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A NOTE ON MULTIPLIER OPERATORS AND DUAL B^*-ALGEBRAS

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Let A be a complex Banach algebra without order. Following Kellogg [4] and Ching and Wong [2], a mapping T of A into itself is called a *right* (*left*) *multiplier* on A if $T(ab)=(Ta)b(T(ab)=a(Tb))$ for all a, b in A.† T is said to be a *multiplier* on A if it is both a right and left multiplier on A. Let $M(A)(RM(A), LM(A))$ be the set of all (right, left) multipliers on A. Then both $RM(A)$ and $LM(A)$ are closed subalgebras of the algebra $L(A)$ of all bounded linear operators on A, and $M(A)$ is a closed commutative subalgebra of $L(A)$ ([4], Theorem 2.1). In a recent paper [5], Malviya and Tomiuk have proved the following result.

THEOREM A ([5], *Corollary* 2.6). *Let A be a dual B^*-algebra, and let $\{I_\lambda : \lambda \in \Lambda\}$ be the family of all minimal closed two-sided ideals of A. For each $T \in LM(A)$ and $\lambda \in \Lambda$, let T_λ be the restriction of T to I_λ, and let $LM_\lambda=\{T_\lambda : T \in LM(A)\}$ Then $LM(A)$ is isometrically isomorphic to the normed full direct sum of the algebras LM_λ.*

In this note we show that, if A is a dual B^*-algebra, and $\Omega(A)$ is the space of minimal closed two-sided ideals of A with its discrete topology, then $M(A)$ is isometrically isomorphic to the algebra of all bounded complex-valued functions on $\Omega(A)$. We also give a similar characterization for the compact multipliers on A. The results obtained are similar to ones established by Kellogg [4] and Ching and Wong [2] for H^*-algebras.

Throughout the remainder of this note, A denotes a dual B^*-algebra.

Let $\{I_\lambda : \lambda \in \Lambda\}$ be the family of all minimal closed two-sided ideals of A. Then $A=(\sum_{\lambda\in\Lambda} I_\lambda)_0$, where $(\sum_{\lambda\in\Lambda} I_\lambda)_0$ denotes the completion of the direct sum $\sum_{\lambda\in\Lambda} I_\lambda$ of the $I_\lambda(\lambda \in \Lambda)$ with respect to the norm $\|\sum_{\lambda\in\Lambda} a_\lambda\|=\sup_{\lambda\in\Lambda}\|a_\lambda\|$ ([1], Theorem 6). By ([1], Theorem 8), ([7], Lemma 4.10.3), and the fact that I_λ is a minimal closed two-sided ideal of A, each I_λ is a simple dual B^*-algebra, and so is isometrically isomorphic to an algebra $LC(H_\lambda)$ of compact operators on some Hilbert space H_λ ([6], p. 334, Theorem 14). Thus $A\cong(\sum_{\lambda\in\Lambda} LC(H_\lambda))_0$. In fact, by combining ([7], Lemma 4.10.1) with the results of ([1], §3), we see that, for each I_λ, there exists a hermitian idempotent e_λ such that I_λ is the closed two-sided ideal generated by Ae_λ, and $I_\lambda\cong LC(Ae_\lambda)$.

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† Kellogg [4] has used the terminology centralizers instead of multipliers.

It follows from a result by Taylor ([8], Lemma 3.1) that $M(A) \simeq (\sum_{\lambda \in \Lambda} M(LC(H_\lambda)))$, the normed full direct sum of the algebras $M(LC(H_\lambda))$. The isomorphism Φ may be defined by the equation

$$\Phi(T) = \mathscr{T}_T, \qquad (T \in M(A))$$

where $\mathscr{T}_T(\lambda) = T_\lambda$ and T_λ denotes the restriction of T to I_λ. Also $\|T\| = \|\mathscr{T}_T\| = \sup_{\lambda \in \Lambda} \|T_\lambda\|$. Each element of $M(LC(H_\lambda))$ is a scalar multiple of the identity operator, as we now show. From ([5], Lemma 2.1) (see also ([3], Theorem 18)), it follows that $M(LC(H_\lambda)) \simeq Z_{LC(H_\lambda)}(L(H_\lambda))$, where $Z_{LC(H_\lambda)}(L(H_\lambda))$ is the centralizer of $LC(H_\lambda)$ in the algebra $L(H_\lambda)$ of all bounded linear operators on H_λ.† Now $D \in Z_{LC(H_\lambda)}(L(H_\lambda))$ if and only if D commutes with the ideal $\mathscr{F}(H_\lambda)$ consisting of operators of finite rank, and it is easy to show that D commutes with $\mathscr{F}(H_\lambda)$ if and only if it is a scalar multiple of the identity operator. Thus, for each $T \in M(A)$ and $\lambda \in \Lambda$, there exists a scalar μ_{T_λ} such that $T_\lambda = \mu_{T_\lambda} I$, as required. For each $T \in M(A)$, we define a function f_T on $\Omega(A)$ by the equation

$$f_T(I_\lambda) = \mu_{T_\lambda}.$$

Then the mapping $T \leftrightarrow f_T$ defines an isometric isomorphism Ψ between $M(A)$ and $C(\Omega(A))$. Collecting our results we have

THEOREM 1. *$M(A)$ is isometrically isomorphic to $C(\Omega(A))$, the algebra of all bounded complex-valued functions on $\Omega(A)$.*

Let $M_c(A)$ denote the compact multipliers on A. If $LC(A)$ is the algebra of all compact operators on A, then $M_c(A) = M(A) \cap LC(A)$, so that $M_c(A)$ is a closed ideal of $M(A)$.

We define Λ_0 to be the set

$$\Lambda_0 = \{\lambda : \lambda \in \Lambda, I_\lambda \text{ is infinite dimensional}\},$$

and let $\mathscr{I}_0$ be the set of all functions f in $C(\Omega(A))$ such that $f(I_\lambda) = 0$ for all $\lambda \in \Lambda_0$; if $\Lambda_0 = \phi$, let $\mathscr{I}_0 = C(\Omega(A)$. Clearly $\mathscr{I}_0$ is a closed ideal of $C(\Omega(A))$. Let $C_0(\Omega(A))$ be the subalgebra of $C(\Omega(A))$ which consists of functions vanishing at infinity.

We now obtain a characterization for $M_c(A)$.

THEOREM 2. *$M_c(A)$ is isometrically isomorphic to $\mathscr{I}_0 \cap C_0(\Omega(A))$.*

Proof. Let Ψ denote the isomorphism between $M(A)$ and $C(\Omega(A))$. Let $C_c(\Omega(A))$ be the subalgebra of $C_0(\Omega(A))$ which consists of functions with compact support, and suppose that $\psi \in \mathscr{I}_0 \cap C_c(\Omega(A))$. Then, since $\Omega(A)$ is discrete, ψ is zero except at a finite number of points $I_{\lambda_1}, \ldots, I_{\lambda_m}$ say, and, since $\psi \in \mathscr{I}_0$, each I_{λ_k} $(1 \leq k \leq m)$ is finite dimensional. Now $\Psi^{-1}(\Psi)$ is an element, T say, of $M(A)$; in fact, $T \in M_c(A)$, as we now show.

† The centralizer of $LC(H_\lambda)$ in $L(H_\lambda)$ is the set of all elements in $L(H_\lambda)$ which commute with all the members of $LC(H_\lambda)$.

Suppose $\tilde{T}$ denotes the restriction of T to $\sum_{\lambda\in\Lambda} I_\lambda$. Then $\tilde{T}$ is defined according to the equation

$$\tilde{T}a = f_T(I_{\lambda_1})a_{\lambda_1} + \cdots + f_T(I_{\lambda_m})a_{\lambda_m},$$

where $a = a_{\lambda_1} + \cdots + a_{\lambda_m}$ and $a_{\lambda_k} \in I_{\lambda_k} (1 \le k \le m)$. The range of $\tilde{T}$ is $I_{\lambda_1} \oplus \cdots \oplus I_{\lambda_m}$, and since $I_{\lambda_k} (1 \le k \le m)$ is finite dimensional, it follows that $\tilde{T}$ is compact. Hence T is compact. Consequently, $\Psi^{-1}(\mathscr{I}_0 \cap C_c(\Omega(A))) \subseteq M_c(A)$. Since $C_c(\Omega(A))$ is dense in $C_0(\Omega(A))$, it follows that $\Psi^{-1}(\mathscr{I}_0 \cap C_0(\Omega(A))) \subseteq M_c(A)$.

Conversely, suppose $T \in M_c(A)$. If $\Psi(T) \notin \mathscr{I}_0$, then there exists a $\lambda \in \Lambda_0$ such that $(\Psi(T))(I_\lambda) \neq 0$. It follows that the restriction of T to I_λ is not compact, and so T itself is not compact. Hence $\Psi(T) \in \mathscr{I}_0$. Also, $\Psi(T) \in C_0(\Omega(A))$, as we now show. Suppose $\Psi(T) \notin C_0(\Omega(A))$. Then there exists $\varepsilon > 0$ such that the set $\Lambda_\varepsilon = \{\lambda : \lambda \in \Lambda, |\Psi(T)(I_\lambda)| \ge \varepsilon\}$ is not finite. For each $\lambda \in \Lambda_\varepsilon$, choose $a_\lambda \in I_\lambda$ such that $\|a_\lambda\| = 1$, and let $(\Psi(T))(I_\lambda) = \mu_{T_\lambda}$. Then $Ta_\lambda = \mu_{T_\lambda} a_\lambda$, and so $\|Ta_\lambda\| = |\mu_{T_\lambda}| \ge \varepsilon$ for all $\lambda \in \Lambda_\varepsilon$. Hence

$$\|Ta_\lambda - Ta_{\lambda'}\| = \max(\|Ta_\lambda\|, \|Ta_{\lambda'}\|) \ge \varepsilon \qquad (\lambda, \lambda^1 e \Lambda_\varepsilon, \lambda \neq \lambda^1).$$

Thus $\{a_\lambda\} (\lambda \in \Lambda_\varepsilon)$ is a bounded subset of A which has no convergent subsequence. This contradicts the compactness of T, and so $\Psi(T) \in C_0(\Omega(A))$. Thus $\Psi(T) \in \mathscr{I}_0 \cap C_0(\Omega(A))$, as required.

If A is commutative, then $I_\lambda = Ae_\lambda$ for each $\lambda \in \Lambda$. Now $Ae_\lambda = e_\lambda A e_\lambda$, and each $e_\lambda A e_\lambda$ is a normed division algebra ([7], Lemma 2.1.5). Consequently, by the Gelfand-Mazur theorem, every element in I_λ is a scalar multiple of e_λ, and so each I_λ is one-dimensional. Therefore, for a commutative dual B^*-algebra, $\Lambda_0 = \phi$, and $\mathscr{I}_0 = C(\Omega(A))$, so that $M_c(A)$ is isometrically isomorphic to $C_0(\Omega(A))$.

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