Canad. Math. Bull. Vol. 30 (4), 1987

THE SPECTRUM OF WEIGHTED MEAN OPERATORS

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ABSTRACT. Recently J. B. Reade determined the spectrum of C, the Cesaro matrix of order 1, considered as an operator on c_0, the space of null sequences. Previously F. P. A. Cass and the author had determined the spectra for a large class of weighted mean operators on c, the space of convergent sequences. Subsequently the author determined the fine spectra of these operators over c. This paper examines the spectra and fine spectra of weighted mean operators on c_0, obtaining the result of Reade as a special case.

In a recent paper Reade [4] determined the spectrum of C, the Cesaro matrix of order 1, regarded as a member of $B(c_0)$; i.e. a bounded linear operator on the space c_0 of null sequences. In 1977 Cass and the author [1] determined the spectra for a large class of weighted mean operators in $B(c)$, c the space of convergent sequences. More recently the author [5] determined the fine spectra of these operators, again in $B(c)$.

This paper extends [1] and [5] to $B(c_0)$, and obtains the result of [4] as a special case.

A weighted mean matrix is a triangular matrix $A = (a_{nk})$ with $a_{nk} = p_k/P_n$, where $p_0 > 0$, $p_n \geqq 0$ for $n > 0$, and $P_n = \sum_{k=0}^{n} p_k$. If $P_n \to \infty$ then $A \in B(c)$ and $B(c_0)$.

Let $\sigma(A)$ denote the spectrum of A in $B(c_0)$.

THEOREM 1. *Let A be weighted mean method with $P_n \to \infty$. Then*

$$\sigma(A) \subseteq \{\lambda : |\lambda - 1/2| \leqq 1/2\}.$$

PROOF. From [1], with $D = (A - \lambda I)^{-1}$, λ satisfying $|\lambda - 1/2| > 1/2$, D has finite norm. That $D \in B(c_0)$ comes from the following lemma.

LEMMA. *Let A be a multiplicative coregular triangle with an inverse B with finite norm. Then $B \in B(c_0)$ and $c_{0_A} = c_0$.*

PROOF. For any matrix A in $B(c)$, $x \in c$, $\lim_A x = \chi(A) \lim x + \sum a_k x_k$, where the a_k are the column limits of A and $\chi(A) = \lim_n \sum_k a_{nk} - \sum_k a_k$. A is

Received by the editors May 12, 1986.
AMS Subject Classification Numbers: 40G99, 47A10.

called multiplicative if each column limit is zero and coregular if $\chi(A) \neq 0$. Thus, the hypotheses on A guarantee that $A \in B(c)$.

The notation c_{0_A} means the set of all sequences that A sums to zero. Let $x \in c_{0_A}$. Then $Ax = y \in c_0$. Therefore y is bounded and $x = BAx = By$ is also bounded. Thus $c_{0_A} \subseteq m$, m the space of bounded sequences. From a lemma of Copping [2] (which is proved for matrices in $B(c)$, but also applies to matrices in $B(c_0)$), $c_{0_A} \subseteq c$.

Suppose there exists an $x \in c \backslash c_0$ with $Ax \in c_0$. Let $\lim x = l$. Then $A(x - l) \in c_0$. Since $A \in B(c)$ and is coregular, $A(x - l) = Ax - Al$. But $Ax \in c_0$ and $\lim_A l = l\chi(A) \neq 0$, a contradiction. Therefore $c_{0_A} = c_0$.

Let $\delta = \overline{\lim}\, p_n/P_n$, $\gamma = \underline{\lim}\, p_n/P_n$.

THEOREM 2. *Let A be a weighted mean method with $P_n \to \infty$. Then*

$$\sigma(A) \supseteq \{\lambda : |\lambda - (2 - \delta)^{-1}| \leqq (1 - \delta)/(2 - \delta)\} \cup S,$$

where

$$S = \overline{\{p_n/P_n : n \geqq 0\}}.$$

The proof is identical to that in [1].

COROLLARY 1. *Let A be a weighted mean method with $P_n \to \infty$ and $\delta = 0$. Then* $\sigma(A) = \{\lambda : |\lambda - 1/2| \leqq 1/2\}$.

PROOF. Combine Theorems 1 and 2, noting that S is contained in the disc.

COROLLARY 2. [4, Theorem 3] $\sigma(C) = \{\lambda : |\lambda - 1/2| \leqq 1/2\}$.

PROOF. C is a weighted mean matrix with each $p_n = 1$.

The remaining theorems of [1] have identical counterparts in $B(c_0)$. For completeness they are stated here.

THEOREM 3. *Let A be a weighted mean method with $P_n \to \infty$ and $\gamma > 0$. Then*

$$\sigma(A) \subseteq \{\lambda : |\lambda - (2 - \gamma)^{-1}| \leqq (1 - \gamma)/(2 - \gamma)\} \cup S.$$

COROLLARY 3. *Let A be a weighted mean method with $P_n \to \infty$ and $\delta = \gamma > 0$. Then*

$$\sigma(A) = \{\lambda : |\lambda - (2 - \gamma)^{-1}| \leqq (1 - \gamma)/(2 - \gamma)\} \cup E,$$

where

$$E = \{p_n/P_n : p_n/P_n < \gamma/(2 - \gamma)\}.$$

THEOREM 4. *Let A be a weighted mean method with $P_n \to \infty$. Then $c_{0_A} = c_0$ if and only if $\theta = \underline{\lim}\, p_{n+1}/P_n > 0$.*

https://doi.org/10.4153/CMB-1987-065-9 Published online by Cambridge University Press

Note that, from Theorems 1 and 2, the spectrum of a weighted mean method in $B(c_0)$ is not determined when $\delta > \gamma$. The examples in [1] which illustrate the pathology that can occur in $B(c)$ apply also to $B(c_0)$.

From Goldberg [3], if $T \in B(X)$, X a Banach space, then there are three possibilities for $R(T)$, the range of T:

(I) $R(T) = X$,
(II) $\overline{R(T)} = X$, but $R(T) \neq X$, and
(III) $\overline{R(T)} \neq X$,

and three possibilities for T^{-1}:

(1) T^{-1} exists and is continuous,
(2) T^{-1} exists but is discontinuous,
(3) T^{-1} does not exist.

In [5] the author analyzed the behavior of each point λ in $\sigma(A)$ relative to these nine classifications. All of the results carry over immediately to $B(c_0)$. For completeness they are stated below.

THEOREM 5. *Let A be a weighted mean method with $P_n \to \infty$ and $\gamma = \delta$. If λ satisfies $|\lambda - (2 - \delta)^{-1}| < (1 - \delta)/(2 - \delta)$ and $\lambda \notin S$, then $\lambda \in III_1\sigma(A)$;* i.e., *$\lambda$ is a point of $\sigma(A)$ for which $\overline{R(T)} \neq X$ and T^{-1} exists and is continuous, $T = \lambda I - A$.*

THEOREM 6. *Let A be a weighted mean method with $P_n \to \infty$ and $\gamma = \delta < 1$. Suppose no diagonal entry of A occurs an infinite number of times. If $\lambda = \delta$ or $\lambda = a_{nn}$, $n > 0$ and $\delta/(2 - \delta) < \lambda < 1$, then $\lambda \in III_1\sigma(A)$.*

THEOREM 7. *Let A be a weighted mean method with $P_n \to \infty$, $\gamma = \delta$ and $p_n/P_n \geqq \delta$ for all n sufficiently large. If λ satisfies $|\lambda - (2 - \delta)^{-1}| = (1 - \delta)/(2 - \delta)$, $\lambda \neq 1, \delta/(2 - \delta)$, then $\lambda \in II_2\sigma(A)$.*

THEOREM 8. *Let A be a weighted mean method with $P_n \to \infty$. Then $1 \in III_3\sigma(A)$.*

THEOREM 9. *Let A be a weighted mean method with $P_n \to \infty$. If there exist values of n such that $0 \leqq p_n/P_n \leqq \gamma/(2 - \gamma)$, then $\lambda = p_n/P_n$ implies $\lambda \in III_3\sigma(A)$.*

$$\text{Let } c_n = p_n/P_n.$$

THEOREM 10. *Let A be a weighted mean method defined by $c_0 = 1$, $c_{2n} = 1/p$, $c_{2n-1} = 1/q$, $n > 0$ where $1 < p < q$. If $\lambda \neq 1/p, 1/q, 1$ and satisfies $(p - 1)(q - 1)|\lambda|^2 > |1 - p\lambda||1 - q\lambda|$, then $\lambda \in III_1\sigma(A)$.*

THEOREM 11. *Let A be as in Theorem 10. If $\lambda = 1/p$ or $1/q$, then $\lambda \in III_1\sigma(A)$.*

THEOREM 12. *Let A be as in Theorem* 10. *If λ satisfies*

$$(p - 1)(q - 1)\,|\lambda|^2 = |1 - p\lambda||1 - q\lambda)\,|,\ \lambda \neq 1,$$

then $\lambda \in II_2\sigma(A)$.

Theorem 8 requires a slightly different proof from its counterpart in [5]. Let $T = I - A$. Then T does not have an inverse. Also $R(T) \subseteq \{e_1, e_2, \ldots\}$, where e_i is the standard coordinate sequence with a 1 in the kth position and zeros elsewhere. Therefore $R(T) \neq c_0$ and $1 \in III_3\sigma(A)$.

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