THE SYMMETRIC GENUS OF 2-GROUPS

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Abstract. Let $G$ be a finite group. The symmetric genus $\sigma(G)$ is the minimum genus of any Riemann surface on which $G$ acts faithfully. We show that if $G$ is a group of order $2^m$ that has symmetric genus congruent to 3 (mod 4), then either $G$ has exponent $2^{m-3}$ and a dihedral subgroup of index 4 or else the exponent of $G$ is $2^{m-2}$. We then prove that there are at most 52 isomorphism types of these 2-groups; this bound is independent of the size of the 2-group $G$. A consequence of this bound is that almost all positive integers that are the symmetric genus of a 2-group are congruent to 1 (mod 4).

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1. Introduction. A finite group $G$ can be represented as a group of automorphisms of a compact Riemann surface. In other words, $G$ acts on a Riemann surface. The symmetric genus $\sigma(G)$ is the minimum genus of any compact Riemann surface on which $G$ acts faithfully.

The origins of this parameter can be traced back over a century to the work of Hurwitz, Poincare, Burnside and others. We use the modern terminology introduced in [16]. There is now a substantial body of work on the symmetric genus parameter.

A natural problem is to determine the positive integers that occur as the symmetric genus of a group (or a particular type of group). Indeed, whether or not there is a group of symmetric genus $n$ for each value of the integer $n$ remains a challenging open question; see the recent, important article [4]. Here, we restrict our attention to 2-groups. The 2-groups are interesting in this context because of the well-known conjecture that, among the finite groups, almost all groups are 2-groups.

The only 2-groups of even genus are the classical 2-groups of genus 0 [11, Theorem 9]. In other words, if $G$ is a 2-group with positive symmetric genus, then $\sigma(G)$ is odd. The 2-groups with positive genus are our focus here, and we show that the 2-groups with symmetric genus congruent to 3 modulo 4 are special indeed. In particular, we show that a group $G$ of order $2^m$ acting on a Riemann surface of genus $g \equiv 3 \pmod{4}$ must contain an element of order $2^{m-3}$ or larger. Further, if $\text{Exp}(G) = 2^{m-3}$, then $G$ contains a dihedral subgroup of index 4. This yields the following result.

**Theorem 1.** Let $G$ be a group of order $2^m$. If $\sigma(G) \equiv 3 \pmod{4}$, then either $\text{Exp}(G) = 2^{m-3}$ and $G$ has a dihedral subgroup of index 4 or else $\text{Exp}(G) = 2^{m-2}$.

Thus, if the symmetric genus $\sigma(G) \equiv 3 \pmod{4}$, then $G$ is a group of one of two types. First, it may be that $G$ has exponent $\text{Exp}(G) = 2^{m-2}$, that is, $G$ has a cyclic subgroup of index 4 but no cyclic subgroup of index 2. The families of 2-groups with
this property were classified, long ago, by Burnside [3] and Miller [13, 14]. There are two abelian groups and 25 non-abelian groups of this type of order $2^m$, as long as $m \geq 6$. It is easy to see that the two abelian groups have symmetric genus 1.

The other possibility for a group $G$ with $\sigma(G) \equiv 3 \pmod{4}$ is that $\text{Exp}(G) = 2^{m-3}$, and further, $G$ has a dihedral subgroup of index 4. These 2-groups are our main focus here, and we obtain a complete classification of the 2-groups of this type. We show that if $m \geq 7$, there are exactly 27 isomorphism types of these 2-groups (There are fewer for small orders.). The important thing here is that this number of isomorphism types is independent of the size of the 2-group $G$.

With this classification and the earlier one of Burnside and Miller, our Theorem 1 gives the following.

**Theorem 2.** Let $G$ be a group of order $2^m$. If $\sigma(G) \equiv 3 \pmod{4}$, then there are at most 52 possible isomorphism types for the group $G$.

Of the 52 possible groups of each order, relatively few actually have genus congruent to 3 (mod 4). We do not attempt to classify those families with genus congruent to 3 (mod 4), but such infinite families exist. A consequence of [4, Theorem 3.1] is that every group in Miller’s family $M_5$ (see [12, Table 2]) has genus congruent to 3 (mod 4). Also, each group in the infinite family $H_6$ (defined in Table 2) has genus congruent to 3 (mod 4).

The upper bound of Theorem 2 allows us to establish some interesting results using the standard notion of density. We consider the general problem of determining whether there is a 2-group of symmetric genus $g$, for each value of $g$. Let $T$ be the set of integers $g \geq 2$ for which there is a 2-group of symmetric genus $g$; we know that $T$ only contains odd integers. Suppose $T_3$ is the subset of $T$ consisting of the integers congruent to 3 (mod 4). Then $T_3$ is infinite, due to the genus formulas for the families $M_5(m)$ and $H_6(m)$. Our main results concerning density are the following.

**Theorem 3.** The set $T_3$ has density 0 in the set of positive integers.

**Theorem 4.** Almost all positive integers that are the symmetric genus of a 2-group are congruent to 1 (mod 4). Further, the density $\delta(T)$ is at most $1/4$.

Theorem 4 has an interesting interpretation in connection with the conjecture that among the finite groups, almost all groups are 2-groups. If this conjecture holds (as it almost certainly does), then our results would imply that almost all groups have symmetric genus congruent to 1 (mod 4).

Not surprisingly, Theorems 3 and 4 agree with the companion results [12] about the strong symmetric genus, a closely related parameter. The general approach in [12] is along similar lines, but, in fact, the proofs there are easier. This is, however, one instance where work on one parameter suggests the companion results about a related parameter.

**2. Preliminaries.** The groups of symmetric genus 0 are the classical, well-known groups that act on the Riemann sphere (possibly reversing orientation) [8, Section 6.3.2]. The groups of symmetric genus 1 have also been classified, at least in a sense. These groups act on the torus and fall into 17 classes, corresponding to quotients of the 17 Euclidean space groups [8, Section 6.3.3]. Each class is characterized by a presentation, typically a partial one.
For each value of the genus $g \geq 2$, there are only a finite number of groups with symmetric genus $g$. This is essentially Hurwitz’s classical bound for the size of the automorphism group of a Riemann surface. We use the standard well-known approach to group actions on surfaces of genus $g \geq 2$. Let the finite group $G$ act on the (compact) Riemann surface $X$ of genus $g \geq 2$. Then represent $X = U/K$, where $K$ is a Fuchsian surface group and obtain an non-Euclidean crystallographic (NEC) group $\Gamma$ and a homomorphism $\phi : \Gamma \rightarrow G$ onto $G$ such that $K = \text{kernel } \phi$. Associated with the NEC group $\Gamma$ are its signature and canonical presentation. It is basic that each period and each link period of $\Gamma$ divide $|G|$. Further, the non-Euclidean area $\mu(\Gamma)$ of a fundamental region for $\Gamma$ can be calculated directly from its signature [15, p. 235]. Then the genus of the surface $X$ on which $G$ acts is given by

$$g = 1 + |G| \cdot \mu(\Gamma)/4\pi. \quad (1)$$

There are four families of non-abelian 2-groups that possess a cyclic subgroup of index 2. A good reference for these groups is [7, Section 5.4]. These families can be constructed using the non-trivial automorphisms of a cyclic 2-group. The automorphism group is well-known; for $n \geq 3$, we have

$$\text{Aut}(\mathbb{Z}_{2^n}) = \langle -1 \rangle \times \langle 5 \rangle \cong \mathbb{Z}_2 \times \mathbb{Z}_{2^n-2}. \quad (2)$$

These power automorphisms are detailed in [7, Lemma 4.1, p. 189]. Three of these families of 2-groups will be needed here, and we describe these three.

For $m \geq 2$, let $D(m)$ be the group with generators $x, y$ and defining relations

$$x^{2^{m-1}} = y^2 = 1, yxy = x^{-1}. \quad (3)$$

The group $D(m)$ is the dihedral group of order $2^m$. Each dihedral group has symmetric genus 0.

For $m \geq 4$, let $QD(m)$ be the group with generators $x, y$ and defining relations

$$x^{2^{m-1}} = y^2 = 1, yxy = x^{-1+2^{m-2}}. \quad (4)$$

The group $QD(m)$ of order $2^m$ is called a quasi-dihedral group (or semi-dihedral group) [7, p. 191]. This group has symmetric genus 1 [10, Theorem 2].

For $m \geq 4$, let $QA(m)$ be the group with generators $x, y$ and defining relations

$$x^{2^{m-1}} = y^2 = 1, y^{-1}xy = x^{1+2^{m-2}}. \quad (5)$$

The group $QA(m)$ is a non-abelian group of order $2^m$ [7, p. 190]; we call this group quasi-abelian [10, p. 237]. This group also has symmetric genus 1 [10, Theorem 3].

The fourth family consists of the dicyclic groups [6, pp. 7, 8]; each dicyclic group has symmetric genus 1 [10, p. 236].

The three automorphisms of order 2 of the maximal cyclic group will be called inversion, the quasi-dihedral action and the quasi-abelian action; these actions are given in (3), (4) and (5), respectively. Inversion is also used to construct a dicyclic group, but the element of the group that gives rise to the inner automorphism which is inversion has order 4.

Two additional families of 2-groups will be important here. Each of these groups has a dihedral subgroup of index 2. First, for $m \geq 4$, let $CD(m)$ be the group with
generators $x, y, z$ and defining relations

$$x^{2^{m-2}} = y^2 = z^4 = (xy)^2 = 1, xz = zx, yz = zy, z^2 = x^{2^{m-3}}.$$  (6)

This group is the central product of the dihedral group $D(m - 1)$ and a cyclic group of order 4. We call $CD(m)$ a $CD$ group. Each of these groups is also toroidal, and $\sigma(CD(m)) = 1$ [11, Theorem 5].

For $m \geq 5$, let $HD(m)$ be the group with generators $x, y, z$ and defining relations

$$x^{2^{m-2}} = y^2 = z^2 = (xy)^2 = (yz)^2 = 1, yz = z^{-1} + 2^{m-3}.$$  (7)

This interesting group of order $2^m$ has a dihedral subgroup $\langle x, y \rangle$ of index 2 as well as a quasi-dihedral subgroup $\langle x, z \rangle$ of index 2. We call $HD(m)$ a *hyperdihedral* group [9, p. 113]. Each group in this family acts on the torus, that is, $\sigma(HD(m)) = 1$ [11, Theorem 4].

Each of the groups $HD(m)$ and $CD(m)$ contains a dihedral subgroup of index 2 and has exponent $2^{m-2}$. Among the 2-groups with exponent $2^{m-2}$, the only other group with a dihedral subgroup of index 2 is the direct product $\mathbb{Z}_2 \times D(m - 1)$. For $m \geq 4$, this group has generators $x, y, z$ and defining relations

$$x^{2^{m-2}} = y^2 = z^2 = (xy)^2 = 1, xz = z, yz = zy.$$  (8)

The following classification is in [9, Theorem 9]; this result will be important here.

**Theorem A.** Let $G$ be a group of order $2^m$ with a dihedral subgroup $M$ of index 2, with $m \geq 5$. If $G$ has no element of order $2^{m-1}$, then $G$ is isomorphic to $\mathbb{Z}_2 \times M$, $HD(m)$ or $CD(m)$.

We established in [11, Theorem 9] that the only 2-groups of even genus are those that act on a Riemann sphere and have genus 0. Important in the proof of the following are the 2-groups with a maximal cyclic subgroup as well as the groups $HD(m)$ and $CD(m)$.

**Theorem B.** Let $G$ be a 2-group with positive symmetric genus. Then $\sigma(G)$ is odd.

### 3. 2-groups of odd genus.

Here, we consider a 2-group $G$ acting on a Riemann surface of genus $g \equiv 3 \pmod{4}$ and obtain a refinement of [11, Theorem 7] in this case.

**Theorem 5.** Let $G$ be a group of order $2^m$ that acts on a Riemann surface $X$ of genus $g \equiv 3 \pmod{4}$. Then $G$ contains an element of order $2^{m-3}$ or larger. If $\text{Exp}(G) = 2^{m-3}$, then, further, $G$ contains a dihedral subgroup of index 4.

**Proof.** Suppose $G$ acts on the Riemann surface $X$ of genus $g \geq 2$ where $g \equiv 3 \pmod{4}$. Represent $X = U/K$, where $K$ is a Fuchsian surface group and obtain an NEC group $\Gamma$ and a homomorphism $\phi : \Gamma \to G$ onto $G$ such that $K = \text{kernel } \phi$. The NEC group $\Gamma$ has signature

$$(\rho; \pm; [\lambda_1, \cdots, \lambda_r]; \{C_1, \cdots, C_k\}),$$

where each period cycle $C_i$ is either empty or contains the link periods $n_{i1}, \cdots, n_{ik}$.

Each link period is the order of a product of involutions in the presentation for $\Gamma$. For more information about signatures, see [15].
Since $K$ is a surface group, each period $\lambda_i$ and each link period $n_j$ must be the order of an element of $G$. The non-Euclidean area is given by

$$
\mu(\Gamma)/2\pi = \varepsilon p - 2 + k + \sum \left(1 - \frac{1}{\lambda_i}\right) + \frac{1}{2} \sum \left(1 - \frac{1}{n_j}\right),
$$

where $\varepsilon = 1$ or $2$ [15, p. 235]. Now write $g = 4t + 3$ for some integer $t$. Then using (1), we have

$$
3 + 4t = 1 + 2^{m-1} \left(\varepsilon p - 2 + k + \sum \left(1 - \frac{1}{\lambda_i}\right) + \frac{1}{2} \sum \left(1 - \frac{1}{n_j}\right)\right),
$$

$$
1 + 2t = 2^{m-2} \left(\varepsilon p - 2 + k + \sum \left(1 - \frac{1}{\lambda_i}\right) + \frac{1}{2} \sum (1 - \frac{1}{n_j})\right).
$$

It follows that the sum

$$
\sum \left(\frac{2^{m-2}}{\lambda_i}\right) (\lambda_i - 1) + \sum \left(\frac{2^{m-3}}{n_j}\right) (n_j - 1)
$$

must be an odd integer. But this clearly would not be the case if $\text{Exp}(G) \leq 2^{m-4}$. Hence, $\text{Exp}(G) \geq 2^{m-3}$.

Suppose that $\text{Exp}(G) = 2^{m-3}$. In this case, an odd number of the link periods must equal to $2^{m-3}$. Then suppose that the specific link period $n_j = 2^{m-3}$. Now in the group $G$, there are generating reflections $c_{i,j-1}$ and $c_{i,j}$ with $n_{i,j} = o(c_{i,j-1} \cdot c_{i,j})$. It follows that $\langle c_{i,j}, c_{i,j-1} \rangle \cong D(m - 2)$ in $G$, and hence, $G$ has a dihedral subgroup of index 4 in this case.

**Proof of Theorem 1.** By the previous result, $\text{Exp}(G)$ must be at least $2^{m-3}$. First, $G$ is not cyclic, since a cyclic group has symmetric genus 0. Suppose then that $G$ contained an element of order $2^{m-1}$. If $G$ were abelian, then $G$ would be isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_{2m-1}$, a group of genus 0. Thus, $G$ must be non-abelian and either dihedral, dicyclic, quasi-dihedral or quasi-abelian [7, Theorem 4.4, p. 193]; but each of these groups has genus 0 or 1. Hence, $\text{Exp}(G)$ is either $2^{m-2}$ or $2^{m-3}$.

Thus, if $\sigma(G) \equiv 3 \pmod{4}$, then $G$ is a group of one of two types. First, the families of 2-groups with exponent $2^{m-2}$ were classified, about a century ago, by Burnside [3] and Miller [13, 14]. There are exactly 27 groups of this type of order $2^m$, as long as $m \geq 6$; two of these are abelian. First, if $G$ is abelian, then $G$ is isomorphic to $\mathbb{Z}_4 \times \mathbb{Z}_{2m-2}$ or $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_{2m-2}$. But each of these groups has symmetric genus 1 [8, pp. 291, 292]; these groups are in classes (a) and (h), respectively. The non-abelian groups of this type were studied in [12]. In particular, Table 1 of [12] gives a presentation for each of the 25 non-abelian groups.

**4. Groups with dihedral subgroups of index 4.** Here, we study the families of 2-groups that have dihedral subgroups of index 4 but no cyclic subgroups of index 4. There are 27 groups of this type of order $2^m$, for each $m \geq 7$.

We use the following notation in all cases. Let $G$ be a group of order $2^m$ with a dihedral subgroup of index 4 such that $\text{Exp}(G) = 2^{m-3}$. Assume that the dihedral subgroup $M \cong D(m - 2)$ has generators $x$ and $y$ satisfying the relations (3), with $H = \langle x \rangle$ a cyclic subgroup of index 8. Then, $G$ has a subgroup $L$ of index 2 that
contains the dihedral subgroup $M$. By Theorem A, $L$ is isomorphic to $CD(m - 1)$, $HD(m - 1)$ or $\mathbb{Z}_2 \times M$. For each of these three possibilities for the subgroup $L$, we determine the number of isomorphism types for $G$.

To construct each group $G$, we use a standard, well-known technique [6, p. 5]. To the group $L$, we adjoin a new element $s$, with conjugation by $s$ transforming the elements of $L$ according to an automorphism of order 2. We identify $s^2$ with a central element $u$ of order $j$. Then the larger group $G$ has order $2|L|$. The defining relations for $G$ consist of the relations for $L$, the relations defining the action of $s$ on each generator of $L$ and the relation $s^2 = u$. This general construction suffices in almost all cases.

**PROPOSITION 1.** Let $G$ be a group of order $2^m$, with $m \geq 7$ and $\text{Exp}(G) = 2^{m-3}$. If $G$ contains a subgroup $L \cong CD(m - 1)$, then $G$ is isomorphic to one of seven groups; each group is an extension of $L$ with an added generator $s$ and added relations listed in Table 1.

**Proof.** The subgroup $L \cong CD(m - 1)$ has generators $x$, $y$ and $z$ satisfying (6). Then the centre $Z(L) = \langle z \rangle$ and $M$ is the unique dihedral subgroup of $L$ with index 2. The group $L$ contains two cyclic subgroups of maximal order. These subgroups are $\langle x \rangle$, which is contained in $M$, and $\langle xz \rangle$, which is contained in the quasi-dihedral subgroup $\langle xz, y \rangle$. Thus, $\langle z \rangle$, $H$ and $M$ are characteristic in $L$, and these three subgroups are normal in $G$. Let $C$ be the centralizer of $H$ in $G$. Clearly, $\langle x, z \rangle \subseteq C$, but $C \neq G$, since $y$ is not in $C$. Hence, $\langle G : C \rangle$ is 2 or 4. In either case, $G/C$ is isomorphic to a subgroup of the automorphism group $\text{Aut}(H)$, and $\text{Aut}(H)$ is isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_{2^{m-5}}$, where the $\mathbb{Z}_2$ factor is generated by the inversion $\alpha(x) = x^{-1}$ [7, p. 189].

**CASE I.** Suppose first that $\langle G : C \rangle = 4$. Then we must have $C = \langle x, z \rangle$. In this case, $G/C \cong \mathbb{Z}_2 \times \mathbb{Z}_2$, with one $\mathbb{Z}_2$ factor generated by inversion and the other $\mathbb{Z}_2$ factor generated by the automorphism $\beta(x) = x^{-1+2^{m-4}}$ (the quasi-dihedral action) [7, p. 189]. Hence, there is an element $s \in G - L$ such that $s^{-1}xs = \beta(x) = x^{-1+2^{m-4}}$. Then, easily, $s^{-2}xz^2 = x$, so that $s^2$ is in the centralizer $C$. Now we have $G = \langle x, y, z, s \rangle$.

Since $\langle z \rangle$ is normal in $G$, we must have either $s^{-1}zs = z$ or $s^{-1}zs = z^{-1}$. Also, since the dihedral subgroup $M = \langle x, y \rangle$ is normal, $s^{-1}ys = x^\ell y$ for some integer $\ell$.

Assume first that $s^{-1}zs = z$. Then $s^2$ commutes with $z$ so that $s^2$ is in $Z((s, x, z)) = \langle x^{2^{m-4}}, z \rangle = \langle z \rangle$. By replacing $s$ with $sz$, if necessary, we may assume that either $s^2 = 1$ or $s^2 = z$. In either case, $s^2$ commutes with $y$. Since $M$ is normal, $s^{-1}ys = x^\ell y$ for some integer $\ell$. Now $y = s^{-2}ys^2 = s^{-1}x^\ell y = x^{2^{m-4}\ell}y$ and so $\ell$ is even. Write $\ell = 2k$, and then replace $y$ by $x^{k}y$, and we get the same relations with either $s^{-1}ys = y$ or $s^{-1}ys = x^{2^{m-4}}y$. In the latter case, replace $y$ by $x^{2^{m-5}}y$ and we get the relation $s^{-1}ys = y$. This gives the two groups $J_1$ and $J_2$, with $s^2 = 1$ and $s^2 = z$, respectively.

Next, assume that $s^{-1}zs = z^{-1}$. Now $s^2 \in Z((s, x, z)) = \langle z^2 \rangle$ and so $s^2 = 1$ or $s^2 = z^2$. In both cases, by the same argument as before, $s^{-1}ys = y$. Then with $s^2 = 1$,
we have the group $J_3$. The group $G$ with $s^2 = z^2$ is isomorphic to $J_3$ by the map $\psi : G \to J_3$ defined by $x \mapsto x$, $y \mapsto x^{-1}y$, $z \mapsto z$ and $s \mapsto sx^{1+2m-5}z$.

**Case II.** Suppose that $[G : C] = 2$. Since inversion is the only non-trivial action on $H$ by any element of $G$, we may choose $s \in G - L$ so that $s \in C$ and $sx = xs$. Now $s^2 \in \langle x, z \rangle$, since $s^2 \in L \cap C$. It is clear that $s^2 = x^{2k}$ or $s^2 = x^{2k}z$, since $o(s) \leq o(x)$. We can replace $s$ by $(x^{-k}s)$ and assume without loss of generality that $s^2 = 1$ or $s^2 = z$. Furthermore, since $\langle z \rangle$ is normal in $G$, we know that $s^{-1}z = z$ or $s^{-1}z = z^{-1}$. Finally, since $M$ is normal in $G$, we have $s^{-1}ys = x^t y$ for some integer $t$. Now $y = s^{-1}ys^2 = s^{-1}x^ty = x^{2t}y$, and $t = 0$ or $t = 2m-4$.

Suppose that $s^{-1}z = z$. There are two possibilities for $s^2$ and two possibilities for the action of $s$ on $y$. Assume first that $s^{-1}ys = y$. Then with $s^2 = 1$ and $s^2 = z$, we obtain the groups $J_4$ and $J_5$, respectively. The group $G$ with $s^2 = 1$ and $s^{-1}ys = x^{2m-4}y$ is isomorphic to $J_4$ by the map $\theta : G \to J_4$ defined by $x \mapsto x$, $y \mapsto y$, $z \mapsto z$ and $s \mapsto sxy^{2m-5}$. The group $G$ with $s^2 = z$ and $s^{-1}ys = x^{2m-4}y$ is isomorphic to $J_5$ by the map $\phi : G \to J_5$ defined by $x \mapsto x$, $y \mapsto y$, $z \mapsto z^{-1}$ and $s \mapsto sx^{2m-5}$.

Suppose that $s^{-1}z = z^{-1}$. It is easy to see that this forces $s^2 = 1$, and we have two groups, $J_6$ with $s^{-1}ys = y$ and $J_7$ with $s^{-1}ys = x^{2m-4}y$. These are the final two groups of this type.

It is not hard to check that in each of the seven presentations, the action of $s$ does define an automorphism of $L$, and it follows that $G$ is a group of order $2m$ by the general construction of [6, p. 5].

No two of these seven groups are isomorphic. These groups can be distinguished using three group invariants. First, the centre and the abelian quotient invariants suffice to distinguish all but $J_3$, $J_6$ and $J_7$; these two invariants agree for these three groups. These groups have different numbers of involutions; these counts are not difficult, just from the presentations.

Next, we consider the second type of group, one with a hyperdihedral subgroup of index 2. We omit some details.

**Proposition 2.** Let $G$ be a group of order $2m$, with $m \geq 7$ and $\text{Exp}(G) = 2^{m-3}$. If $G$ contains a subgroup $L \cong \text{HD}(m - 1)$, then $G$ is isomorphic to one of 10 groups; each group is an extension of $L$ with an added generator $s$ and added relations listed in Table 2.
Proof. The subgroup $L$ has generators $x$, $y$ and $z$ satisfying (7). Then $L$ has two cyclic subgroups of maximal order, namely, $\langle x \rangle$ and $\langle xyz \rangle$. The proof splits into two cases, depending upon whether or not these subgroups are normal in $G$.

CASE I. Suppose $\langle x \rangle$ is normal in $G$. Thus, $G/C_G(x)$ has order 4 or 8.
Assume first that $|G/C_G(x)| = 4$. We can find an element $s$ in $G - L$ that centralizes $x$. Since $s^2 \in C_L(x)$, we see that $s^2 \in \langle x \rangle$ and $k$ must be even. Since $\langle x, s \rangle$ is abelian, choose $s$ so that $s^2 = 1$. A consideration of the possible actions of $s$ on $y$ and of $s$ on $z$ gives four possible presentations and the groups $H_1$ to $H_4$ in Table 2.

Suppose that $|G/C_G(x)| = 8$, and let $s$ be any element of $G - L$. Then the automorphism of $\langle x \rangle$ given by conjugation by $s$ has order 4. By multiplying $s$ by the appropriate element of $L$, we may suppose that $s^{-1}xs = x^{1 + 2m - 3}$. It follows that we must have $s^{-1}ys = xy^k$ and also $s^{-1}zs = zy^k$, with $k = 0$ or $k = 2m - 4$. The two choices for $k$ yield two presentations, but both lead to the group $H_5$.

CASE II. Suppose that $\langle x \rangle$ is not normal in $G$. Any element $s$ in $G - L$ must intercalate the subgroups $\langle x \rangle$ and $\langle xyz \rangle$, and hence $s^{-1}xs = x^{k}yz$, where $k$ is odd. Therefore, $s^{-1}x^2s = (xyz)^2 = (x^2)^{(1 + 2m - 5)}$ and $\langle x^2 \rangle$ is normal in $G$. It follows that $k \equiv 1, -1, 1 + 2m - 5$ or $-1 + 2m - 5 \pmod{2m - 4}$. By choosing a suitable element $s$ in $G - L$, we have either $s^{-1}xs = xy^z$ or $s^{-1}xs = x^{1 + 2m - 5}yz$. Either way, we may assume that $s^2 = x^{2\ell}$, where $\ell = 1$ or $\ell$ is even, and also that $s^{-1}ys = zy^{2\ell}$.

Suppose that $s^{-1}xs = xy^z$. If $s^2 = x^2$, we can derive a contradiction. Therefore, $s^2 = x^{4k}$ for some integer $k$. By replacing $s$ with the appropriate element, we have $s^2 = 1$ and $1 = x^{-4r + 12m - 4}$. Consequently, $G$ either has relations $s^{-1}ys = zy^{2m - 4}$ and $s^{-1}zs = yx^{2m - 4}$ or else $s^{-1}ys = z$ and $s^{-1}zs = y$. We now have two complete presentations; each defines the group $H_6$ in Table 2.

Suppose that $s^{-1}xs = x^{1 + 2m - 5}yz$. We also know that $s^{-1}ys = zy^{2\ell}$ and $s^2 = x^{2\ell}$, where $\ell = 1$ or $\ell$ is even. In either case, we have $s^{-1}zs = yx^{2\ell - 2m - 4}$.

First, suppose that $s^2 = x^2$. It can be shown that $s^{-1}ys = zy^{2 + 2m - 5}$ and $s^{-1}zs = yx^{2 + 2m - 5}$. This leads to groups $H_7$ and $H_8$ in Table 2. Finally, if $s^2 = x^{4k}$, the same type of calculation gives the final two groups, $H_9$ and $H_{10}$, in Table 2.

In seven of the presentations, the action of $s$ defines an automorphism of $L$, and the general construction of [6, p. 5] shows that we have a group of order $2^m$. The general construction will also handle the groups $H_7$ and $H_8$, if we first replace $s$ by $s_1 = sxy$. This gives alternate presentations of $H_7$ and $H_8$. Finally, by eliminating the redundant generator $z$ from the presentation for $H_5$, we see that $H_5$ is isomorphic to a semi-direct product of $D(m - 2)$ by $\mathbb{Z}_4$.

No two of these 10 groups are isomorphic. The first five groups can be distinguished using the centre, the abelian quotient invariants and the fact that $H_2$ and $H_3$ are isomorphic to $J_6$ and $J_3$, respectively. These invariants also distinguish the first five groups from the second five groups. Among the remaining five groups, $H_7$ is the only one not generated by involutions and only $H_9(m)$ has a subgroup isomorphic to $\mathbb{Z}_2 \times D(m - 2)$. The groups $H_6$ and $H_{10}$ can be distinguished from $H_8$ because the quotients of each by $\langle x \rangle$ are different groups of order 32. The group $\langle x \rangle$ is contained in the intersection of all of the cyclic subgroups of maximal order. Finally, the third centre of $H_6$ is an abelian subgroup of order 32, and the third centre of $H_{10}$ is a non-abelian subgroup of order 32.
Now we consider the third type of index 2 subgroup. Again, we provide an outline of the proof but omit quite a few details.

**Proposition 3.** Let $G$ be a group of order $2^m$, with $m \geq 7$ and $\text{Exp}(G) = 2^{m-3}$. If $G$ contains a subgroup $L \cong \mathbb{Z}_2 \times D(m-2)$, then $G$ is isomorphic to one of 17 groups; each group is an extension of $L$ with an added generator $s$ and added relations listed in Table 3.

**Proof.** The subgroup $L$ has generators $x$, $y$, and $z$ satisfying (8). The centre $Z(L) = \langle x^{2^{m-4}}, z \rangle$ is isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$. There are two maximal cyclic subgroups of $L$, namely $\langle x \rangle$ and $\langle zx \rangle$. The proof splits into two cases, depending upon whether or not these subgroups are normal in $G$.

**Case I.** Suppose that $\langle x \rangle$ is normal in $G$. Now $C_L(x) = \langle x, z \rangle$ and it follows that $G/C_L(x)$ has order 2 or order 4. We consider these two possibilities in two subcases.

**Subcase a.** Suppose that $G/C_L(x)$ has order 4. Then we can find an element $s$ in $G/L$ such that $s^{-1}xs = x^{1+2^{m-4}}$. It is easy to see that either $s^{-1}zs = z$ or $s^{-1}zs = zx^{2^{m-4}}$. Further, by replacing $s$ by an element of the form $x^{-i}s$ if necessary, we may assume that either $s^2 = 1$ or $s^2 = z$.

Now suppose that $s^{-1}zs = z$. First, it is clear that $s^2 = z$ is not possible and so $s^2 = 1$. Again, it follows that $s^{-1}ys = x^ky$ or $s^{-1}ys = x^kyz$. These two relations (with appropriate values for $k$) lead to groups isomorphic to one of these four.

**Subcase b.** Suppose that $G/C_L(x)$ has order 2. Then we can find an element $s$ in $G/L$ that $s^{-1}xs = x$. Also, $s \in C_L(x) = \langle x, z \rangle$, and therefore $s^2 = x^4$ or $s^2 = x^4z$ for

<table>
<thead>
<tr>
<th>Name</th>
<th>$s^{-1}xs =</th>
<th>$s^{-1}ys =</th>
<th>$s^{-1}zs =</th>
<th>$s^2 =</th>
</tr>
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<tr>
<td>$A_1$</td>
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<td>$y$</td>
<td>$z$</td>
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<tr>
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<tr>
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<td>$zy$</td>
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<tr>
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<td>$zy$</td>
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</tr>
<tr>
<td>$A_5$</td>
<td>$x^{1+2^{m-4}}$</td>
<td>$y$</td>
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</tr>
<tr>
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<td>$1$</td>
</tr>
<tr>
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<tr>
<td>$A_{12}$</td>
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<td>$x^{2^{m-4}}yz$</td>
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</tr>
<tr>
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<td>$x^{-1}z$</td>
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<td>$1$</td>
</tr>
<tr>
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<td>$z$</td>
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</tr>
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<td>$y$</td>
<td>$z^{x^{2^{m-4}}}$</td>
<td>$1$</td>
</tr>
</tbody>
</table>
some integer $\ell$. As before, we can get $s^2 = 1$ or $s^2 = z$. It is also easy to see that either $s^{-1}zs = z$ or $s^{-1}zs = z\chi^{2m-4}$. If $s^{-1}zs = z\chi^{2m-4}$, then with some work, we are back in subcase (a) and so $s^{-1}zs = z$.

As before, $s^{-1}ys = x^ky$ or $s^{-1}ys = x^kyz$ and $k = 0$ or $k = 2m-4$. So, there are four possible actions of $s$ on $y$ and with the two possibilities for $s^2$, again there are eight presentations. In this subcase, there are six different groups, the groups $A_7$–$A_{12}$.

**Case II.** Suppose that $\langle x \rangle$ is not normal in $G$. Any element $s$ in $G - L$ conjugates $\langle x \rangle$ to $\langle xz \rangle$. As before, $\langle x^2 \rangle$ is normal in $G$. Choosing $s$ carefully, we get $s^{-1}xs = x^{-1+2m-5}z$ for $\ell$ equal to 0, 1, 2 or 3. As before, $s^{-1}zs = z$ or $s^{-1}zs = z\chi^{2m-4}$.

If $s^{-1}zs = z$, without loss of generality, $s^{-1}xs = x^{-1}z$ and we have $s^2 \in C_{G}(x,z) = \langle x^{2m-4}, z \rangle$. We only need to consider two actions of $s$ on $y$, $s^{-1}ys = y$ and $s^{-1}ys = yz$. With the relation $s^{-1}ys = y$, we obtain the groups $A_{13}$–$A_{16}$ in Table 3. The other action $s^{-1}ys = yz$ yields the same four groups.

Finally, suppose that $s^{-1}zs = z\chi^{2m-4}$. By replacing generators, if needed, we may assume that $s^{-1}xs = x^{-1+2m-5}z$ and $s^{-1}ys = y$, and there are two presentations, depending on the value of $s^2$. Both presentations yield $A_{17}$. This completes the listing of the groups in Table 3.

In each of the 17 presentations, the action of $s$ defines an automorphism of $L$, and consequently, in each case, we obtain a group of order $2^m$ by the general construction of [6, p. 5].

Finally, we need to check that no two of these 17 groups are isomorphic. First, the centre and the abelian quotient invariants distinguish $A_1$, $A_2$, $A_3$, $A_7$, $A_8$ and $A_{11}$. Further, these two invariants separate the other 11 into two sets, the pair $A_6$ and $A_{17}$ and the remaining nine. Case I and case II groups cannot be isomorphic. This distinguishes $A_6$ from $A_{17}$ and helps with the others. Of these nine groups, only $A_{12}$ (case I) and $A_{16}$ (case II) are not generated by involutions. This leaves seven groups to consider. Then using the second centre and separating groups by the two cases distinguishes $A_{15}$ and divides the others into three pairs, $A_3$ and $A_9$, $A_4$ and $A_{10}$, and the pair $A_{13}$ and $A_{14}$. In the groups $A_3$ and $A_4$, the centralizer of a maximal order cyclic subgroup that is normal has index 4, whereas in $A_9$ and $A_{10}$, it has index 2. Finally, to separate the pair $A_{13}$ and $A_{14}$, we use quotient groups by the three central subgroups of order 2. Two of the corresponding subgroups give isomorphic quotients. However, the group $A_{13}$ has a third quotient group isomorphic to $Z_2 \times D(m-2)$, but the corresponding quotient of $A_{14}$ is isomorphic to $CD(m-1))$. These are the quotients by $\langle z \rangle$ in our presentations.

**Theorem 6.** Let $G$ be a group of order $2^m$, with $m \geq 7$. If $G$ has a dihedral subgroup of index 4 such that $\text{Exp}(G) = 2^{m-3}$, then $G$ is isomorphic to one of 27 groups, independent of $m$.

**Proof.** Theorem A and Propositions 1–3 show that there are at most 34 groups. Table 4 gives seven isomorphisms among these three types of groups.

Finally, it is necessary to show that there are no further isomorphisms among the remaining 27 groups. A careful consideration of the centre and the abelian quotient invariants for these groups distinguishes nine of the groups and separates the others into three sets, the trio $J_3$, $J_6$ and $J_7$, the set of nine $A_{6}$s considered in the proof of Proposition 3, and a final set of six, $A_6$ and the set of five $H_s$s considered in the proof of Proposition 2. Since the groups of each type have been classified, the only possible
remaining isomorphism is between \( A_6 \) and some \( H_j \). But it is not hard to see that the group \( A_6 \) does not have a hyperdihedral subgroup of index 2. This completes the classification. □

Now Theorem 1, the classification of Burnside and Miller, and the classification of Theorem 6 combine to establish Theorem 2.

Of the 52 possible 2-groups of each order, relatively few actually have genus congruent to 3 (mod 4). Some have symmetric genus 1, and some have higher genus \( \sigma(G) \equiv 1 \) (mod 4). For example, among the groups of order 128, there are 10 groups with genus congruent to 3 (mod 4); the symmetric genus of each group was calculated using MAGMA.

Among the 52 infinite families, there are some containing groups with genus congruent to 3 (mod 4). In [4], Conder and Tucker define the following group of order \( 16n \).

\[
V_n = \langle x, y | x^4 = y^4 = [x^2, y] = [y^2, x] = 1, (xy)^{2n} = x^2 \rangle.
\]

They prove that \( \sigma(V_n) = 4n - 1 \) for all \( n > 1 \) [4, Theorem 3.1]. This gives examples of order \( 2^m \) with genus congruent to 3 (mod 4), for all \( m \geq 7 \). A little bit of work suffices to show that

\[
V_{2^{m-4}} \cong M_5(m),
\]

one of Miller’s 2-groups from [13]. The family \( H_6 \) is another family of groups with genus congruent to 3 (mod 4). We omit the proof.

**PROPOSITION 4.** Suppose that \( G \) is the group \( H_6(m) \) of order \( 2^m \), where \( m \geq 7 \). Then \( G \) has symmetric genus \( \sigma(G) = 2^{m-4} - 1 \).

### 5. Density

Now we consider the general problem of determining whether there is a 2-group of symmetric genus \( g \) for each value of \( g \), and describe our results using the standard notion of density.

Let \( T \) be the set of integers \( g \geq 2 \) for which there is a 2-group of symmetric genus \( g \). By Theorem B, all the integers in \( T \) are odd. For an integer \( n \), let \( f(n) \) denote the number of integers in \( T \) that are less than or equal to \( n \). Then the natural density \( \delta(T) \) of \( T \) in the set of positive integers is

\[
\delta(T) = \lim_{n \to \infty} \frac{f(n)}{n}.
\]
Also, let $T_3$ be the subset of $T$ consisting of the integers congruent to 3 (mod 4), with the companion “counting” function denoted by $f_3$.

Although the set $T_3$ is infinite, the upper bound of Theorem 2 suffices to prove that the density of $T_3$ in the set of positive integers is zero.

**Proof of Theorem 3.** First, among the 2-groups of order 64 or less, there are exactly 11 groups with genus congruent to 3 (mod 4). Assume $n = 2^m$, with $m \geq 7$, and let $G$ be a 2-group with genus $n$ or less such $\sigma(G) \equiv 3 \pmod{4}$. From the basic lower bound for the genus of a 2-group, we have

$$|G| \leq 32(\sigma(G) - 1) \leq 32(n - 1) < 2^{m+5},$$

so that $|G| \leq 2^{m+4}$. For each of the possible $m - 2$ orders in the range 128, 256, ..., $2^{m+4}$, there are at most 52 groups with genus congruent to 3 (mod 4), by Theorem 2. Thus, $f_3(n) = f_3(2^m) \leq 52(m - 2) + 11$, counting the 11 groups of small order. Hence, $\delta(T_3) = 0$. □

Together, Theorems B and 3 clearly imply Theorem 4.

Another interesting interpretation of our results is possible by considering group counting functions, together with the abundance of 2-groups. Here, see the recent survey article [5], together with [1] and the book [2].

For the positive integer $n$, the *group number* of $n$, denoted by $\text{gnu}(n)$, is the number of distinct abstract groups of order $n$ [5]. The values of $\text{gnu}(n)$ are given for all $n < 2,048$ in the Appendix of [5].

Let $F$ be a family of finite groups. For a positive integer $n$, let $f(n)$ denote the number of groups in the family $F$ that have order $n$ or less, and let $t(n)$ be the total number of groups of these orders. Then the natural *group density* $\Delta(F)$ of the family $F$ in the collection of finite groups is

$$\Delta(F) = \lim_{n \to \infty} \frac{f(n)}{t(n)}.$$

For small $n$, values of the counting function $t$ may be obtained by summing values of the $\text{gnu}$ function, of course.

Now let $F_2$ be the family of finite 2-groups, with companion counting function $f_2$. As is well understood, the number of 2-groups simply overwhelms the number of other groups. In fact, the following conjecture is well-known.

**Conjecture.** The group density of the 2-groups is 1, that is, $\Delta(F_2) = 1$.

We call this conjecture the density of 2-groups (D2G) conjecture. If the D2G conjecture holds, then in this sense, almost all finite groups are 2-groups.

Theorem 2 can now be given another interpretation. Let $F_1$ be the family of finite groups, each of which has symmetric genus congruent to 1 (mod 4). Now the following clearly holds; for a detailed proof of a very similar result, see [12, Theorem 9].

**Theorem 7.** If the D2G conjecture holds, then $\Delta(F_1) = 1$.

Among the finite groups, then, almost all groups would have symmetric genus congruent to 1 (mod 4) (assuming that the D2G conjecture holds). On the other hand,
there is the conjecture that for every integer \( g \geq 0 \), there is a group \( G \) with symmetric genus \( \sigma(G) = g \) [4, p. 273].

Finally, we would like to thank the referees for several helpful comments.

REFERENCES

11. C. L. May and J. Zimmerman, Groups of symmetric genus \( \sigma \leq 8 \), *Comm. Algebra* 36 (2008), 4078–4095.
13. G. A. Miller, Determination of all the groups of order \( p^m \) which contain the abelian group of type (m-2,1), \( p \) being any prime, *Trans. Am. Math. Soc.* 2 (1901), 259–272.