,v=-\infty}^{\infty} q^{8u^2-2u+8v^2} - q^2 \sum_{u,v=-\infty}^{\infty} q^{8u^2+2u+8v^2+8v}\end{aligned}$$

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$$+q\sum_{u,v=-\infty}^{\infty} q^{8u^2-6u+8v^2} - q^3 \sum_{u,v=-\infty}^{\infty} q^{8u^2+6u+8v^2+8v}$$

$$= \sum_{-\infty}^{\infty} q^{8u^2-2u}\left\{\sum_{-\infty}^{\infty} q^{8v^2} - \sum_{-\infty}^{\infty} q^{8v^2+8v+2}\right\}$$

$$+ q\sum_{-\infty}^{\infty} q^{8u^2-6u}\left\{\sum_{-\infty}^{\infty} q^{8v^2} - \sum_{-\infty}^{\infty} q^{8v^2+8v+2}\right\}$$

$$= \sum_{-\infty}^{\infty} (-1)^v q^{2v^2}\left\{\sum_{-\infty}^{\infty} q^{8u^2-2u} + \sum_{-\infty}^{\infty} q^{8u^2-6u+1}\right\}$$

$$= \sum_{-\infty}^{\infty} (-1)^v q^{2v^2} \sum_{-\infty}^{\infty} q^{2u^2-u}$$

$$= \prod_{n\geqslant 1} \frac{1-q^{2n}}{1+q^{2n}} \prod_{n\geqslant 1} \frac{(1-q^{2n})^2}{1-q^n}$$

$$= \frac{(q^2)_\infty^4}{(q)_\infty (q^4)_\infty}$$

as claimed, and the other identity is proved similarly.

If we divide (2) by

$$(q,q^3,q^4,q^5,q^7,q^8;q^8)_\infty = (q,q^3,q^4;q^4)_\infty = \frac{(q)_\infty (q^4)_\infty}{(q^2)_\infty}$$

we obtain

$$\text{(3)} \qquad \begin{pmatrix} q^3,q^5,q^8 \\ q,\ q^4,q^7 \end{pmatrix};q^8\Big)_\infty + q\begin{pmatrix} q,\ q^7,q^8 \\ q^3,q^4,q^5 \end{pmatrix};q^8\Big)_\infty = \frac{(q^2)_\infty^5}{(q)_\infty^2 (q^4)_\infty^2} = \sum_{-\infty}^{\infty} q^{n^2}$$

and

$$\begin{pmatrix} q^3,q^5,q^8 \\ q,\ q^4,q^7 \end{pmatrix};q^8\Big)_\infty - q\begin{pmatrix} q,\ q^7,q^8 \\ q^3,q^4,q^5 \end{pmatrix};q^8\Big)_\infty = \frac{(q^4)_\infty^5}{(q^2)_\infty^2 (q^8)_\infty^2} = \sum_{-\infty}^{\infty} q^{2n^2}.$$

Theorem 1 follows. □

3. Proof of Theorem 2

We start by proving

Corollary to Theorem 1.

$$\text{(4)} \qquad \begin{pmatrix} q^3,q^5,q^8 \\ q,\ q^4,q^7 \end{pmatrix};q^8\Big)_\infty = \begin{pmatrix} q^6,q^{10},q^{16} \\ q^2,q^8,\ q^{14} \end{pmatrix};q^{16}\Big)_\infty + q\begin{pmatrix} q^{16} \\ q^8 \end{pmatrix};q^{16}\Big)_\infty,$$

$$\begin{pmatrix} q,\ q^7,q^8 \\ q^3,q^4,q^5 \end{pmatrix};q^8\Big)_\infty = \begin{pmatrix} q^{16} \\ q^8 \end{pmatrix};q^{16}\Big)_\infty - q\begin{pmatrix} q^2,q^{14},q^{16} \\ q^6,q^8,\ q^{10} \end{pmatrix};q^{16}\Big)_\infty.$$

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PROOF: We have

$$2\begin{pmatrix} q^3, q^5, q^8 \\ q,\ q^4, q^7 \end{pmatrix}; q^8\Big)_\infty = \sum_{-\infty}^{\infty} q^{n^2} + \sum_{-\infty}^{\infty} q^{2n^2}$$
$$= \sum_{-\infty}^{\infty} q^{4n^2} + \sum_{-\infty}^{\infty} q^{(2n+1)^2} + \sum_{-\infty}^{\infty} q^{2n^2}$$
$$= \left\{\sum_{-\infty}^{\infty} q^{2n^2} + \sum_{-\infty}^{\infty} q^{4n^2}\right\} + 2q\sum_{n\geqslant 0} q^{4n^2+4n}$$
$$= 2\left(\begin{matrix} q^6, q^{10}, q^{16} \\ q^2, q^8,\ q^{14} \end{matrix}; q^{16}\right)_\infty + 2q\left(\begin{matrix} q^{16} \\ q^8 \end{matrix}; q^{16}\right)_\infty$$

as claimed. The other identity is proved similarly. □

Next we show that

(5) $$(q)_\infty = (q^2, q^{12}, q^{14}, q^{16}, q^{18}, q^{20}, q^{30}, q^{32}; q^{32})_\infty - q(q^4, q^6, q^{10}, q^{16}, q^{22}, q^{26}, q^{28}, q^{32}; q^{32})_\infty.$$

We have

$$(q)_\infty = \sum_{-\infty}^{\infty} (-1)^n q^{(3n^2-n)/2}$$
$$= \sum_{-\infty}^{\infty} q^{(3(4n)^2-(4n))/2} - \sum_{-\infty}^{\infty} q^{(3(4n-1)^2-(4n-1))/2} - \sum_{-\infty}^{\infty} q^{(3(4n+1)^2-(4n+1))/2} + \sum_{-\infty}^{\infty} q^{(3(4n+2)^2-(4n+2))/2}$$
$$= \left\{\sum_{-\infty}^{\infty} q^{24n^2-2n} - \sum_{-\infty}^{\infty} q^{24n^2-14n+2}\right\} - \left\{\sum_{-\infty}^{\infty} q^{24n^2+10n+1} - \sum_{-\infty}^{\infty} q^{24n^2+22n+5}\right\}$$
$$= \left(\begin{matrix} q^4, q^{12}, q^{16} \\ -q^2, -q^{14} \end{matrix}; q^{16}\right)_\infty - q\left(\begin{matrix} q^4, q^{12}, q^{16} \\ -q^6, -q^{10} \end{matrix}; q^{16}\right)_\infty$$
$$= (q^2, q^{12}, q^{14}, q^{16}, q^{18}, q^{20}, q^{30}, q^{32}; q^{32})_\infty - q(q^4, q^6, q^{10}, q^{16}, q^{22}, q^{26}, q^{28}, q^{32}; q^{32})_\infty.$$

PROOF OF THEOREM 2: From (5) and (4),

$$2(q)_\infty/(q^4, q^6, q^8, q^{10}, q^{22}, q^{24}, q^{26}, q^{28}; q^{32})_\infty$$
$$= 2\left(\begin{matrix} q^2, q^{12}, q^{14}, q^{16}, q^{18}, q^{20}, q^{30}, q^{32} \\ q^4, q^6,\ q^8,\ q^{10}, q^{22}, q^{24}, q^{26}, q^{28} \end{matrix}; q^{32}\right)_\infty - 2q\left(\begin{matrix} q^{16}, q^{32} \\ q^8,\ q^{24} \end{matrix}; q^{32}\right)_\infty$$

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$$= 2\begin{pmatrix} q^{12}, q^{20} \\ q^4, \ q^{28} \end{pmatrix}; q^{32}\Big)_\infty \begin{pmatrix} q^2, q^{14}, q^{16} \\ q^6, q^8, \ q^{10} \end{pmatrix}; q^{16}\Big)_\infty - 2q\begin{pmatrix} q^{16} \\ q^8 \end{pmatrix}; q^{16}\Big)_\infty$$

$$= 2\begin{pmatrix} q^{12}, q^{20} \\ q^4, \ q^{28} \end{pmatrix}; q^{32}\Big)_\infty \left\{ \begin{pmatrix} q^{32} \\ q^{16} \end{pmatrix}; q^{32}\Big)_\infty - q^2 \begin{pmatrix} q^4, \ q^{28}, q^{32} \\ q^{12}, q^{16}, q^{20} \end{pmatrix}; q^{32}\Big)_\infty \right\} - 2q\begin{pmatrix} q^{16} \\ q^8 \end{pmatrix}; q^{16}\Big)_\infty$$

$$= 2\begin{pmatrix} q^{12}, q^{20}, q^{32} \\ q^4, \ q^{16}, q^{28} \end{pmatrix}; q^{32}\Big)_\infty - 2q^2\begin{pmatrix} q^{32} \\ q^{16} \end{pmatrix}; q^{32}\Big)_\infty - 2q\begin{pmatrix} q^{16} \\ q^8 \end{pmatrix}; q^{16}\Big)_\infty$$

$$= \left\{ \sum_{-\infty}^{\infty} q^{4n^2} + \sum_{-\infty}^{\infty} q^{8n^2} \right\} - \sum_{-\infty}^{\infty} q^{2(2n+1)^2} - \sum_{-\infty}^{\infty} q^{(2n+1)^2}$$

$$= \left\{ \sum_{-\infty}^{\infty} q^{4n^2} - \sum_{-\infty}^{\infty} q^{(2n+1)^2} \right\} + \left\{ \sum_{-\infty}^{\infty} q^{8n^2} - \sum_{-\infty}^{\infty} q^{2(2n+1)^2} \right\}$$

$$= \sum_{-\infty}^{\infty} (-1)^n q^{n^2} + \sum_{-\infty}^{\infty} (-1)^n q^{2n^2}.$$

as claimed. The other identity is proved in similar fashion. ∎

4. Proofs of Theorems 3 and 4

Proof of Theorem 3: We start by obtaining the 3-dissections

(6)

$$\begin{pmatrix} q^2, q^6, q^{10}, q^{12} \\ q, \ q^3, q^9, \ q^{11} \end{pmatrix}; q^{12}\Big)_\infty = \begin{pmatrix} q^6, q^{18}, q^{30}, q^{36} \\ q^3, q^9, \ q^{27}, q^{33} \end{pmatrix}; q^{36}\Big)_\infty + q\begin{pmatrix} q^6, q^{18}, q^{30}, q^{36} \\ q^3, q^{15}, q^{21}, q^{33} \end{pmatrix}; q^{36}\Big)_\infty,$$

$$\begin{pmatrix} q^2, q^6, q^{10}, q^{12} \\ q^3, q^5, q^7, \ q^9 \end{pmatrix}; q^{12}\Big)_\infty = \begin{pmatrix} q^6, q^{18}, q^{30}, q^{36} \\ q^3, q^{15}, q^{21}, q^{33} \end{pmatrix}; q^{36}\Big)_\infty - q^2\begin{pmatrix} q^6, q^{18}, q^{30}, q^{36} \\ q^9, q^{15}, q^{21}, q^{27} \end{pmatrix}; q^{36}\Big)_\infty$$

We have

$$\begin{pmatrix} q^2, q^6, q^{10}, q^{12} \\ q, \ q^3, q^9, \ q^{11} \end{pmatrix}; q^{12}\Big)_\infty = \begin{pmatrix} q^6 \\ q^3, q^9 \end{pmatrix}; q^{12}\Big)_\infty \begin{pmatrix} q^2, q^{10}, q^{12}, q^{14}, q^{22}, q^{24} \\ q, \ q^{11}, q^{13}, q^{23} \end{pmatrix}; q^{24}\Big)_\infty$$

$$= \frac{(-q^3; q^6)_\infty}{(q^{24})_\infty} \prod_{n\geqslant 1} (1 + q^{12n-11})(1 + q^{12n-1})(1 - q^{12n}) \times \prod_{n\geqslant 1} (1 - q^{24n-14})(1 - q^{24n-10})(1 - q^{24n})$$

$$= \frac{(-q^3; q^6)_\infty}{(q^{24})_\infty} \sum_{-\infty}^{\infty} q^{6n^2-5n} \sum_{-\infty}^{\infty} (-1)^n q^{12n^2-2n}$$

$$= \frac{(-q^3; q^6)_\infty}{(q^{24})_\infty} \sum_{m,n=-\infty}^{\infty} (-1)^n q^{6m^2-5m+12n^2-2n}$$

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We now split the sum into three, according to the residue modulo 3 of $m+n$. If $m+n \equiv 0 \pmod 3$, $m-2n \equiv 0$, let $s=(m+n)/3$, $t=(m-2n)/3$, $m=2s+t$, $n=s-t$;
if $m+n\equiv 1$, $s=(m+n-1)/3$, $t=(m-2n-1)/3$, $m=2s+t+1$, $n=s-t$;
if $m+n\equiv -1$, $s=(m+n+1)/3$, $t=(m-2n+1)/3$, $m=2s+t-1$, $n=s-t$.
We obtain

$$\begin{aligned}
&\left(\begin{matrix} q^2, q^6, q^{10}, q^{12} \\ q, q^3, q^9, q^{11} \end{matrix}; q^{12}\right)_\infty \\
&= \frac{(-q^3;q^6)_\infty}{(q^{24})_\infty}\left\{\sum_{s,t=-\infty}^{\infty}(-1)^{s-t}q^{6(2s+t)^2-5(2s+t)+12(s-t)^2-2(s-t)}\right. \\
&\qquad + \sum_{s,t=-\infty}^{\infty}(-1)^{s-t}q^{6(2s+t+1)^2-5(2s+t+1)+12(s-t)^2-2(s-t)} \\
&\qquad \left. + \sum_{s,t=-\infty}^{\infty}(-1)^{s-t}q^{6(2s+t-1)^2-5(2s+t-1)+12(s-t)^2-2(s-t)}\right\} \\
&= \frac{(-q^3;q^6)_\infty}{(q^{24})_\infty}\left\{\sum_{s,t=-\infty}^{\infty}(-1)^{s+t}q^{36s^2-12s+18t^2-3t}\right. \\
&\qquad + \sum_{s,t=-\infty}^{\infty}(-1)^{s+t}q^{36s^2+12s+18t^2+9t+1} \\
&\qquad \left. + \sum_{s,t=-\infty}^{\infty}(-1)^{s+t}q^{36s^2-36s+18t^2-15t+11}\right\} \\
&= (-q^3;q^6)_\infty\left\{(q^{15},q^{21},q^{36};q^{36})_\infty + q(q^9,q^{27},q^{36};q^{36})_\infty\right\} \\
&= \left(\begin{matrix} q^6, q^{18}, q^{30} \\ q^3, q^9, q^{15}, q^{21}, q^{27}, q^{33} \end{matrix}; q^{36}\right)_\infty \left\{(q^{15},q^{21},q^{36};q^{36})_\infty + q(q^9,q^{27},q^{36};q^{36})_\infty\right\} \\
&= \left(\begin{matrix} q^6, q^{18}, q^{30}, q^{36} \\ q^3, q^9, q^{27}, q^{33} \end{matrix}; q^{36}\right)_\infty + q\left(\begin{matrix} q^6, q^{18}, q^{30}, q^{36} \\ q^3, q^{15}, q^{21}, q^{33} \end{matrix}; q^{36}\right)_\infty,
\end{aligned}$$

as claimed. The other identity is proved in similar fashion.

Now

$$q\begin{pmatrix} q^6, q^{18}, q^{30}, q^{36} \\ q^3, q^{15}, q^{21}, q^{33} \end{pmatrix}; q^{36}\Big)_\infty = q\prod_{n\geqslant 1}(1+q^{18n-15})(1+q^{18n-3})(1-q^{18n})$$

$$= q\sum_{-\infty}^{\infty} q^{9n^2-6n} = \sum_{-\infty}^{\infty} q^{(3n-1)^2} = \frac{1}{2}\sum_{3\nmid n} q^{n^2}$$

$$= \frac{1}{2}\left\{\sum_{-\infty}^{\infty} q^{n^2} - \sum_{-\infty}^{\infty} q^{9n^2}\right\}.$$

If we write

$$P_1(q) = \left(\begin{matrix} q^2, q^6, q^{10}, q^{12} \\ q, q^3, q^9, q^{11} \end{matrix}; q^{12}\right)_\infty, \quad P_2(q) = \left(\begin{matrix} q^2, q^6, q^{10}, q^{12} \\ q^3, q^5, q^7, q^9 \end{matrix}; q^{12}\right)_\infty,$$

then (6) becomes

$$P_1(q) = P_1(q^3) + \frac{1}{2}\left\{\sum_{-\infty}^{\infty} q^{n^2} - \sum_{-\infty}^{\infty} q^{9n^2}\right\},$$

$$qP_2(q) = \frac{1}{2}\left\{\sum_{-\infty}^{\infty} q^{n^2} - \sum_{-\infty}^{\infty} q^{9n^2}\right\} - q^3 P_2(q^3).$$

Theorem 3 follows by iteration, since $P_1\left(q^{3^n}\right), P_2\left(q^{3^n}\right) \to 1$ as $n \to \infty$. □

PROOF OF THEOREM 4: We start by obtaining the 5-dissections

(7)

$$\left(\begin{matrix} q^2, q^8, q^{10}, q^{12}, q^{18}, q^{20} \\ q, q^4, q^9, q^{11}, q^{16}, q^{19} \end{matrix}; q^{20}\right)_\infty = \left(\begin{matrix} q^{10}, q^{40}, q^{50}, q^{60}, q^{90}, q^{100} \\ q^5, q^{20}, q^{45}, q^{55}, q^{80}, q^{95} \end{matrix}; q^{100}\right)_\infty + q\left(\begin{matrix} q^{30}, q^{50}, q^{70}, q^{100} \\ q^{15}, q^{35}, q^{65}, q^{85} \end{matrix}; q^{100}\right)_\infty + q^4\left(\begin{matrix} q^{10}, q^{50}, q^{90}, q^{100} \\ q^5, q^{45}, q^{55}, q^{95} \end{matrix}; q^{100}\right)_\infty,$$

$$\left(\begin{matrix} q^4, q^6, q^{10}, q^{14}, q^{16}, q^{20} \\ q^3, q^7, q^8, q^{12}, q^{13}, q^{17} \end{matrix}; q^{20}\right)_\infty = \left(\begin{matrix} q^{30}, q^{50}, q^{70}, q^{100} \\ q^{15}, q^{35}, q^{65}, q^{85} \end{matrix}; q^{100}\right)_\infty + q^3\left(\begin{matrix} q^{10}, q^{50}, q^{90}, q^{100} \\ q^5, q^{45}, q^{55}, q^{95} \end{matrix}; q^{100}\right)_\infty - q^4\left(\begin{matrix} q^{20}, q^{30}, q^{50}, q^{70}, q^{80}, q^{100} \\ q^{15}, q^{35}, q^{40}, q^{60}, q^{65}, q^{85} \end{matrix}; q^{100}\right)_\infty$$

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We have

$$\begin{aligned}
&\left(\begin{matrix} q^2, q^8, q^{10}, q^{12}, q^{18}, q^{20} \\ q, q^4, q^9, q^{11}, q^{16}, q^{19} \end{matrix}; q^{20}\right)_\infty \\
&= \frac{1}{(q^{20})_\infty}\left(\begin{matrix} q^8, q^{12}, q^{20} \\ q^4, q^{16} \end{matrix}; q^{20}\right)_\infty \left(\begin{matrix} q^2, q^{10}, q^{18}, q^{20} \\ q, q^9, q^{11}, q^{19} \end{matrix}; q^{20}\right)_\infty \\
&= \frac{1}{(q^{20})_\infty}\left(\begin{matrix} q^8, q^{12}, q^{20} \\ q^4, q^{16} \end{matrix}; q^{20}\right)_\infty \prod_{n\geqslant 1}\left(1+q^{10n-9}\right)\left(1+q^{10n-1}\right)\left(1-q^{10n}\right) \\
&= \frac{1}{(q^{20})_\infty}\left\{\sum_{-\infty}^{\infty}(-1)^n q^{30n^2-2n} + \sum_{-\infty}^{\infty}(-1)^n q^{30n^2-22n+4}\right\}\sum_{-\infty}^{\infty} q^{5n^2-4n} \\
&= \frac{1}{(q^{20})_\infty}\left\{\sum_{m,n=-\infty}^{\infty}(-1)^m q^{30m^2-2m+5n^2-4n}\right. \\
&\qquad\qquad \left. + \sum_{m,n=-\infty}^{\infty}(-1)^m q^{30m^2-22m+5n^2-4n+4}\right\}.
\end{aligned}$$

We now split each sum into five, according to the residue modulo 5 of $m+2n$.

If $m+2n \equiv 0 \pmod 5$, $2m-n\equiv 0$, $3m+n\equiv 0$, let

$$s = \frac{1}{5}(3m+n), \quad t = \frac{1}{5}(2m-n), \quad m = s+t, \quad n = 2s-3t;$$

if $m+2n\equiv 1$, $s=(3m+n+2)/5$, $t=(2m-n-2)/5$, $m=s+t$, $n=2s-3t-2$;
if $m+2n\equiv 2$, $s=(3m+n-1)/5$, $t=(2m-n+1)/5$, $m=s+t$, $n=2s-3t+1$;
if $m+2n\equiv -1$, $s=(3m+n-2)/5$, $t=(2m-n+2)/5$, $m=s+t$, $n=2s-3t+2$;
if $m+2n\equiv -2$, $s=(3m+n+1)/5$, $t=(2m-n-1)/5$, $m=s+t$, $n=2s-3t-1$.
If we make these substitutions into the two sums, we obtain ten sums, each of which separates into the product of two sums, each summable via the triple product identity. Two of the ten sums are zero and two cancel, leaving six which can be grouped in pairs in an obvious way, and each pair summed by the quintuple product identity, leading to the stated identity. The other identity is proved in similar fashion.

Now,

$$\begin{aligned}
&q\left(\begin{matrix} q^{30}, q^{50}, q^{70}, q^{100} \\ q^{15}, q^{35}, q^{65}, q^{85} \end{matrix}; q^{100}\right)_\infty + q^4\left(\begin{matrix} q^{10}, q^{50}, q^{90}, q^{100} \\ q^5, q^{45}, q^{55}, q^{95} \end{matrix}; q^{100}\right)_\infty \\
&\quad = q\prod_{n\geqslant 1}\left(1+q^{50n-35}\right)\left(1+q^{50n-15}\right)\left(1-q^{50n}\right) \\
&\qquad\qquad + q^4\prod_{n\geqslant 1}\left(1+q^{50n-45}\right)\left(1+q^{50n-5}\right)\left(1-q^{50n}\right)
\end{aligned}$$

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$$= q\sum_{-\infty}^{\infty} q^{25n^2-10n} + q^4\sum_{-\infty}^{\infty} q^{25n^2-20n} = \sum_{-\infty}^{\infty} q^{(5n-1)^2} + \sum_{-\infty}^{\infty} q^{(5n-2)^2} = \frac{1}{2}\sum_{5\nmid n} q^{n^2}$$

$$= \frac{1}{2}\left\{\sum_{-\infty}^{\infty} q^{n^2} - \sum_{-\infty}^{\infty} q^{25n^2}\right\}.$$

If we write

$$P_1(q) = \left(\begin{matrix} q^2, q^8, q^{10}, q^{12}, q^{18}, q^{20} \\ q,\ q^4, q^9,\ q^{11}, q^{16}, q^{19} \end{matrix}; q^{20}\right)_\infty, \quad P_2(q) = \left(\begin{matrix} q^4, q^6, q^{10}, q^{14}, q^{16}, q^{20} \\ q^3, q^7, q^8,\ q^{12}, q^{13}, q^{17} \end{matrix}; q^{20}\right)_\infty$$

then (7) becomes

$$P_1(q) = P_1(q^5) + \frac{1}{2}\left\{\sum_{-\infty}^{\infty} q^{n^2} - \sum_{-\infty}^{\infty} q^{25n^2}\right\},$$

$$qP_2(q) = \frac{1}{2}\left\{\sum_{-\infty}^{\infty} q^{n^2} - \sum_{-\infty}^{\infty} q^{25n^2}\right\} - q^5P_2(q^5).$$

Theorem 4 follows by iteration, since $P_1\left(q^{5^n}\right), P_2\left(q^{5^n}\right) \to 1$ as $n \to \infty$. □

5. Proofs of Theorems 5 and 6

Proof of Theorem 5: We start by proving

(8)

$$\prod_{n\geqslant 1}(1+q^{12n-11})(1+q^{12n-9})(1+q^{12n-3})(1+q^{12n-1})(1-q^{12n})^2$$
$$= \prod_{n\geqslant 1}(1+q^{24n-20})(1+q^{24n-14})(1+q^{24n-10})(1+q^{24n-4})(1-q^{24n})^2$$
$$+ q\prod_{n\geqslant 1}(1+q^{24n-22})(1+q^{24n-16})(1+q^{24n-8})(1+q^{24n-2})(1-q^{24n})^2.$$

We have

$$\prod_{n\geqslant 1}(1+q^{12n-11})(1+q^{12n-9})(1+q^{12n-3})(1+q^{12n-1})(1-q^{12n})^2$$
$$= \sum_{m,n=-\infty}^{\infty} q^{6m^2-5m+6n^2-3n}$$
$$= \sum_{m\equiv n \pmod 2} q^{6m^2-5m+6n^2-3n} + \sum_{m\not\equiv n \pmod 2} q^{6m^2-5m+6n^2-3n}$$

$$= \sum_{u,v=-\infty}^{\infty} q^{6(u+v)^2-5(u+v)+6(u-v)^2-3(u-v)}$$

$$+ \sum_{u,v=-\infty}^{\infty} q^{6(u+v+1)^2-5(u+v+1)+6(u-v)^2-3(u-v)}$$

$$= \sum_{u,v=-\infty}^{\infty} q^{12u^2-8u+12v^2-2v} + q \sum_{u,v=-\infty}^{\infty} q^{12u^2+4u+12v^2+10v}$$

$$= \prod_{n\geqslant 1} (1+q^{24n-20})(1+q^{24n-14})(1+q^{24n-10})(1+q^{24n-4})(1-q^{24n})^2$$

$$+ q \prod_{n\geqslant 1} (1+q^{24n-22})(1+q^{24n-16})(1+q^{24n-8})(1+q^{24n-2})(1-q^{24n})^2.$$

We can write (8) as follows.

$$\begin{pmatrix} q^2, q^6, q^{12}, q^{12}, q^{18}, q^{22}, q^{24}, q^{24} \\ q, \ q^3, q^9, \ q^{11}, q^{13}, q^{15}, q^{21}, q^{23} \end{pmatrix}; q^{24} \Big)_\infty$$

$$= \begin{pmatrix} q^8, q^{24}, q^{24}, q^{40}, q^{48}, q^{48} \\ q^4, q^{10}, q^{14}, q^{34}, q^{38}, q^{44} \end{pmatrix}; q^{48} \Big)_\infty + q \begin{pmatrix} q^4, q^{24}, q^{24}, q^{44}, q^{48}, q^{48} \\ q^2, q^8, \ q^{22}, q^{26}, q^{40}, q^{46} \end{pmatrix}; q^{48} \Big)_\infty$$

If we multiply by $\left(\begin{matrix} q^{10}, q^{14} \\ q^{12}, q^{24} \end{matrix}; q^{24}\right)_\infty$ we obtain

(9) $$\left(\begin{matrix} q^2, q^6, q^{10}, q^{12} \\ q, \ q^3, q^9, \ q^{11} \end{matrix}; q^{12}\right)_\infty = \left(\begin{matrix} q^8, q^{24}, q^{40}, q^{48} \\ q^4, q^{12}, q^{36}, q^{44} \end{matrix}; q^{48}\right)_\infty + q \left(\begin{matrix} q^4, q^{10}, q^{14}, q^{24}, q^{34}, q^{38}, q^{44}, q^{48} \\ q^2, q^8, \ q^{12}, q^{22}, q^{26}, q^{36}, q^{40}, q^{46} \end{matrix}; q^{48}\right)_\infty$$

Combining Theorem 3 and (9), we find

$$\sum_{-\infty}^{\infty} q^{n^2} + \sum_{-\infty}^{\infty} q^{3n^2} = \sum_{-\infty}^{\infty} q^{4n^2} + \sum_{-\infty}^{\infty} q^{12n^2} + 2q \left(\begin{matrix} q^4, q^{10}, q^{14}, q^{24}, q^{34}, q^{38}, q^{44}, q^{48} \\ q^2, q^8, \ q^{12}, q^{22}, q^{26}, q^{36}, q^{40}, q^{46} \end{matrix}; q^{48}\right)_\infty.$$

If we extract odd powers, divide by q and replace q^2 by q we obtain

$$\sum_{-\infty}^{\infty} q^{2n(n+1)} + \sum_{-\infty}^{\infty} q^{6n(n+1)+1} = 2 \left(\begin{matrix} q^2, q^5, q^7, q^{12}, q^{17}, q^{19}, q^{22}, q^{24} \\ q, \ q^4, q^6, q^{11}, q^{13}, q^{18}, q^{20}, q^{23} \end{matrix}; q^{24}\right)_\infty$$

If we put $-q$ for q we find

$$\sum_{-\infty}^{\infty} q^{2n(n+1)} - \sum_{-\infty}^{\infty} q^{6n(n+1)+1} = 2 \left(\begin{matrix} q, \ q^{10}, q^{11}, q^{12}, q^{13}, q^{14}, q^{23}, q^{24} \\ q^4, q^5, \ q^6, \ q^7, \ q^{17}, q^{18}, q^{19}, q^{20} \end{matrix}; q^{24}\right)_\infty \qquad \square$$

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PROOF OF THEOREM 6: We shall start by proving

$$(10)\quad \prod_{n\geqslant 1}(1+q^{10n-9})(1+q^{10n-1})(1-q^{10n}) = \prod_{n\geqslant 1}(1+q^{40n-28})(1+q^{40n-12})(1-q^{40n}) + q\prod_{n\geqslant 1}(1+q^{40n-32})(1+q^{40n-8})(1-q^{40n}).$$

We have

$$\begin{aligned}\prod_{n\geqslant 1}&(1+q^{10n-9})(1+q^{10n-1})(1-q^{10n})\\ &= \sum_{-\infty}^{\infty} q^{5n^2-4n} = \sum_{-\infty}^{\infty} q^{20n^2-8n} + q\sum_{-\infty}^{\infty} q^{20n^2+12n}\\ &= \prod_{n\geqslant 1}(1+q^{40n-28})(1+q^{40n-12})(1-q^{40n})\\ &\qquad + q\prod_{n\geqslant 1}(1+q^{40n-32})(1+q^{40n-8})(1-q^{40n}).\end{aligned}$$

We can write (10) as follows.

$$\begin{pmatrix} q^2, q^{10}, q^{18}, q^{20} \\ q, \ q^9, \ q^{11}, q^{19} \end{pmatrix}; q^{20}\Bigg)_\infty = \begin{pmatrix} q^{24}, q^{40}, q^{56}, q^{80} \\ q^{12}, q^{28}, q^{52}, q^{68} \end{pmatrix}; q^{80}\Bigg)_\infty + q\begin{pmatrix} q^{16}, q^{40}, q^{64}, q^{80} \\ q^{8}, \ q^{32}, q^{48}, q^{72} \end{pmatrix}; q^{80}\Bigg)_\infty.$$

If we multiply by $\left(\begin{matrix} q^8, q^{12} \\ q^4, q^{16} \end{matrix}; q^{20}\right)_\infty$ we obtain

$$(11)\quad \left(\begin{matrix} q^2, q^8, q^{10}, q^{12}, q^{18}, q^{20} \\ q, \ q^4, q^9, \ q^{11}, q^{16}, q^{19} \end{matrix}; q^{20}\right)_\infty = \left(\begin{matrix} q^8, q^{32}, q^{40}, q^{48}, q^{72}, q^{80} \\ q^4, q^{16}, q^{36}, q^{44}, q^{64}, q^{76} \end{matrix}; q^{80}\right)_\infty + q\left(\begin{matrix} q^{12}, q^{28}, q^{40}, q^{52}, q^{68}, q^{80} \\ q^4, \ q^{24}, q^{36}, q^{44}, q^{56}, q^{76} \end{matrix}; q^{80}\right)_\infty.$$

Combining Theorem 4 and (11) we obtain

$$\sum_{-\infty}^{\infty} q^{n^2} + \sum_{-\infty}^{\infty} q^{5n^2} = \sum_{-\infty}^{\infty} q^{4n^2} + \sum_{-\infty}^{\infty} q^{20n^2} + 2q\left(\begin{matrix} q^{12}, q^{28}, q^{40}, q^{52}, q^{68}, q^{80} \\ q^4, \ q^{24}, q^{36}, q^{44}, q^{56}, q^{76} \end{matrix}; q^{80}\right)_\infty.$$

If we extract odd powers, divide by q and replace q^4 by q we obtain

$$\sum_{-\infty}^{\infty} q^{n(n+1)} + \sum_{-\infty}^{\infty} q^{5n(n+1)+1} = 2\left(\begin{matrix} q^3, q^7, q^{10}, q^{13}, q^{17}, q^{20} \\ q, \ q^6, q^9, \ q^{11}, q^{14}, q^{19} \end{matrix}; q^{20}\right)_\infty.$$

If we put $-q$ for q we find

$$\sum_{-\infty}^{\infty} q^{n(n+1)} - \sum_{-\infty}^{\infty} q^{5n(n+1)+1} = 2\left(\begin{matrix} q, \ q^9, q^{10}, q^{11}, q^{19}, q^{20} \\ q^2, q^3, q^7, \ q^{13}, q^{17}, q^{18} \end{matrix}; q^{20}\right)_\infty. \qquad \square$$

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