Then \( OA = OC \) and \( \angle ACP = 90^\circ \). \( \therefore \) \( OA = OP \).

\[ \therefore \] the circumscribing circle of \( \triangle ABC \) passes through \( P \). But

\[ \angle P = \frac{1}{2} \angle AOC = \angle B = \text{constant}. \]

\[ \therefore B \] lies on the fixed circle which circumscribes the fixed right-angled triangle \( ACP \) in which \( \angle P = \text{given} \angle B \).

If \( \angle B \) is obtuse (Fig. 2), \( B \) lies on the circumscribing circle of the fixed right-angled triangle \( ACP \) in which \( \angle P = 180^\circ - \angle B \)

(2) If in the quadrilateral \( ABCD \) the angles \( B \) and \( D \) are supplementary, \( D \) being acute (Fig. 2), then by the previous theorem \( B \) and \( D \) both lie on the fixed circle which circumscribes the fixed right-angled triangle \( ACP \) in which \( \angle ACP = \angle D \).

R. F. Blades.

An Elementary Proof of Feuerbach's Theorem.

Let \( O \) be the centre of the circumscribing circle of \( \triangle ABC \), \( A_1 \) the middle point of \( BC \), and \( EA, OF \) the diameter at right angles to \( BC \). Draw \( AX \) perpendicular to \( BC \) and produce it to meet the circle in \( K \). Let \( H \) be the orthocentre of \( \triangle ABC \); join \( OH \) and bisect it in \( N \), the centre of the nine-point circle.

Draw \( OY \) perpendicular to and bisecting \( AK \).

Join \( EA \), which bisects \( \angle BAC \) and contains the incentre \( I \); draw \( ID, NM \) perpendicular to \( BC \). Join \( AF \) and draw \( AG \) perpendicular to \( EF \); also draw \( PIQ \) parallel to \( BC \) and meeting \( KF \) in \( P \) and \( AX \) in \( Q \).

Then we have \( AH = 2O_1A_1, HK = 2HX, AI, IE = 2Rr \).

Also from similar triangles \( \frac{PI}{IE} = \frac{FG}{AF} \) and \( \frac{IQ}{AI} = \frac{AF}{FE} \).

Thus \( \frac{PI \cdot IQ}{AI \cdot IE} = \frac{FG}{FE} \), so that \( \frac{PI \cdot IQ}{2R \cdot r} = \frac{FG}{2R} \), and \( PI \cdot IQ = r \cdot FG \).

Now the projection of \( IN \) on \( FE = ID - NM = r - \frac{1}{2}(OA_1 + HX) \)

\( = r - \frac{1}{4}(AH + HK) = r - \frac{1}{2}AY \).

(11)
Hence the square of this projection \[ r^2 - r \cdot AY + \frac{1}{4}AY^2 \]
\[ = r^2 - r \cdot GO + \frac{1}{4}AY^2 \] ............(1)

Again, the square of the projection of \( IN \) on \( BC \) \[ = DM^2 = A_M^2 - A_D \cdot DX \]
\[ = \frac{1}{4}A_X^2 - PL \cdot IQ \]
\[ = \frac{1}{4}OY^2 - r \cdot FG \] .................(2)

Adding the results (1) and (2) we get
\[ I_N^2 = \frac{1}{4} (AY^2 + OY^2) - r (FG + GO) + r^2 \]
\[ = \frac{1}{4}R^2 - r \cdot R + r^2. \]

Thus \( IN = \frac{1}{2}R - r \), and the theorem is proved for the incircle. The proof for an excircle proceeds on exactly similar lines.

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