

Then $OA = OC$ and $\angle ACP = 90^\circ$. $\therefore OA = OP$.

\therefore the circumscribing circle of $\triangle ABC$ passes through P . But $\angle P = \frac{1}{2} \angle AOC = \angle B = \text{constant}$.

$\therefore B$ lies on the fixed circle which circumscribes the fixed right-angled triangle ACP in which $\angle P = \text{given } \angle B$.

If $\angle B$ is obtuse (Fig. 2), B lies on the circumscribing circle of the fixed right-angled triangle ACP in which $\angle P = 180^\circ - \angle B$

(2) If in the quadrilateral $ABCD$ the angles B and D are supplementary, D being acute (Fig. 2), then by the previous theorem B and D both lie on the fixed circle which circumscribes the fixed right-angled triangle ACP in which $\angle ACP = \angle D$.

R. F. BLADES.

An Elementary Proof of Feuerbach's Theorem.

Let O be the centre of the circumscribing circle of $\triangle ABC$, A_1 the middle point of BC , and EA_1OF the diameter at right angles to BC . Draw AX perpendicular to BC and produce it to meet the circle in K . Let H be the orthocentre of $\triangle ABC$; join OH and bisect it in N , the centre of the nine-point circle.

Draw OY perpendicular to and bisecting AK .

Join EA , which bisects $\angle BAC$ and contains the incentre I ; draw ID , NM perpendicular to BC . Join AF and draw AG perpendicular to EF ; also draw PIQ parallel to BC and meeting EF in P and AX in Q .

Then we have $AH = 2OA_1$, $HK = 2HX$, $AI \cdot IE = 2Rr$.

Also from similar triangles $\frac{PI}{IE} = \frac{FG}{AF}$ and $\frac{IQ}{AI} = \frac{AF}{FE}$.

Thus $\frac{PI \cdot IQ}{AI \cdot IE} = \frac{FG}{FE}$, so that $\frac{PI \cdot IQ}{2R \cdot r} = \frac{FG}{2R}$, and $PI \cdot IQ = r \cdot FG$.

Now the projection of IN on $FE = ID - NM = r - \frac{1}{2}(OA_1 + HX)$
 $= r - \frac{1}{4}(AH + HK) = r - \frac{1}{2}AY$.

(11)

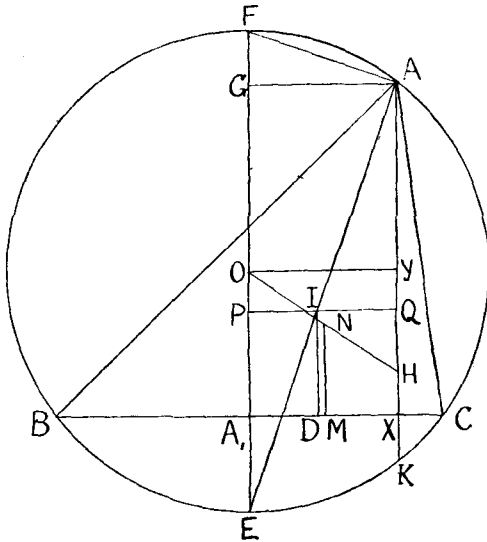
Hence the square of this projection = $r^2 - r \cdot AY + \frac{1}{4}AY^2$
 $= r^2 - r \cdot GO + \frac{1}{4}AY^2 \dots\dots\dots(1)$

Again, the square of the projection of

$$IN \text{ on } BC = DM^2 = A_1M^2 - A_1D \cdot DX$$

$$= \frac{1}{4}A_1X^2 - PI \cdot IQ$$

$$= \frac{1}{4}OY^2 - r \cdot FG \dots\dots\dots(2)$$



Adding the results (1) and (2) we get

$$I_1N^2 = \frac{1}{4}(AY^2 + OY^2) - r(FG + GO) + r^2$$

$$= \frac{1}{4}R^2 - r \cdot R + r^2.$$

Thus $IN = \frac{1}{2}R - r$, and the theorem is proved for the incircle.
 The proof for an excircle proceeds on exactly similar lines.

K. J. SANJANA.