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A TRANSFORMATION FORMULA FOR A CLASS OF ARITHMETIC SUMS

BY

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1. Introduction. Arithmetic sums of the form

$$\sum_{n\le x} f(n)\left[\frac{x}{n}\right]$$

where f is an arithmetic function and $[\,]$ is the greatest integer function frequently occur in various situations in the theory of numbers and have much of interest in their own right. Two instances appear in the well-known results

$$\sum_{n\le x}\left[\frac{x}{n}\right]=\sum_{n\le x}\tau(n) \qquad \text{(Dirichlet)}$$

$$\sum_{n\le x}\mu(n)\left[\frac{x}{n}\right]^2=2\sum_{n\le x}\phi(n)-1 \qquad \text{(Mertens)}$$

where τ is the number of divisors function, μ is Moebius' function and ϕ is Euler's totient.

These two results are used, for example, in obtaining the average orders of τ and ϕ.

However, sums of the form

$$\sum_{\substack{n\le x\\(n,m)=1}} f(n)\left[\frac{x}{n}\right], \qquad x \text{ real} \ge 1$$

do not seem to have appeared in the literature until quite recently. In 1968, Gupta [1] showed

$$\sum_{\substack{n\le x\\(n,m)=1}} \mu(n)\left[\frac{x}{n}\right]=\sum_{1\le n\le x;\, n\mid m^h} 1$$

so that the sum equals $\tau(m^h, x)$ defined to be the number of divisors of m^h which are $\le x$, the exponent h chosen so that $m^h \ge 2^h \ge x$.

Subsequently, Harris and Subbarao [2] in 1973 discussed certain sums of the

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form

$$\sum_{\substack{n\le x\\(n,T)=1}} a^*(n)\left[\frac{x}{n}\right]$$

where T is a non-empty set of positive integers and $(n, T)=1$ means n is relatively prime to each element of T. They showed that, if A is any set of positive integers whose characteristic function $a(n)$ is multiplicative, and if

$$a^*(n)=\sum_{d\mid n} a(d)\mu\left(\frac{n}{d}\right),$$

then

$$\sum_{\substack{n\le x\\(n,T)=1}} a^*(n)\left[\frac{x}{n}\right]=Q_{A;T}(x),$$

namely, the number of positive integers $\le x$ whose largest factors relatively prime to T are elements of A. (For $A=\{1\}$ and $T=\{m\}$ it results that $a^*(n)=\mu(n)$ and the result of Gupta is obtained.) They used the sums in a more general form to obtain the average orders of quasi k-free and k-free integers.

In this paper another sum is introduced and evaluated, reducing in special cases to Gupta's and other known results. Certain special cases of the new sum give results identical to certain special cases of the sum $Q_{A;T}(x)$.

2. The arithmetic sum S. Let n, k be positive integers, s, x be real numbers with $x\ge 1$. Also, let m be a squarefree integer so that $m=1$ or $m=m_1m_2\cdots m_z$, the product of $z\ge 1$ distinct primes. Further, let f be any arithmetic function, and $(n, m^k)_k$ be the greatest kth power common divisor of n and m^k. Then we define the sum

$$S=\sum_{\substack{n\le x\\(n,m^k)_k=1}} f(n)\left(1^s+2^s+\cdots+\left[\frac{x}{n}\right]^s\right)$$

3. The transformation formula. If $\sigma_s(n, x)$ is the sum of the divisors of n which are $\le x$, and h is a sufficiently large positive integer such that $m^h\ge 2^h\ge x$ for $m>1$ and further if $\gamma=\gamma(m, r)$ is the product of all those m_i, $i=1, 2, \ldots, z$, for which $m_i^{k-1}\mid r$, or $\gamma=\gamma(m, r)=1$ if no such m_i exist, and also if

$$f^*(r)=\sum_{d\mid r} f(d)\left(\frac{r}{d}\right)^s,$$

then we have the following transformation formula:

THEOREM

$$S=\sum_{n\mid m^{k-1}} f(n)\sigma_s\left(m^h, \frac{x}{n}\right)+\sum_{\substack{r=2\\(r,m^k)_k=1\\r\nmid m^h}}^{[x]} f^*(r)\sigma_s\left(\gamma^h, \frac{x}{r}\right)$$

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Proof. We note that we can write

$$S=\sum_{r=1}^{[x]}\sum_{\substack{n\le r\\(n,m^k)_k=1}} f(n)\left\{\left(1^s+2^s+\cdots+\left[\frac{r}{n}\right]^s\right)-\left(1^s+2^s+\cdots+\left[\frac{r-1}{n}\right]^s\right)\right\}$$

The quantity in braces is $(r/n)^s$ or 0 according as $n \mid r$ or $n \nmid r$. Hence we find

$$S=\sum_{r=1}^{[x]}\sum_{\substack{n\le r\\(n,m^k)_k=1\\n\mid r}} f(n)\left(\frac{r}{n}\right)^s$$

For those n such that $n \mid r$ there are two cases.

(i) $n \mid r$ and $r \mid m^h$. Then $n \mid m^h$ and $(n, m^k)_k=1$—for the term to appear in S—so that $n \mid m^{k-1}$. The contribution to S in this case is

$$\sum_{r\le x}\sum_{\substack{n\mid r\\r\mid m^h\\(n,m^k)_k=1}} f(n)\left(\frac{r}{n}\right)^s=\sum_{r\le x}\sum_{\substack{n\mid r\\n\mid m^{k-1}\\r\mid m^h}} f(n)\left(\frac{r}{n}\right)^s$$

By expanding this sum and rearranging according to multiples of $f(n)$ for n fixed, we find the contribution to S from case (i) to be

$$\sum_{n\mid m^{k-1}} f(n)\sigma_s\left(m^h,\frac{x}{n}\right)$$

(ii) $n \mid r$ and $r \nmid m^h$ for any positive integer h. Let $r=r_1r_2$ where r_1 is the largest divisor of r such that $(r_1, m^k)_k=1$. Then we have

$$r=q\prod_{0\le a_i\le k-2} m_i^{a_i}\prod_{k-1\le b_i} m_i'^{b_i}$$

$$r_1=q\prod_{0\le a_i\le k-2} m_i^{a_i}\prod m_j'^{k-1}$$

$$r_2=\prod_{k-1\le b_j} m_j'^{b_j-(k-1)}$$

where $q>1$, $(q, m)=1, \prod m_i \prod m_j'=m$

Thus $\gamma(m, r)=$ core of the $(k-1)$ full part of that part of r that is not relatively prime to m

$=\gamma(m, r_1)$

The contribution of these terms to S is

$$\sum_{2\le r\le x}\sum_{\substack{n\mid r\\r\nmid m^h\\(n,m^k)_k=1}} f(n)\left(\frac{r}{n}\right)^s=\sum_{2\le{}_1r_2\le x}\sum_{\substack{n\mid r_1\\(r_1,m^k)_k=1\\r_1\nmid m^h}} f(n)\left(\frac{r_1}{n}\right)^s r_2^s.$$

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Note that $n \mid r_1$, since, if $n \mid r$ but $n \nmid r_1$, then $(n, m^k)_k \neq 1$. By fixing r_1 and summing, we find the contribution to S in case (ii) is

$$\sum_{\substack{2 \leq r_1 \leq x \\ (r_1, m^k)_k = 1 \\ r_1 \nmid m^h}} \sum_{d \mid r_1} f(d)\left(\frac{r_1}{d}\right)^s \sigma_s\left(\gamma^h, \frac{x}{r_1}\right).$$

By using the definition of $f^*(r_1)$ and dropping subscripts on r as unnecessary, we find for case (ii) the contribution to S is

$$\sum_{\substack{r=2 \\ (r, m^k)_k = 1 \\ r \nmid m^h}}^{[x]} f^*(r)\sigma_s\left(\gamma^h, \frac{x}{r}\right).$$

Adding the results for the two cases gives the result.

4. Specializations. We give several known and previously unknown resulting relations. Reference to Dickson will be to Vol. I, History of the Theory of Numbers.

(a) For $f(n) = \mu(n)$ and $k = 1$, $J_s(r)$ denoting the Jordan totient,

$$\sum_{\substack{n \leq x \\ (n,m)=1}} \mu(n)\left(1^s + 2^s + \cdots + \left[\frac{x}{n}\right]^s\right) = \sigma_s(m^h, x) + \sum_{\substack{2 \leq r \leq x \\ (r,m)=1}} J_s(r)\sigma_s\left(m^h, \frac{x}{r}\right).$$

For $s = 0$, since $J_0(r) = 1$ if $r = 1$ and $= 0$ otherwise, this reduces to the result of Gupta previously given. For $m = 1$ the result becomes the known equality

$$\sum_{n \leq x} \mu(n)\left[\frac{x}{n}\right] = 1$$

For $s = 1$ we get

$$\sum_{\substack{n \leq x \\ (n,m)=1}} \mu(n)\left(\frac{\left[\frac{x}{n}\right]\left(\left[\frac{x}{n}\right]+1\right)}{2}\right) = \sum_{\substack{r \leq x \\ (r,m)=1}} \phi(r)\sigma\left(m^h, \frac{x}{r}\right)$$

Solving and using a previous result for

$$\sum_{\substack{n \leq x \\ (n,m)=1}} \mu(n)[x/n],$$

$$\sum_{\substack{n \leq x \\ (n,m)=1}} \mu(n)\left[\frac{x}{n}\right]^2 = 2\sum_{\substack{r \leq x \\ (r,m)=1}} \phi(r)\sigma\left(m^h, \frac{x}{r}\right) - \tau(m^h, x)$$

Now setting $m = 1$ gives the result of Mertens:

$$\sum_{n \leq x} \mu(n)\left[\frac{x}{n}\right]^2 = 2\sum_{r \leq x} \phi(r) - 1 \qquad \text{[Dickson p. 122]}$$

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Or, setting $m=1$, we have, for all s,

$$\sum_{n\le x}\mu(n)\left(1^s+2^s+\cdots+\left[\frac{x}{n}\right]^s\right)=\sum_{r=1}^{[x]}J_s(r)$$

(b) Let $f(n)=n^{a+s}$ and $k=1$. Then we obtain

$$\sum_{\substack{n\le x\\(n,m)=1}} n^{a+s}\left(1^s+2^s+\cdots+\left[\frac{x}{n}\right]^s\right)=\sum_{\substack{r\le x\\(r,m)=1}} r^s\sigma_a(r)\sigma_s\left(m^h,\frac{x}{r}\right)$$

For $s=0$ and $m=1$ this reduces to a result of Lerch:

$$\sum_{n\le x} n^a\left[\frac{x}{n}\right]=\sum_{r\le x}\sigma_a(r) \qquad \text{[Dickson 313]}$$

The result for the case $a=0$ is

$$\sum_{\substack{n\le x\\(n,m)=1}} n^s\left(1^s+2^s+\cdots+\left[\frac{x}{n}\right]^s\right)=\sum_{\substack{r\le x\\(r,m)=1}} r^s\tau(r)\sigma_s\left(m^h,\frac{x}{r}\right)$$

and for $a=-s$:

$$\sum_{\substack{n\le x\\(n,m)=1}} \left(1^s+2^s+\cdots+\left[\frac{x}{n}\right]^s\right)=\sum_{\substack{r\le x\\(r,m)=1}} \sigma_s(r)\sigma_s(\left(m^h,\frac{x}{r}\right)$$

(c) $f(n)=n^s\mu(n)$ and $k=1$.
Then

$$\sum_{\substack{n\le x\\(n,m)=1}} n^s\mu(n)\left(1^s+2^s+\cdots+\left[\frac{x}{n}\right]^s\right)=\sigma_s(m^h,x)$$

(d) $f(n)=1$, $m=a$ prime p, $k=2$, and $s=0$:

$$\sum_{\substack{n\le x\\(n,p^2)_2=1}} \left[\frac{x}{n}\right]=1+\sum_{\substack{2\le r\le x\\(r,p^2)_2=1}} \tau(r)\left\{\left[\frac{\log x/r}{\log p}\right]+1\right\}$$

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2. V. C. Harris and M. V. Subbarao, 'An arithmetic sum with an application to quasi k-free integers', Journal Australian Math. Soc. XV part 3 (1973) 272–278.

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