Then $OA = OC$ and $\angle ACP = 90°$. $\therefore OA = OP$.

$\therefore$ the circumscribing circle of $\triangle ABC$ passes through P. But $\angle P = \frac{1}{2} \angle AOC = \angle B =$ constant.

$\therefore B$ lies on the fixed circle which circumscribes the fixed right-angled triangle ACP in which $\angle P =$ given $\angle B$.

If $\angle B$ is obtuse (Fig. 2), B lies on the circumscribing circle of the fixed right-angled triangle ACP in which $\angle P = 180° - \angle B$

(2) If in the quadrilateral $ABCD$ the angles B and D are supplementary, D being acute (Fig. 2), then by the previous theorem B and D both lie on the fixed circle which circumscribes the fixed right-angled triangle ACP in which $\angle ACP = \angle D$.

R. F. BLADES.

An Elementary Proof of Feuerbach's Theorem.

Let O be the centre of the circumscribing circle of $\triangle ABC$, A_1 the middle point of BC, and EA_1OF the diameter at right angles to BC. Draw AX perpendicular to BC and produce it to meet the circle in K. Let H be the orthocentre of $\triangle ABC$; join OH and bisect it in N, the centre of the nine-point circle.

Draw OY perpendicular to and bisecting AK.

Join EA, which bisects $\angle BAC$ and contains the incentre I; draw ID, NM perpendicular to BC. Join AF and draw AG perpendicular to EF; also draw PIQ parallel to BC and meeting EF in P and AX in Q.

Then we have $AH = 2OA_1$, $HK = 2HX$, $AI \,.\, IE = 2Rr$.

Also from similar triangles $\dfrac{PI}{IE} = \dfrac{FG}{AF}$ and $\dfrac{IQ}{AI} = \dfrac{AF}{FE}$.

Thus $\dfrac{PI \,.\, IQ}{AI \,.\, IE} = \dfrac{FG}{FE}$, so that $\dfrac{PI \,.\, IQ}{2R \,.\, r} = \dfrac{FG}{2R}$, and $PI \,.\, IQ = r \,.\, FG$.

Now the projection of IN on $FE = ID - NM = r - \frac{1}{2}(OA_1 + HX)$

$$= r - \tfrac{1}{4}(AH + HK) = r - \tfrac{1}{2}AY.$$

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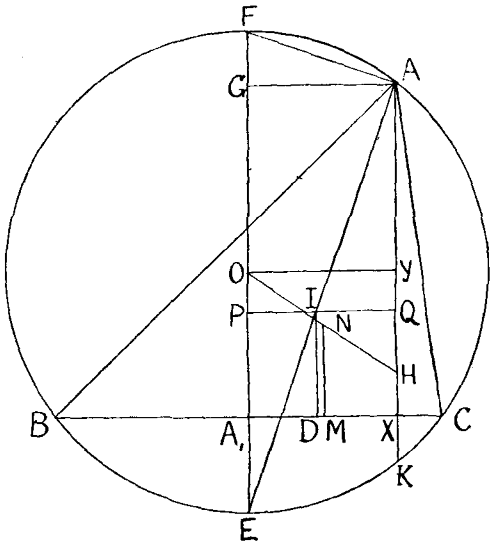
Hence the square of this projection $= r^2 - r \,.\, AY + \frac{1}{4} AY^2$

$$= r^2 - r \,.\, GO + \tfrac{1}{4} AY^2 \quad \text{.........(1)}$$

Again, the square of the projection of

$$IN \text{ on } BC = DM^2 = A_1M^2 - A_1D \,.\, DX$$
$$= \tfrac{1}{4} A_1X^2 - PI \,.\, IQ$$
$$= \tfrac{1}{4} OY^2 - r \,.\, FG \quad \text{.........................(2)}$$



Adding the results (1) and (2) we get

$$I_1N^2 = \tfrac{1}{4}(AY^2 + OY^2) - r(FG + GO) + r^2$$
$$= \tfrac{1}{4}R^2 - r \,.\, R + r^2.$$

Thus $IN = \frac{1}{2}R - r$, and the theorem is proved for the incircle. The proof for an excircle proceeds on exactly similar lines.

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