

Then  $OA = OC$  and  $\angle ACP = 90^\circ$ .  $\therefore OA = OP$ .

$\therefore$  the circumscribing circle of  $\triangle ABC$  passes through  $P$ . But  $\angle P = \frac{1}{2} \angle AOC = \angle B = \text{constant}$ .

$\therefore B$  lies on the fixed circle which circumscribes the fixed right-angled triangle  $ACP$  in which  $\angle P = \text{given } \angle B$ .

If  $\angle B$  is obtuse (Fig. 2),  $B$  lies on the circumscribing circle of the fixed right-angled triangle  $ACP$  in which  $\angle P = 180^\circ - \angle B$

(2) If in the quadrilateral  $ABCD$  the angles  $B$  and  $D$  are supplementary,  $D$  being acute (Fig. 2), then by the previous theorem  $B$  and  $D$  both lie on the fixed circle which circumscribes the fixed right-angled triangle  $ACP$  in which  $\angle ACP = \angle D$ .

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### An Elementary Proof of Feuerbach's Theorem.

Let  $O$  be the centre of the circumscribing circle of  $\triangle ABC$ ,  $A_1$  the middle point of  $BC$ , and  $EA_1OF$  the diameter at right angles to  $BC$ . Draw  $AX$  perpendicular to  $BC$  and produce it to meet the circle in  $K$ . Let  $H$  be the orthocentre of  $\triangle ABC$ ; join  $OH$  and bisect it in  $N$ , the centre of the nine-point circle.

Draw  $OY$  perpendicular to and bisecting  $AK$ .

Join  $EA$ , which bisects  $\angle BAC$  and contains the incentre  $I$ ; draw  $ID, NM$  perpendicular to  $BC$ . Join  $AF$  and draw  $AG$  perpendicular to  $EF$ ; also draw  $PIQ$  parallel to  $BC$  and meeting  $EF$  in  $P$  and  $AX$  in  $Q$ .

Then we have  $AH = 2OA_1$ ,  $HK = 2HX$ ,  $AI \cdot IE = 2Rr$ .

Also from similar triangles  $\frac{PI}{IE} = \frac{FG}{AF}$  and  $\frac{IQ}{AI} = \frac{AF}{FE}$ .

Thus  $\frac{PI \cdot IQ}{AI \cdot IE} = \frac{FG}{FE}$ , so that  $\frac{PI \cdot IQ}{2R \cdot r} = \frac{FG}{2R}$ , and  $PI \cdot IQ = r \cdot FG$ .

Now the projection of  $IN$  on  $FE = ID - NM = r - \frac{1}{2}(OA_1 + HX)$   
 $= r - \frac{1}{4}(AH + HK) = r - \frac{1}{2}AY$ .

(11)

