THE SPECTRUM OF WEIGHTED MEAN OPERATORS

BY

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ABSTRACT. Recently J. B. Reade determined the spectrum of $C$, the Cesaro matrix of order 1, considered as an operator on $c_0$, the space of null sequences. Previously F. P. A. Cass and the author had determined the spectra for a large class of weighted mean operators on $c$, the space of convergent sequences. Subsequently the author determined the fine spectra of these operators over $c$. This paper examines the spectra and fine spectra of weighted mean operators on $c_0$, obtaining the result of Reade as a special case.

In a recent paper Reade [4] determined the spectrum of $C$, the Cesaro matrix of order 1, regarded as a member of $B(c_0)$; i.e. a bounded linear operator on the space $c_0$ of null sequences. In 1977 Cass and the author [1] determined the spectra for a large class of weighted mean operators in $B(c)$, $c$ the space of convergent sequences. More recently the author [5] determined the fine spectra of these operators, again in $B(c)$.

This paper extends [1] and [5] to $B(c_0)$, and obtains the result of [4] as a special case.

A weighted mean matrix is a triangular matrix $A = (a_{nk})$ with $a_{nk} = p^k / P_n$, where $p_0 > 0$, $p_n \geq 0$ for $n > 0$, and $P_n = \sum_{k=0}^{n} p_k$. If $P_n \to \infty$ then $A \in B(c)$ and $B(c_0)$.

Let $\sigma(A)$ denote the spectrum of $A$ in $B(c_0)$.

THEOREM 1. Let $A$ be weighted mean method with $P_n \to \infty$. Then

$$\sigma(A) \subseteq \{ \lambda : |\lambda - 1/2| \leq 1/2 \}.$$

PROOF. From [1], with $D = (A - \lambda I)^{-1}$, $\lambda$ satisfying $|\lambda - 1/2| > 1/2$, $D$ has finite norm. That $D \in B(c_0)$ comes from the following lemma.

LEMMA. Let $A$ be a multiplicative coregular triangle with an inverse $B$ with finite norm. Then $B \in B(c_0)$ and $c_{0,A} = c_0$.

PROOF. For any matrix $A$ in $B(c)$, $x \in c$, $\lim_A x = \chi(A) \lim x + \sum a_k x_k$, where the $a_k$ are the column limits of $A$ and $\chi(A) = \lim_n \sum_k a_{nk} - \sum_k a_k$. $A$ is
called multiplicative if each column limit is zero and coregular if \( \chi(A) \neq 0 \). Thus, the hypotheses on \( A \) guarantee that \( A \in B(c) \).

The notation \( c_{0,1} \) means the set of all sequences that \( A \) sums to zero. Let \( x \in c_{0,1} \). Then \( Ax = y \in c_0 \). Therefore \( y \) is bounded and \( x = BAx = By \) is also bounded. Thus \( c_{0,1} \subseteq m, m \) the space of bounded sequences. From a lemma of Copping [2] (which is proved for matrices in \( B(c) \), but also applies to matrices in \( B(c_0) \)), \( c_{0,1} \subseteq c \).

Suppose there exists an \( x \in c \setminus c_0 \) with \( Ax \in c_0 \). Let \( \lim x = l \). Then \( A(x - l) \in c_0 \). Since \( A \in B(c) \) and is coregular, \( A(x - l) = Ax - Al \). But \( Ax \in c_0 \) and \( \lim_A l = l(A) \neq 0 \), a contradiction. Therefore \( c_{0,1} = c_0 \).

**Theorem 2.** Let \( A \) be a weighted mean method with \( P_n \to \infty \). Then

\[
\sigma(A) \supseteq \{ \lambda : |\lambda - (2 - \delta)^{-1}| \leq (1 - \delta)/(2 - \delta) \} \cup S,
\]

where

\[
S = \{ p_n/P_n, n \geq 0 \}.
\]

The proof is identical to that in [1].

**Corollary 1.** Let \( A \) be a weighted mean method with \( P_n \to \infty \) and \( \delta = 0 \). Then

\[
\sigma(A) = \{ \lambda : |\lambda - 1/2| \leq 1/2 \}.
\]

**Proof.** Combine Theorems 1 and 2, noting that \( S \) is contained in the disc.

**Corollary 2.** [4, Theorem 3] \( \sigma(C) = \{ \lambda : |\lambda - 1/2| \leq 1/2 \} \).

**Proof.** \( C \) is a weighted mean matrix with each \( p_n = 1 \).

The remaining theorems of [1] have identical counterparts in \( B(c_0) \). For completeness they are stated here.

**Theorem 3.** Let \( A \) be a weighted mean method with \( P_n \to \infty \) and \( \gamma > 0 \). Then

\[
\sigma(A) \supseteq \{ \lambda : |\lambda - (2 - \gamma)^{-1}| \leq (1 - \gamma)/(2 - \gamma) \} \cup S.
\]

**Corollary 3.** Let \( A \) be a weighted mean method with \( P_n \to \infty \) and \( \delta = \gamma > 0 \). Then

\[
\sigma(A) = \{ \lambda : |\lambda - (2 - \gamma)^{-1}| \leq (1 - \gamma)/(2 - \gamma) \} \cup E,
\]

where

\[
E = \{ p_n/P_n : p_n/P_n < \gamma/(2 - \gamma) \}.
\]

**Theorem 4.** Let \( A \) be a weighted mean method with \( P_n \to \infty \). Then \( c_{0,1} = c_0 \) if and only if \( \theta = \lim p_{n+1}/P_n > 0 \).
Note that, from Theorems 1 and 2, the spectrum of a weighted mean method in $B(c_0)$ is not determined when $\delta > \gamma$. The examples in [1] which illustrate the pathology that can occur in $B(c)$ apply also to $B(c_0)$.

From Goldberg [3], if $T \in B(X)$, $X$ a Banach space, then there are three possibilities for $R(T)$, the range of $T$:

(I) $R(T) = X$.
(II) $R(T) = X$, but $R(T) \neq X$, and
(III) $R(T) \neq X$,

and three possibilities for $T^{-1}$:

(1) $T^{-1}$ exists and is continuous,
(2) $T^{-1}$ exists but is discontinuous,
(3) $T^{-1}$ does not exist.

In [5] the author analyzed the behavior of each point $\lambda$ in $\sigma(A)$ relative to these nine classifications. All of the results carry over immediately to $B(c_0)$. For completeness they are stated below.

**Theorem 5.** Let $A$ be a weighted mean method with $P_n \to \infty$ and $\gamma = \delta$. If $\lambda$ satisfies $|\lambda - (2 - \delta)^{-1}| < (1 - \delta)/(2 - \delta)$ and $\lambda \notin S$, then $\lambda \in III_1\sigma(A)$; i.e., $\lambda$ is a point of $\sigma(A)$ for which $R(T) \neq X$ and $T^{-1}$ exists and is continuous, $T = \lambda I - A$.

**Theorem 6.** Let $A$ be a weighted mean method with $P_n \to \infty$ and $\gamma = \delta < 1$. Suppose no diagonal entry of $A$ occurs an infinite number of times. If $\lambda = \delta$ or $\lambda = a_{nn}$, $n > 0$ and $\delta/(2 - \delta) < \lambda < 1$, then $\lambda \in III_1\sigma(A)$.

**Theorem 7.** Let $A$ be a weighted mean method with $P_n \to \infty$, $\gamma = \delta$ and $p_n/P_n \equiv \delta$ for all $n$ sufficiently large. If $\lambda$ satisfies $|\lambda - (2 - \delta)^{-1}| = (1 - \delta)/(2 - \delta)$, $\lambda \neq 1, \delta/(2 - \delta)$, then $\lambda \in II_2\sigma(A)$.

**Theorem 8.** Let $A$ be a weighted mean method with $P_n \to \infty$. Then $1 \in III_3\sigma(A)$.

**Theorem 9.** Let $A$ be a weighted mean method with $P_n \to \infty$. If there exist values of $n$ such that $0 \leq p_n/P_n \leq \gamma/(2 - \gamma)$, then $\lambda = p_n/P_n$ implies $\lambda \in III_3\sigma(A)$.

Let $c_n = p_n/P_n$.

**Theorem 10.** Let $A$ be a weighted mean method defined by $c_0 = 1, c_{2n} = 1/p, c_{2n-1} = 1/q, n > 0$ where $1 < p < q$. If $\lambda \neq 1/p, 1/q, 1$ and satisfies $(p - 1)(q - 1)|\lambda|^2 > |1 - p\lambda||1 - q\lambda|$, then $\lambda \in III_1\sigma(A)$.

**Theorem 11.** Let $A$ be as in Theorem 10. If $\lambda = 1/p$ or $1/q$, then $\lambda \in III_1\sigma(A)$. 

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THEOREM 12. Let $A$ be as in Theorem 10. If $\lambda$ satisfies

$$(p - 1)(q - 1)|\lambda|^2 = |1 - p\lambda||1 - q\lambda|, \lambda \neq 1,$$

then $\lambda \in II_3\sigma(A)$.

Theorem 8 requires a slightly different proof from its counterpart in [5]. Let $T = I - A$. Then $T$ does not have an inverse. Also $R(T) \subseteq \{e_1, e_2, \ldots\}$, where $e_i$ is the standard coordinate sequence with a 1 in the $k$th position and zeros elsewhere. Therefore $R(T) \neq c_0$ and $1 \in III_3\sigma(A)$.

REFERENCES


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