

The range of the Douglas–Rachford operator in infinite-dimensional Hilbert spaces

Walaa M. Moursi*

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Abstract

The Douglas–Rachford algorithm is one of the most prominent splitting algorithms for solving convex optimization problems. Recently, the method has been successful in finding a generalized solution (provided that one exists) for optimization problems in the inconsistent case, i.e., when a solution does not exist. The convergence analysis of the inconsistent case hinges on the study of the range of the displacement operator associated with the Douglas–Rachford splitting operator and the corresponding minimal displacement vector. A comprehensive study of this range has been developed in finite-dimensional Hilbert spaces. In this paper, we provide a formula for the range of the Douglas–Rachford splitting operator in (possibly) infinite-dimensional Hilbert spaces under mild assumptions on the underlying operators. Our new results complement known results in finite-dimensional Hilbert spaces. Several examples illustrate and tighten our conclusions.

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1 Introduction

Throughout, we assume that

$$X \text{ is a real Hilbert space with inner product } \langle \cdot, \cdot \rangle : X \times X \rightarrow \mathbb{R}, \quad (1)$$

*Department of Combinatorics and Optimization, University of Waterloo, Waterloo, Ontario N2L 3G1, Canada. E-mail: walaa.moursi@uwaterloo.ca.

and induced norm $\|\cdot\|$. Let $A: X \rightrightarrows X$. Recall that A is *monotone* if $\{(x, u), (y, v)\} \subseteq \text{gra } A$ implies that $\langle x - y, u - v \rangle \geq 0$ and that A is *maximally monotone* if any proper extension of $\text{gra } A$ destroys its monotonicity. The *resolvent* of A is $J_A = (\text{Id} + A)^{-1}$ and the *reflected resolvent* of A is $R_A = 2J_A - \text{Id}$, where $\text{Id}: X \rightarrow X: x \mapsto x$. We recall the well-known inverse resolvent identity (see [19, Lemma 12.14])

$$J_A + J_{A^{-1}} = \text{Id}, \quad (2)$$

and the following, useful description of the graph of A thanks to Minty (see [15]).

Fact 1.1 (Minty). *Let $A: X \rightrightarrows X$ be monotone. Then*

$$\text{gra } A = \{(J_A x, J_{A^{-1}} x) \mid x \in \text{ran}(\text{Id} + A)\}. \quad (3)$$

Moreover, A is *maximally monotone* $\Leftrightarrow \text{ran}(\text{Id} + A) = X$.

In the following, we assume that

$$A \text{ and } B \text{ are maximally monotone on } X. \quad (4)$$

The Douglas–Rachford splitting operator [14] associated with the ordered pair (A, B) is

$$T := T_{A,B} := \text{Id} - J_A + J_B R_A. \quad (5)$$

It is critical to observe that $\text{zer}(A + B) \neq \emptyset$ if and only if $\text{Fix } T \neq \emptyset$, see, e.g., [4, Proposition 26.1(iii)(b)]. Let $x_0 \in X$. The splitting operator in (5) defines the so-called *governing* sequence via iterating T at x_0 to obtain $(x_n)_{n \in \mathbb{N}} = (T^n x_0)_{n \in \mathbb{N}}$ and $(x_n)_{n \in \mathbb{N}}$ converges weakly to a point in $\text{Fix } T$ provided that the latter is nonempty. In this case, the *shadow* sequence $(y_n)_{n \in \mathbb{N}} = (J_A x_n)_{n \in \mathbb{N}}$ converges weakly to a point in $\text{zer}(A + B)$. Note that

$$\text{Id} - T = J_A - J_B R_A = J_{A^{-1}} + J_{B^{-1}} R_A. \quad (6)$$

The range of the displacement operator $\text{Id} - T$ is the central focus of this paper. Indeed, $\text{Fix } T \neq \emptyset$ if and only if $0 \in \text{ran}(\text{Id} - T)$. Because T is firmly nonexpansive, we learn that (see, e.g., [22, Theorem 31.2])

$$\text{Id} - T \text{ is maximally monotone, hence } \overline{\text{ran}}(\text{Id} - T) \text{ is convex.} \quad (7)$$

This beautiful topological property of $\overline{\text{ran}}(\text{Id} - T)$ allows us to work with best approximation properties of nonempty closed convex sets. In the following we set

$$D = \text{dom } A - \text{dom } B \text{ and } R = \text{ran } A + \text{ran } B. \quad (8)$$

It is always true that (see [13, Corollary 4.1] or [5, Corollary 2.14])

$$\text{ran}(\text{Id} - T) = \{a - b \mid (a, a^*) \in \text{gra } A, (b, b^*) \in \text{gra } B, a - b = a^* + b^*\} \subseteq D \cap R. \quad (9)$$

Hence

$$\overline{\text{ran}}(\text{Id} - T) \subseteq \overline{D \cap R}. \quad (10)$$

When X is finite-dimensional, the authors in [6] proved that under mild assumptions on A and B we have

$$\overline{\text{ran}}(\text{Id} - T) = \overline{D \cap R}. \quad (11)$$

It is natural to ask how would the Douglas–Rachford algorithm perform if the problem is inconsistent, i.e., if it does not have a solution. The formula in (11) was the cornerstone to study such a performance in the case of finite-dimensional Hilbert spaces. Therefore, this motivated the question:

$$\text{Does the formula in (11) extend to infinite-dimensional Hilbert spaces?} \quad (12)$$

Contribution. In this paper we provide a positive answer to the question in (12). Our proof techniques are completely independent and significantly distinct from the tools used in the finite-dimensional case: Indeed the analysis techniques used in [6] hinge on the well-developed calculus of the relative interior of convex and *nearly convex* sets (see [7], [16] and [20]). Not only does this powerful calculus notably fails to extend in infinite-dimensional Hilbert spaces but also there exists no analogously useful notion that can replace it. This highlights the infeasibility of extending any of the proof techniques in [6] to infinite-dimensional settings. Instead, our analysis hinges on a novel and powerful approach that connects the graph of $\text{Id} - T$ and the graphs of the individual operators A and B . This allows to exploit any additional properties of the operators A and B and demonstrate how this reflects on the displacement mapping $\text{Id} - T$. As a byproduct of our analysis we were able to refine the topological properties of the sets D and R as well as the corresponding minimal norm vectors.

Organization and notation. The remainder of this paper is organized as follows: Section 2 presents the main results of the paper. In Section 3 we provide sharper conclusions when specializing the main results to subdifferential operators. In Section 4 We explore some properties of the sets D and R as well as $D \cap R$. Finally, in Section 5 we provide a detailed study of the minimal norm vector in the range of the displacement map. An application of our results to the product space setting is also presented.

Our notation is standard and follows largely, e.g., [4], [19] and [21].

2 The range of $\text{Id} - T$

We start with the following key lemma.

Lemma 2.1. *Let $(a, b, a^*, x) \in X^4$. Then the following holds:*

$$\begin{aligned} \langle x - (a + a^*), x - Tx - (a - b) \rangle &\geq \langle J_A x - a, J_{A^{-1}} x - a^* \rangle \\ &\quad + \langle J_B R_A x - b, J_{B^{-1}} R_A x - (a - a^* - b) \rangle. \end{aligned} \quad (13)$$

Proof. Recalling (6), we have

$$\begin{aligned} & \langle x - (a + a^*), x - Tx - (a - b) \rangle - \langle J_A x - a, J_{A^{-1}} x - a^* \rangle \\ & - \langle J_B R_A x - b, J_{B^{-1}} R_A x - (a - a^* - b) \rangle \\ = & \langle x - (a + a^*), J_A x - a - (J_B R_A x - b) \rangle - \langle J_A x - a, x - (a + a^*) - (J_A x - a) \rangle \\ & - \langle J_B R_A x - b, 2(J_A x - a) - J_B R_A x - (x - a - a^* - b) \rangle \end{aligned} \quad (14a)$$

$$\begin{aligned} = & \langle x - (a + a^*), J_A x - a \rangle - \langle x - (a + a^*), J_B R_A x - b \rangle \\ & - \langle J_A x - a, x - (a + a^*) \rangle + \|J_A x - a\|^2 - 2 \langle J_A x - a, J_B R_A x - b \rangle \\ & + \|J_B R_A x - b\|^2 + \langle J_B R_A x - b, x - (a + a^*) \rangle \end{aligned} \quad (14b)$$

$$= \|J_A x - a\|^2 - 2 \langle J_A x - a, J_B R_A x - b \rangle + \|J_B R_A x - b\|^2 \quad (14c)$$

$$= \|x - Tx - (a - b)\|^2 \geq 0. \quad (14d)$$

The proof is complete. ■

Proposition 2.2. Suppose that

$$(\text{dom } A - \text{dom } A) \perp (\text{ran } A - \text{ran } A) \text{ and } (\text{dom } B - \text{dom } B) \perp (\text{ran } B - \text{ran } B). \quad (15)$$

Then $\text{ran } (\text{Id} - T) = (\text{dom } A - \text{dom } B) \cap (\text{ran } A + \text{ran } B)$.

Proof. In view of (9) we only need to show that $(\text{dom } A - \text{dom } B) \cap (\text{ran } A + \text{ran } B) \subseteq \text{ran } (\text{Id} - T)$. Indeed, let $w \in (\text{dom } A - \text{dom } B) \cap (\text{ran } A + \text{ran } B)$. Then $(\exists(a, b, a^*, b^*) \in \text{dom } A \times \text{dom } B \times \text{ran } A \times \text{ran } B$ such that $w = a - b = a^* + b^*$. Lemma 2.1 implies that $(\forall x \in X)$

$$\langle x - (a + a^*), x - Tx - w \rangle \geq \langle J_A x - a, J_{A^{-1}} x - a^* \rangle + \langle J_B R_A x - b, J_{B^{-1}} R_A x - b^* \rangle. \quad (16)$$

Recalling Fact 1.1, we learn that

$$(J_A x - a, J_{A^{-1}} x - a^*) \in (\text{dom } A - \text{dom } A) \times (\text{ran } A - \text{ran } A) \quad (17a)$$

$$(J_B R_A x - b, J_{B^{-1}} R_A x - b^*) \in (\text{dom } B - \text{dom } B) \times (\text{ran } B - \text{ran } B). \quad (17b)$$

Combining (16) and (17) in view of (15) yields $\langle x - (a + a^*), x - Tx - w \rangle \geq 0$. In view of (7) we conclude that $(a + a^*, w) \in \text{gra}(\text{Id} - T)$. This completes the proof. ■

Example 2.3. Let U and V be closed linear subspaces of X , let $(u, v) \in U \times V$ and let $(u^\perp, v^\perp) \in U^\perp \times V^\perp$. Set¹ $(A, B) = (u + N_{u^\perp + U}, v + N_{v^\perp + V})$. Then the following hold:

- (i) $(\text{dom } A, \text{dom } B, \text{ran } A, \text{ran } B) = (u^\perp + U, v^\perp + V, u + U^\perp, v + V^\perp)$.
- (ii) $(\text{dom } A - \text{dom } A, \text{dom } B - \text{dom } B, \text{ran } A - \text{ran } A, \text{ran } B - \text{ran } B) = (U, V, U^\perp, V^\perp)$.
- (iii) $(\forall C \in \{A, B\})$ we have $(\text{dom } C - \text{dom } C) \perp (\text{ran } C - \text{ran } C)$.

¹Let C be a nonempty closed convex subset of X . Here and elsewhere we shall use ι_C (respectively N_C) to denote the indicator function (respectively the normal cone operator) associated with C .

$$(iv) \operatorname{ran}(\operatorname{Id} - T) = (u^\perp - v^\perp + U + V) \cap (u + v + U^\perp + V^\perp).$$

Proof. (i)–(ii): This is clear. (iii): This is a direct consequence of (ii).

(iv): Combine (iii) and Proposition 2.2. ■

We are now ready to derive an analogous formula to the conclusion of Proposition 2.2 in a more general setting. Let $D: X \rightrightarrows X$ be monotone. Recall that D is 3^* monotone (this is also known as *rectangular*) if $(\forall(y, z^*) \in \operatorname{dom} D \times \operatorname{ran} D)$ we have $\inf_{(x, x^*) \in \operatorname{gra} D} \langle x - y, x^* - z^* \rangle > -\infty$. It is well-known that (see, e.g., [9, page 167] and [22, page 127]) for a proper lower semicontinuous convex function $f: X \rightarrow]-\infty, +\infty]$ we have

$$\partial f \text{ is } 3^* \text{ monotone.} \quad (18)$$

Proposition 2.4. *Let $w \in (\operatorname{dom} A - \operatorname{dom} B) \cap (\operatorname{ran} A + \operatorname{ran} B)$. Suppose that one of the following hold:*

- (i) A and B are 3^* monotone.
- (ii) $\operatorname{dom} A \subseteq (w + \operatorname{dom} B)$ and B is 3^* monotone.
- (iii) $\operatorname{dom} B \subseteq (-w + \operatorname{dom} A)$ and A is 3^* monotone.

Then $w \in \overline{\operatorname{ran}}(\operatorname{Id} - T)$.

Proof. Let $n \geq 1$ and observe that Fact 1.1 applied with A replaced by $\operatorname{Id} - T$, in view of (7) implies that $\operatorname{ran}\left(\left(1 + \frac{1}{n^2}\right)\operatorname{Id} - T\right) = X$. Consequently, $(\forall n \geq 1) (\exists x_n \in X)$ such that $w = \left(1 + \frac{1}{n^2}\right)x_n - Tx_n$. Then $(x_n, w - \frac{1}{n^2}x_n)_{n \geq 1}$ lies in $\operatorname{gra}(\operatorname{Id} - T)$. It is sufficient to show that

$$\frac{1}{n^2}x_n \rightarrow 0. \quad (19)$$

To this end, let $x \in X$. First suppose that (i) holds. By assumption $(\exists(a, b, a^*, b^*) \in \operatorname{dom} A \times \operatorname{dom} B \times \operatorname{ran} A \times \operatorname{ran} B)$ such that

$$w = a - b = a^* + b^*. \quad (20)$$

Now Lemma 2.1 and (20) imply that $(\forall x \in X)$

$$\langle x - (a + a^*), x - Tx - w \rangle \geq \langle J_A x - a, J_{A^{-1}} x - a^* \rangle + \langle J_B R_A x - b, J_{B^{-1}} R_A x - b^* \rangle. \quad (21)$$

It follows from the 3^* monotonicity of A and B that

$$\inf_{x \in X} \langle J_A x - a, J_{A^{-1}} x - a^* \rangle > -\infty \text{ and } \inf_{x \in X} \langle J_B R_A x - b, J_{B^{-1}} R_A x - b^* \rangle > -\infty. \quad (22)$$

Combining (22), (21) and (20), we learn that

$$\inf_{x \in X} \langle x - (a + a^*), x - Tx - (a - b) \rangle$$

$$\geq \inf_{x \in X} (\langle J_A x - a, J_{A^{-1}} x - a^* \rangle + \langle J_B R_A x - b, J_{B^{-1}} R_A x - b^* \rangle) \quad (23a)$$

$$\geq \inf_{x \in X} \langle J_A x - a, J_{A^{-1}} x - a^* \rangle + \inf_{x \in X} \langle J_B R_A x - b, J_{B^{-1}} R_A x - b^* \rangle > -\infty. \quad (23b)$$

It follows from (23) and the fact that $(x_n, w - \frac{1}{n^2} x_n)_{n \geq 1}$ lies in $\text{gra}(\text{Id} - T)$ that $(\exists M \in \mathbb{R})$ such that $(\forall n \geq 1)$ we have $\langle x_n - (a + a^*), -\frac{1}{n^2} x_n \rangle = \langle x_n - (a + a^*), w - \frac{1}{n^2} x_n - w \rangle \geq M$. Therefore, we learn that $\frac{1}{n^2} \|x_n\|^2 \leq \langle \frac{1}{n^2} x_n, a + a^* \rangle - M \leq \frac{1}{2n^2} (\|x_n\|^2 + \|a + a^*\|^2) - M$. Simplifying yields $\frac{1}{n^2} \|x_n\|^2 \leq \frac{1}{n^2} \|a + a^*\|^2 - 2M \leq \|a + a^*\|^2 - 2M$ and we learn that the sequence $(\frac{x_n}{n})_{n \geq 1}$ is bounded. Consequently, $\frac{1}{n^2} x_n \rightarrow 0$ and (19) is verified.

Now suppose that (ii) holds. By assumption $(\exists(a^*, b^*) \in \text{ran } A \times \text{ran } B)$ such that

$$w = a^* + b^*. \quad (24)$$

Let $(a, b) \in \text{dom } A \times \text{dom } B$ be such that $(a, a^*) \in \text{gra } A$ and $a - b = w$. Lemma 2.1 and (24) imply that $(\forall x \in X)$

$$\langle x - (a + a^*), x - Tx - w \rangle \geq \langle J_A x - a, J_{A^{-1}} x - a^* \rangle + \langle J_B R_A x - b, J_{B^{-1}} R_A x - b^* \rangle. \quad (25)$$

It follows from the monotonicity of A and the 3^* monotonicity of B that

$$\langle J_A x - a, J_{A^{-1}} x - a^* \rangle \geq 0 \text{ and } \inf_{x \in X} \langle J_B R_A x - b, J_{B^{-1}} R_A x - b^* \rangle > -\infty. \quad (26)$$

Combining this with (25) we learn that

$$\begin{aligned} & \inf_{x \in X} \langle x - (a + a^*), x - Tx - (a - b) \rangle \\ & \geq \inf_{x \in X} (\langle J_A x - a, J_{A^{-1}} x - a^* \rangle + \langle J_B R_A x - b, J_{B^{-1}} R_A x - b^* \rangle) \end{aligned} \quad (27a)$$

$$\geq \inf_{x \in X} \langle J_A x - a, J_{A^{-1}} x - a^* \rangle + \inf_{x \in X} \langle J_B R_A x - b, J_{B^{-1}} R_A x - b^* \rangle > -\infty. \quad (27b)$$

Now proceed similar to the above. The case when (iii) holds is treated similarly to the case when (ii) holds. The proof is complete. ■

Let $D: X \rightrightarrows X$ be monotone. Before we proceed we recall (see, e.g., [4, Proposition 25.19(i)]) that

$$D \text{ is } 3^* \text{ monotone} \Leftrightarrow D^{-1} \text{ is } 3^* \text{ monotone} \Leftrightarrow D^{-\odot} \text{ is } 3^* \text{ monotone}, \quad (28)$$

where $D^{-\odot} := (-\text{Id}) \circ D^{-1} \circ (-\text{Id})$.

We also recall that T is self-dual (see, e.g., [13, Lemma 3.6 on page 133]), i.e.,

$$T_{(A,B)} = T_{(A^{-1}, B^{-\odot})}. \quad (29)$$

The following result provides a positive answer to the question in (12).

Theorem 2.5 (the range of $\text{Id} - T$). *The following implications hold:*

- (i) A and B are 3^* monotone $\Rightarrow \overline{\text{ran}}(\text{Id} - T) = \overline{D \cap R}$.
- (ii) $(\exists C \in \{A, B\}) \text{ dom } C = X$ and C is 3^* monotone $\Rightarrow \overline{\text{ran}}(\text{Id} - T) = \overline{D \cap R} = \overline{R}$.
- (iii) $(\exists C \in \{A, B\}) \text{ ran } C = X$ and C is 3^* monotone $\Rightarrow \overline{\text{ran}}(\text{Id} - T) = \overline{D \cap R} = \overline{D}$.

Proof. (i): It follows from Proposition 2.4(i) that $\overline{D \cap R} \subseteq \overline{\text{ran}}(\text{Id} - T)$. Now combine this with (10). (ii): Observe that in this case $D = \text{dom } A - \text{dom } B = X$, hence $D \cap R = R$. First suppose that $C = B$. It follows from Proposition 2.4(ii) that $\overline{R} = \overline{D \cap R} \subseteq \overline{\text{ran}}(\text{Id} - T)$. Now combine this with (10). To prove the claim in the case $C = A$ proceed as above but use Proposition 2.4(iii). (iii): Observe that (28) implies $[(\exists C \in \{A, B\}) \text{ ran } C = X \text{ and } C \text{ is } 3^* \text{ monotone}] \Leftrightarrow [(\exists \tilde{C} \in \{A^{-1}, B^{-\odot}\}) \text{ dom } \tilde{C} = X \text{ and } \tilde{C} \text{ is } 3^* \text{ monotone}]$ by, e.g., [4, Proposition 25.19(i)]. Now combine this with (ii) applied with (A, B) replaced by $(A^{-1}, B^{-\odot})$ in view of (29). ■

Theorem 2.6 (the range of T). *The following implications hold:*

- (i) A and B are 3^* monotone $\Rightarrow \overline{\text{ran}} T = \overline{(\text{dom } A - \text{ran } B) \cap (\text{ran } A + \text{dom } B)}$.
- (ii) $(\exists C \in \{A, B^{-1}\}) \text{ dom } C = X$ and C is 3^* monotone $\Rightarrow \overline{\text{ran}} T = \overline{\text{ran } A + \text{dom } B}$.
- (iii) $(\exists C \in \{A, B^{-1}\}) \text{ ran } C = X$ and C is 3^* monotone $\Rightarrow \overline{\text{ran}} T = \overline{\text{dom } A - \text{ran } B}$.

Proof. It is straightforward to verify that $T = \text{Id} - (\text{Id} - J_A + J_{B^{-1}} R_A) = \text{Id} - T_{(A, B^{-1})}$. Now apply Theorem 2.5(i)–(iii) with B replaced by B^{-1} in view of (28) applied with D replaced by B . ■

The assumptions in Theorem 2.5 are critical as we illustrate below.

Example 2.7. Suppose that $S: X \rightarrow X$ is continuous, linear, and single-valued such that S and $-S$ are monotone and $S^2 = -\gamma \text{Id}$ where $\gamma > 0$. Set $(A, B) = (S, -S)$. Then the following hold:

- (i) $(\forall x \in X) \langle x, Sx \rangle = 0$.
- (ii) Neither S nor S^* is 3^* monotone.
- (iii) $J_A = \frac{1}{1+\gamma}(\text{Id} - S)$ and $J_B = \frac{1}{1+\gamma}(\text{Id} + S)$.
- (iv) $R_A = \frac{1}{1+\gamma}((1 - \gamma)\text{Id} - 2S)$ and $R_B = \frac{1}{1+\gamma}((1 - \gamma)\text{Id} + 2S)$.
- (v) $R_B R_A = \text{Id}$. Hence, $T = \text{Id}$ and consequently $\text{Id} - T \equiv 0$.
- (vi) $\{0\} = \text{ran}(\text{Id} - T) = \overline{\text{ran}}(\text{Id} - T) \subsetneq X = \overline{D \cap R}$.

Proof. (i): This is clear. (ii): This follows from, e.g., combining [4, Example 20.53 and Proposition 25.11]. (iii): Indeed, observe that $J_A J_B = (\text{Id} + S)^{-1}(\text{Id} - S)^{-1} = ((\text{Id} - S)(\text{Id} + S))^{-1} = ((1 + \gamma) \text{Id})^{-1} = (1 + \gamma)^{-1} \text{Id}$. Therefore, $J_A = (1 + \gamma)^{-1}(\text{Id} - S)$ and $J_B = (1 + \gamma)^{-1}(\text{Id} + S)$ as claimed. (iv): This is a direct consequence of (iii). (v): Using (iv) we have $R_B R_A = (1 + \gamma)^{-2}((1 - \gamma) \text{Id} + 2S)((1 - \gamma) \text{Id} - 2S) = (1 + \gamma)^{-2}((1 - \gamma)^2 \text{Id} - 4S^2) = (1 + \gamma)^{-2}((1 - \gamma)^2 \text{Id} + 4\gamma \text{Id}) = \text{Id}$. (vi): Clearly, $\text{dom } A = \text{dom } B = X$. Moreover, by assumption $A^{-1} = -B^{-1} = -\gamma^{-1}S$. Hence, A and B are surjective and we conclude that $\text{ran } A = \text{ran } B = X$ and the conclusion follows. ■

Example 2.8. Suppose that $X = \mathbb{R}^2$. Let $u \in \mathbb{R}^2$ be such that $\|u\| = 1$. Let $A: \mathbb{R}^2 \rightarrow \mathbb{R}^2: (\xi_1, \xi_2) \mapsto (-\xi_2, \xi_1)$ (the counter-clockwise rotator in the plane by $\pi/2$) and set $B = N_{\mathbb{R} \cdot u}$. Then A is not 3^* monotone, $J_A = \frac{1}{2}(\text{Id} - A)$, $R_A = -A$ and $J_B = P_{\mathbb{R} \cdot u} = \langle \cdot, u \rangle u$. Moreover, the following hold:

- (i) $\text{dom } A = \text{ran } A = \mathbb{R}^2$, $\text{dom } B = \mathbb{R} \cdot u$, $\text{ran } B = \{u\}^\perp$.
- (ii) $\overline{D \cap R} = D \cap R = \mathbb{R}^2$.
- (iii) $\text{Id} - T = P_{\mathbb{R} \cdot J_A u}$.
- (iv) $\text{ran } (\text{Id} - T) = \overline{\text{ran}} (\text{Id} - T) = \mathbb{R} \cdot J_A u$.
- (v) $\mathbb{R} \cdot J_A u = \overline{\text{ran}} (\text{Id} - T) \subsetneq \overline{D \cap R} = \mathbb{R}^2$.

Proof. It is clear that A is not 3^* monotone (by applying Example 2.7(i) with S replaced by A), that $B = \partial_{\ell_{\mathbb{R} \cdot u}}$ is 3^* monotone, that $\text{dom } A = \text{ran } A = \mathbb{R}^2$ and that $\text{dom } B = \mathbb{R} \cdot u$. The formulae for J_A and R_A follow from applying Example 2.7(iii)&(iv) with $\gamma = 1$. The formula for J_B follows from, e.g., [4, Example 23.4].

(i)&(ii): This is clear.

(iii): Set $u = (\alpha, \beta)$ and observe that $\alpha^2 + \beta^2 = 1$. It is straightforward to verify that

$$J_B = P_{\mathbb{R} \cdot u} = \langle \cdot, u \rangle u = \begin{pmatrix} \alpha^2 & \alpha\beta \\ \alpha\beta & \beta^2 \end{pmatrix} \text{ hence } R_B = \begin{pmatrix} 2\alpha^2 - 1 & 2\alpha\beta \\ 2\alpha\beta & 2\beta^2 - 1 \end{pmatrix}. \quad (30)$$

On the one hand we have

$$\text{Id} - T = \frac{1}{2}(\text{Id} - R_B R_A) = \begin{pmatrix} \frac{1}{2} + \alpha\beta & \frac{1}{2} - \alpha^2 \\ -\frac{1}{2} + \beta^2 & \frac{1}{2} - \alpha\beta \end{pmatrix} = \frac{1}{2} \begin{pmatrix} (\alpha + \beta)^2 & -(\alpha^2 - \beta^2) \\ -(\alpha^2 - \beta^2) & (\alpha - \beta)^2 \end{pmatrix}. \quad (31)$$

On the other hand, observe that $J_A u = \frac{1}{2}(u - Au) = \frac{1}{2}(\alpha + \beta, \beta - \alpha)$, hence $\|J_A u\|^2 = \frac{1}{2}$. Consequently,

$$P_{\mathbb{R} \cdot J_A u} = \frac{1}{\|J_A u\|^2} \langle \cdot, J_A u \rangle J_A u = \frac{1}{2} \begin{pmatrix} (\alpha + \beta)^2 & -(\alpha^2 - \beta^2) \\ -(\alpha^2 - \beta^2) & (\alpha - \beta)^2 \end{pmatrix}, \quad (32)$$

and the conclusion follows.

(iv): This is a direct consequence of (iii). (v): Combine (ii) and (iv). ■

We conclude this section with the following example which shows in a strong way that the closures of the sets in Theorem 2.5 are critical.

Let U and V be nonempty closed convex subsets of X such that U is bounded. We recall that (see, e.g., [4, Proposition 3.42])

$$U + V \text{ is closed.} \quad (33)$$

Example 2.9. Suppose that $\dim X \geq 2$. Let $C = \text{ball}(0; 1)$ be the closed unit ball and let U be a closed linear subspace of X such that $U \neq \{0\}$. Set $(A, B) = (N_C, N_U)$. Then the following hold:

- (i) $(\text{dom } A, \text{dom } B, \text{ran } A, \text{ran } B) = (C, U, X, U^\perp)$.
- (ii) $\overline{\text{dom } A - \text{dom } B} = \overline{C + U} = C + U = \text{dom } A - \text{dom } B$.
- (iii) $\overline{\text{ran } (\text{Id} - T)} = C + U$.
- (iv) Set $S = \{u + u^\perp \mid u \in U \setminus \{0\}, u^\perp \in U^\perp, \|u^\perp\| = 1\}$. Then $S \subseteq (C + U) \setminus \text{ran } (\text{Id} - T)$. Consequently, $\text{ran } (\text{Id} - T) \subsetneq C + U = \text{dom } A - \text{dom } B$.

Proof. (i): This is clear in view of, e.g., [4, Corollary 21.25]. (ii): The first and the third identities follow from (i). The second identity follows from (33) applied with V replaced by C . (iii): Combine (i), (ii) and Theorem 2.5(iii). (iv): Clearly, $S \subseteq C + U$. Now let $s \in S$. Then $s = u + u^\perp, u \in U \setminus \{0\}, u^\perp \in U^\perp, \|u^\perp\| = 1$. Suppose for eventual contradiction that $s \in \text{ran } (\text{Id} - T)$. Let $y \in X$ be such that $s = y - Ty$. Because P_U is linear by, e.g., [12, 5.13(1) on page 79], we have in view of (6)

$$u + u^\perp = P_C y - 2P_U P_C y + P_U y. \quad (34)$$

We proceed by examining the following cases.

CASE 1: $y \in C$. Then (34) yields $u + u^\perp = y - 2P_U y + P_U y = P_{U^\perp} y \in U^\perp$. That is, $u \in (-u^\perp + U^\perp) \cap U = U^\perp \cap U$, hence $u = 0$ which is absurd.

CASE 2: $y \notin C$. In this case $P_C y = y/\|y\|$ by, e.g., [4, Example 3.18]. Therefore, (34) yields

$$u + u^\perp = \frac{1}{\|y\|} y - 2\frac{1}{\|y\|} P_U y + P_U y = \frac{1}{\|y\|} P_{U^\perp} y + \left(1 - \frac{1}{\|y\|}\right) P_U y. \quad (35)$$

That is, $u^\perp = P_{U^\perp} y/\|y\|$. On the one hand, this implies that $1 = \|u^\perp\| = \|P_{U^\perp} y\|/\|y\|$. Therefore, $\|P_{U^\perp} y\|^2 = \|y\|^2 = \|P_U y\|^2 + \|P_{U^\perp} y\|^2$. Hence, $P_U y = 0$. On the other hand, (35) implies that $u = (1 - (1/\|y\|))P_U y$. Altogether, we conclude that $u = 0$ which is absurd. Therefore, $s \notin \text{ran } (\text{Id} - T)$. The proof is complete. ■

3 The case $(A, B) = (\partial f, \partial g)$

In the remainder of this paper we assume that

$$f \text{ and } g \text{ are proper lower semicontinuous convex functions on } X. \quad (36)$$

We use the abbreviations

$$(P_f, P_{f^*}, P_g, R_f) = (\text{Prox}_f, \text{Prox}_{f^*}, \text{Prox}_g, 2\text{Prox}_f - \text{Id}). \quad (37)$$

In this case

$$T_{(\partial f, \partial g)} = \text{Id} - J_{\partial f} + J_{\partial g} R_{\partial f} = \text{Id} - P_f + P_g R_f. \quad (38)$$

The following simple lemma is stated in [9, page 167]. We provide the short proof for the sake of completeness.

Lemma 3.1. *Let $(y, z^*) \in \text{dom } f \times \text{dom } f^*$. Then*

$$\inf_{(x, x^*) \in \text{gra } \partial f} \langle x - y, x^* - z^* \rangle > -\infty. \quad (39)$$

Consequently, ∂f is 3^* monotone.

Proof. Let $(x, x^*) \in \text{gra } \partial f$. It follows from the subgradient inequality that $(\forall y \in X)$ $f(y) \geq f(x) + \langle x^*, y - x \rangle = f(x) + \langle x^* - z^*, y - x \rangle + \langle z^*, y - x \rangle$. Rearranging yields

$$\langle x - y, x^* - z^* \rangle \geq f(x) - f(y) - \langle z^*, x \rangle + \langle y, z^* \rangle \quad (40a)$$

$$= -f(y) - (\langle z^*, x \rangle - f(x)) + \langle y, z^* \rangle \quad (40b)$$

$$\geq -f(y) - f^*(z^*) + \langle y, z^* \rangle. \quad (40c)$$

This verifies (39). The 3^* monotonicity of ∂f follows from combining (39) and the fact that $\text{dom } \partial f \subseteq \text{dom } f$ and $\text{dom } \partial f^* \subseteq \text{dom } f^*$. ■

Proposition 3.2. *Let $w \in (\text{dom } f - \text{dom } g) \cap (\text{dom } f^* + \text{dom } g^*)$. Set $(A, B) = (\partial f, \partial g)$. Then $w \in \overline{\text{ran}}(\text{Id} - T)$.*

Proof. Proceeding similar to the proof Proposition 2.4, let $(x_n)_{n \geq 1}$ be such that $(x_n, w - \frac{1}{n^2}x_n)_{n \geq 1}$ lies in $\text{gra}(\text{Id} - T)$. Our goal is to show that

$$\frac{1}{n^2}x_n \rightarrow 0. \quad (41)$$

Obtain (a, b, a^*, b^*) from $\text{dom } f \times \text{dom } g \times \text{dom } f^* \times \text{dom } g^*$ such that

$$w = a - b = a^* + b^*. \quad (42)$$

Recall that Lemma 2.1 and (42) imply $(\forall x \in X)$

$$\langle x - (a + a^*), x - Tx - w \rangle \geq \langle P_f x - a, P_{f^*} x - a^* \rangle + \langle P_g R_f x - b, P_{g^*} R_f x - b^* \rangle. \quad (43)$$

It follows from Lemma 3.1 applied to f and g respectively that

$$\inf_{x \in X} \langle P_f x - a, P_{f^*} x - a^* \rangle > -\infty \text{ and } \inf_{x \in X} \langle P_g R_A x - b, P_{g^*} R_A x - b^* \rangle > -\infty. \quad (44)$$

Combining (44) with (43) in view of (42) we learn that

$$\begin{aligned} & \inf_{x \in X} \langle x - (a + a^*), x - Tx - (a - b) \rangle \\ & \geq \inf_{x \in X} (\langle P_f x - a, P_{f^*} x - a^* \rangle + \langle P_g R_f x - b, P_{g^*} R_f x - b^* \rangle) \end{aligned} \quad (45a)$$

$$\geq \inf_{x \in X} \langle P_f x - a, P_{f^*} x - a^* \rangle + \inf_{x \in X} \langle P_g R_f x - b, P_{g^*} R_f x - b^* \rangle > -\infty. \quad (45b)$$

Now proceed similarly to the proof of Proposition 2.4(i) to conclude that $(x_n/n)_{n \geq 1}$ is bounded hence (41) holds. ■

The following result further refines the formula obtained in Theorem 2.5 and Theorem 2.6 when the operators are subdifferential operators.

Theorem 3.3. Set $T = T_{(\partial f, \partial g)}$. Then the following hold:

$$(i) \quad \overline{\text{ran}} (\text{Id} - T) = \overline{(\text{dom } f - \text{dom } g) \cap (\text{dom } f^* + \text{dom } g^*)}.$$

$$(ii) \quad \overline{\text{ran}} T = \overline{(\text{dom } f - \text{dom } g^*) \cap (\text{dom } f^* + \text{dom } g)}.$$

Proof. (i): Indeed, by Proposition 3.2 and (10) applied with (A, B) replaced with $(\partial f, \partial g)$ we have

$$\overline{(\text{dom } f - \text{dom } g) \cap (\text{dom } f^* + \text{dom } g^*)} \quad (46a)$$

$$\subseteq \overline{\text{ran}} (\text{Id} - T) \quad (46b)$$

$$\subseteq \overline{(\text{dom } \partial f - \text{dom } \partial g) \cap (\text{dom } \partial f^* + \text{dom } \partial g^*)} \quad (46c)$$

$$\subseteq \overline{(\text{dom } f - \text{dom } g) \cap (\text{dom } f^* + \text{dom } g^*)}. \quad (46d)$$

(ii): Observe that $T = \text{Id} - T_{(\partial f, \partial g^*)}$. Now combine with (i) applied with g replaced by g^* . This completes the proof. ■

As a byproduct of the last proof we obtain the following corollary.

Corollary 3.4. We have

$$\overline{(\text{dom } \partial f - \text{dom } \partial g) \cap (\text{dom } \partial f^* + \text{dom } \partial g^*)} = \overline{(\text{dom } f - \text{dom } g) \cap (\text{dom } f^* + \text{dom } g^*)}. \quad (47)$$

Proof. This is a direct consequence of (46). ■

Example 3.5. Suppose that U and V are nonempty closed convex subsets of X . Set $f = \iota_U$ and $g = \iota_V$. Suppose that U is bounded. Then the following hold:

$$(i) \overline{\text{ran}}(\text{Id} - T) = U - V.$$

$$(ii) \overline{\text{ran}} T = U - (\text{rec } V)^\ominus.$$

Proof. Observe that $(\text{dom } f, \text{dom } g) = (\overline{\text{dom } f}, \overline{\text{dom } g}) = (U, V)$. Moreover, $(\text{dom } f^*, \overline{\text{dom } g^*}) = (X, (\text{rec } V)^\ominus)$. (i): It follows from Theorem 3.3(i) that $\overline{\text{ran}}(\text{Id} - T) = \overline{(U - V) \cap X} = \overline{U - V}$. Now combine with (33) applied with V replaced by $-V$. (ii): It follows from Theorem 3.3(ii) that $\overline{\text{ran}} T = \overline{\text{dom } f - \text{dom } g^*} = \overline{\text{dom } f} - \overline{\text{dom } g^*} = U - (\text{rec } V)^\ominus$. Now combine with (33) applied with V replaced by $-(\text{rec } V)^\ominus$. ■

4 The sets D and R and their corresponding minimal norm vectors

Because A , B and $\text{Id} - T$ are maximally monotone, we know that the sets $\text{dom } A$, $\text{dom } B$, $\text{ran } A$, $\text{ran } B$ and $\text{ran}(\text{Id} - T)$ have *convex closures* (see [22, Theorem 31.2]) i.e.,

$$\overline{D}, \overline{R} \text{ and } \overline{\text{ran}}(\text{Id} - T) \text{ are convex.} \quad (48)$$

Consequently, the following vectors

$$v = P_{\overline{\text{ran}}(\text{Id} - T)}(0), \quad v_D = P_{\overline{\text{dom } A - \text{dom } B}}(0), \quad v_R = P_{\overline{\text{ran } A + \text{ran } B}}(0) \quad (49)$$

are well defined.

Remark 4.1.

- (i) If X is finite-dimensional, then we know more (see, e.g., [6, Lemma 5.1(i)]): D and R are *nearly convex*²; so in particular by [7, Theorem 2.16], $\overline{D} = \overline{\text{ri } D}$ and $\overline{R} = \overline{\text{ri } R}$. Moreover, Minty's theorem applied to B yields $\text{ri } D - \text{ri } R = \text{ri}(D - R) = \text{ri}(\text{dom } A - \text{dom } B - \text{ran } A - \text{ran } B) = \text{ri } X = X$. Hence, $\text{ri } D \cap \text{ri } R \neq \emptyset$. Consequently, in view of [6, Lemma 3.9(iv)], we learn that

$$\overline{D \cap R} = \overline{D} \cap \overline{R}. \quad (50)$$

- (ii) We do not know whether or not the identity (50) survives in infinite-dimensional Hilbert spaces. Indeed, in view of Theorem 4.2 below, on the one hand, any counterexample must feature that neither of the operators is an affine relation or that none of the operators has a bounded domain or a bounded range. On the other hand, Theorem 4.4 implies that one has to avoid scenarios when both D and R has a nonempty interior.

²Let E be a subset of X . We say that E is *nearly convex* if there exists a convex subset C of X such that $C \subseteq E \subseteq \overline{C}$. (For a detailed discussion on the algebra of convex and nearly convex sets we refer the reader to [18, Section 3], [7] and [16].)

The next result provides some sufficient conditions where (50) holds in infinite-dimensional Hilbert spaces.

Theorem 4.2. *Suppose that one of the following conditions holds:*

- (i) $(\exists C \in \{A, B\})$ such that $\text{dom } C$ and $\text{ran } C$ are affine.
- (ii) $(A, B) = (N_K, N_L)$ where K and L are nonempty closed convex cones of X and $K^\ominus + L^\ominus$ is closed.
- (iii) $(\exists C \in \{A, B\})$ $\text{dom } C$ is bounded or $\text{ran } C$ is bounded.
- (iv) $(\exists C \in \{A, B\})$ $\text{dom } C = X$ or $\text{ran } C = X$.

Then $\overline{D \cap R} = \overline{D} \cap \overline{R}$.

Proof. It is obvious that $\overline{D \cap R} \subseteq \overline{D} \cap \overline{R}$. Therefore, we only need to verify that $\overline{D} \cap \overline{R} \subseteq \overline{D \cap R}$. (i): Indeed, suppose that $C = A$. Let $w \in \overline{D} \cap \overline{R}$ and let $(a_n, b_n, a_n^*, b_n^*)_{n \in \mathbb{N}}$ be a sequence in $\text{dom } A \times \text{dom } B \times \text{ran } A \times \text{ran } B$ such that $a_n - b_n \rightarrow w$ and $a_n^* + b_n^* \rightarrow w$. Set $(\forall n \in \mathbb{N}) w_n = J_A(a_n^* + b_n^*) + J_{A^{-1}}(a_n - b_n)$. Because J_A , as is $J_{A^{-1}}$, is firmly nonexpansive, it is continuous. Therefore

$$w_n \rightarrow J_A w + J_{A^{-1}} w = w. \quad (51)$$

We claim that

$$(w_n)_{n \in \mathbb{N}} \text{ lies in } D \cap R. \quad (52)$$

Indeed, on the one hand because $\text{dom } A = \text{ran } J_A$ is affine we have

$$w_n = J_A(a_n^* + b_n^*) + a_n - b_n - J_A(a_n - b_n) \quad (53a)$$

$$= J_A(a_n^* + b_n^*) + a_n - J_A(a_n - b_n) - b_n \in \text{dom } A - \text{dom } B = D. \quad (53b)$$

On the other hand, because $\text{ran } A = \text{ran } J_{A^{-1}}$ is affine we have

$$w_n = J_{A^{-1}}(a_n - b_n) + a_n^* + b_n^* - J_{A^{-1}}(a_n^* + b_n^*) \quad (54a)$$

$$= J_{A^{-1}}(a_n - b_n) + a_n^* - J_{A^{-1}}(a_n^* + b_n^*) + b_n^* \in \text{ran } A + \text{ran } B = R. \quad (54b)$$

This proves (52). Now combine with (51). The proof in the case $C = B$ is similar.

(ii): Let $w \in \overline{D} \cap \overline{R} = \overline{(K - L) \cap (K^\ominus + L^\ominus)}$ and let $(k_n, l_n)_{n \in \mathbb{N}}$ be a sequence in $K \times L$ such that $k_n - l_n \rightarrow w$. Set $(\forall n \in \mathbb{N}) w_n = P_{(K \cap L)^\ominus}(k_n - l_n)$. Observe that because $(K \cap L)^\ominus = \overline{K^\ominus + L^\ominus} = K^\ominus + L^\ominus$ by, e.g., [12, remarks on page 48], we have

$$w_n \rightarrow P_{(K \cap L)^\ominus} w = w. \quad (55)$$

On the one hand, by construction $(w_n)_{n \in \mathbb{N}}$ lies in $K^\ominus + L^\ominus = R$. On the other hand we have $(\forall n \in \mathbb{N}) w_n = P_{(K \cap L)^\ominus}(k_n - l_n) = k_n - l_n - P_{K \cap L}(k_n - l_n) = k_n - (l_n + P_{K \cap L}(k_n - l_n)) \in K - L = D$. Hence, $(w_n)_{n \in \mathbb{N}}$ lies in $(K - L) \cap (K^\ominus + L^\ominus) = D \cap R$. Combining this

with (55) we learn that $w \in \overline{(K - L) \cap (K^\ominus + L^\ominus)}$. (iii): If $\text{dom } C$ is bounded then [4, Corollary 21.25] implies that $\text{ran } C = X$, hence $R = X$. Therefore, $\overline{D \cap R} = \overline{D} = \overline{D} \cap X = \overline{D} \cap \overline{R}$. The case when $\text{ran } C$ is bounded follows similarly by applying the previous argument to C^{-1} . (iv): If $\text{dom } C = X$, then $D = X$. Therefore, $\overline{D \cap R} = \overline{R} = X \cap \overline{R} = \overline{D} \cap \overline{R}$. The case when $\text{ran } C = X$ follows similarly. ■

Before we proceed we recall the following facts.

Fact 4.3 (Simons). Suppose that $\text{int } D \neq \emptyset$. Then $\overline{D} = \overline{\text{int } D}$.

Proof. See [21, Theorem 22.1(c) and Theorem 22.2(a)]. ■

When the sets D and R are reasonably *fat*; namely, when $\text{int } D \neq \emptyset$ and $\text{int } R \neq \emptyset$ we obtain another sufficient condition for the conclusion $\overline{D \cap R} = \overline{D} \cap \overline{R}$ as we see in Theorem 4.4 below.

Theorem 4.4. Suppose that A and B are 3^* monotone. Suppose that $\text{int } D \neq \emptyset$ and $\text{int } R \neq \emptyset$. Then $\overline{D \cap R} = \overline{D} \cap \overline{R}$.

Proof. It follows from Fact 4.3 applied to (A, B) (respectively $(A^{-1}, B^{-\odot})$) that $\overline{D} = \overline{\text{int } D}$ (respectively $\overline{R} = \overline{\text{int } R}$). Let $w \in \overline{D \cap R}$. By Fact 4.3 $\overline{D \cap R} = \overline{\text{int } D} \cap \overline{\text{int } R}$, hence $w \in \overline{\text{int } D} \cap \overline{\text{int } R}$ and therefore

there exists a sequence $(d_n, r_n)_{n \in \mathbb{N}}$ in $\text{int } D \times \text{int } R$ such that $(d_n, r_n) \rightarrow (w, w)$. (56)

Observe that (56) implies that $(\forall n \in \mathbb{N}) 0 \in \text{int}(\text{dom } A - (d_n + \text{dom } B)) = \text{int}(\text{dom } A - \text{dom } B(\cdot - d_n))$. Therefore, by e.g., [4, Corollary 25.5(iii)] we learn that $(\forall n \in \mathbb{N}) A + B(\cdot - d_n)$ is maximally monotone. On the one hand, it follows from the 3^* monotonicity of A and B in view of the celebrated Brezis–Haraux theorem (see [9, Théorème 3] and also [22, Corollary 31.6]) that $(\forall n \in \mathbb{N})$

$$r_n \in \text{int}(\text{ran } A + \text{ran } B) = \text{int}(\text{ran } A + \text{ran } B(\cdot - d_n)) \quad (57a)$$

$$= \text{int } \text{ran } (A + B(\cdot - d_n)). \quad (57b)$$

Therefore, there exists a sequence $((x_n, u_n))_{n \in \mathbb{N}}$ in $\text{gra } A$ and $(x_n - d_n, -u_n + r_n)_{n \in \mathbb{N}}$ in $\text{gra } B$. Using Minty's theorem Fact 1.1 we rewrite this as

$$(x_n, u_n) = (J_A(x_n + u_n), J_{A^{-1}}(x_n + u_n)) \quad (58a)$$

$$(x_n - d_n, -u_n + r_n) = (J_B(x_n - u_n - d_n + r_n), J_{B^{-1}}(x_n - u_n - d_n + r_n)). \quad (58b)$$

It follows from (58) that

$$d_n = J_A(x_n + u_n) - J_B(x_n - u_n - d_n + r_n) \quad (59a)$$

$$r_n = J_{A^{-1}}(x_n + u_n) + J_{B^{-1}}(x_n - u_n - d_n + r_n). \quad (59b)$$

Now, set $(\forall n \in \mathbb{N})$

$$z_n = J_A(x_n + u_n) - J_B R_A(x_n + u_n) \in \text{ran } (\text{Id} - T). \quad (60)$$

We claim that $(\forall n \in \mathbb{N})$

$$\|z_n - r_n\|^2 + \|z_n - d_n\|^2 \leq \|d_n - r_n\|^2. \quad (61)$$

Indeed, using (60), (59b), the firm nonexpansiveness of $J_{B^{-1}}$, (58), (60) and (59a) we obtain

$$\|z_n - r_n\|^2 = \|J_A(x_n + u_n) - J_B R_A(x_n + u_n) - J_{A^{-1}}(x_n + u_n) - J_{B^{-1}}(x_n - u_n - d_n + r_n)\|^2 \quad (62a)$$

$$= \|J_{B^{-1}} R_A(x_n + u_n) - J_{B^{-1}}(x_n - u_n - d_n + r_n)\|^2 \quad (62b)$$

$$\leq \|R_A(x_n + u_n) - (x_n - u_n - d_n + r_n)\|^2 - \|J_B R_A(x_n + u_n) - J_B(x_n - u_n - d_n + r_n)\|^2 \quad (62c)$$

$$= \|(x_n - u_n) - (x_n - u_n - d_n + r_n)\|^2 - \|(x_n - z_n) - (x_n - d_n)\|^2 \quad (62d)$$

$$= \|d_n - r_n\|^2 - \|d_n - z_n\|^2. \quad (62e)$$

This proves (61). Taking the limit as $n \rightarrow \infty$ in (61) in view of (56) we learn that $z_n \rightarrow w$. Therefore, in view of (60), we learn that $w \in \overline{\text{ran}}(\text{Id} - T)$. Hence, $\overline{D} \cap \overline{R} \subseteq \overline{\text{ran}}(\text{Id} - T)$. Now combine this with (10) and recall that $\overline{D} \cap \overline{R} \subseteq \overline{D} \cap \overline{R}$. The proof is complete. ■

5 The vectors v , v_D and v_R

We now turn to the minimal norm vectors in the sets $\overline{D}$, $\overline{R}$ and $\overline{\text{ran}}(\text{Id} - T)$; namely, v_D , v_R and v respectively. Before we proceed we recall the following useful facts.

Fact 5.1. *Let U and V be nonempty closed convex subsets of X . Then*

$$P_{\overline{U-V}}(0) \in \overline{(P_U - \text{Id})(V)} \cap \overline{(\text{Id} - P_V)(U)} \subseteq (-\text{rec } U)^\ominus \cap (\text{rec } V)^\ominus. \quad (63)$$

Proof. This follows from [2, Corollary 4.6] and [23, Theorem 3.1]. ■

Fact 5.2. *Let $A: X \rightrightarrows X$ be maximally monotone. Then the following hold:*

$$(i) \quad (\text{rec } \overline{\text{dom } A})^\ominus \subseteq \text{rec}(\overline{\text{ran } A}).$$

$$(ii) \quad (\text{rec } \overline{\text{ran } A})^\ominus \subseteq \text{rec}(\overline{\text{dom } A}).$$

Proof. See [8, Lemma 3.2] ■

Lemma 5.3. *Let S_1 and S_2 be nonempty closed convex subsets of X and set $(\forall i \in \{1, 2\}) v_i = P_{S_i}(0)$. Suppose that $\langle v_1, v_2 \rangle \leq 0$ and that $v_1 + v_2 \in S_1 \cap S_2$. Then $v_1 + v_2 = P_{S_1 \cap S_2}(0)$.*

Proof. Let $s \in S_1 \cap S_2$. In view of the projection theorem see, e.g., [4, Theorem 3.16] it suffices to show that $(\forall s \in S_1 \cap S_2) \langle v_1 + v_2 - 0, v_1 + v_2 - s \rangle \leq 0$. Indeed, we have $\langle v_1 + v_2 - 0, v_1 + v_2 - s \rangle = \langle v_1, v_1 + v_2 - s \rangle + \langle v_2, v_1 + v_2 - s \rangle \leq \langle v_1, v_1 - s \rangle + \langle v_2, v_2 - s \rangle \leq 0 + 0 = 0$. ■

Parts of the following proposition were proved in [8]. We reiterate the short proof for the sake of completeness and to avoid any confusion with the standing assumptions in [8].

Proposition 5.4. *The following hold:*

- (i) $v_D \in (-\text{rec } \overline{\text{dom}} A)^\ominus \cap (\text{rec } \overline{\text{dom}} B)^\ominus = (-\text{rec } \overline{\text{dom}} A)^\ominus \cap (\text{rec } \overline{\text{dom}} B)^\ominus$.
- (ii) $v_R \in (-\text{rec } \overline{\text{ran}} A)^\ominus \cap (-\text{rec } \overline{\text{ran}} B)^\ominus = -((\text{rec } \overline{\text{ran}} A)^\ominus \cap (\text{rec } \overline{\text{ran}} B)^\ominus)$.
- (iii) $v_D \in (-\text{rec } \overline{\text{ran}} A) \cap (\text{rec } \overline{\text{ran}} B)$.
- (iv) $v_R \in (-\text{rec } \overline{\text{dom}} A) \cap (-\text{rec } \overline{\text{dom}} B) = -(\text{rec } \overline{\text{dom}} A \cap \text{rec } \overline{\text{dom}} B)$.
- (v) $\langle v_D, v_R \rangle = 0$.
- (vi) $v_D + v_R \in \overline{D} \cap \overline{R}$.
- (vii) $v_D + v_R = P_{\overline{D} \cap \overline{R}}(0)$.

Proof. (i)&(ii): Apply Fact 5.1 with (U, V) replaced by $(\overline{\text{dom}} A, \overline{\text{dom}} B)$ (respectively $(\overline{\text{ran}} A, -\overline{\text{ran}} B)$).

(iii)&(iv): Combine (i) (respectively (ii)) and Fact 5.2(i) (respectively Fact 5.2(ii)).

(v): It follows from (i) and (iv) that $(-v_D, -v_R) \in (\text{rec } \overline{\text{dom}} A)^\ominus \times \text{rec } \overline{\text{dom}} A$. Hence $\langle v_D, v_R \rangle = \langle -v_D, -v_R \rangle \leq 0$. Similarly, (iii) and (ii) imply that $(v_D, -v_R) \in \text{rec } \overline{\text{ran}} B \times (\text{rec } \overline{\text{ran}} B)^\ominus$. Hence, $-\langle v_D, v_R \rangle = \langle v_D, -v_R \rangle \leq 0$. Altogether, $\langle v_D, v_R \rangle = 0$. (vi): Indeed, in view of (iv) we have $-v_R \in \text{rec } \overline{\text{dom}} B$. Therefore, $v_D + v_R \in \overline{\text{dom}} A - \overline{\text{dom}} B + v_R = \overline{\text{dom}} A - (-v_R + \overline{\text{dom}} B) \subseteq \overline{\text{dom}} A - \overline{\text{dom}} B = \overline{\text{dom}} A - \overline{\text{dom}} B$. Similarly, in view of (iii) we have $v_D \in \text{rec } \overline{\text{ran}} B$. Therefore $v_D + v_R \in \overline{\text{ran}} A + v_D + \overline{\text{ran}} B \subseteq \overline{\text{ran}} A + \overline{\text{ran}} B = \overline{\text{ran}} A + \overline{\text{ran}} B$. (vii): Combine (v), (vi) and Lemma 5.3 applied with (S_1, S_2, v_1, v_2) replaced by $(\overline{D}, \overline{R}, v_D, v_R)$. ■

Before we proceed we recall the following useful fact.

Fact 5.5. *Let C be a nonempty closed convex subset of X and let $w = P_C(0)$. Suppose that $(u_n)_{n \in \mathbb{N}}$ is a sequence in C such that $\|u_n\| \rightarrow \|w\|$. Then $u_n \rightarrow w$.*

Proof. See [17, Lemma 2]. ■

Lemma 5.6. *Suppose that $\overline{\text{ran}}(\text{Id} - T) = \overline{D} \cap \overline{R}$ (see Theorem 2.5 for sufficient conditions). Then we have*

- (i) $v = v_D + v_R \Leftrightarrow v_D + v_R \in \overline{D \cap R}$.
- (ii) Suppose $\overline{D \cap R} = \overline{D} \cap \overline{R}$. Then $v = v_D + v_R$.

Proof. (i): " $\Rightarrow$ ": This is clear in view of (49). " $\Leftarrow$ ": Observe that (11) implies that $\|v\| \leq \|v_D + v_R\|$. It follows from Proposition 5.4(v), the definition of v_D that $\|v_D\|^2 \leq \langle v_D, \overline{D} \rangle$, hence $\|v_D\|^2 \leq \langle v_D, \overline{D \cap R} \rangle$. Similarly, $\|v_R\|^2 \leq \langle v_R, \overline{D \cap R} \rangle$. Therefore using Cauchy-Schwarz and Proposition 5.4(v) we learn that $\|v_D + v_R\|^2 = \|v_D\|^2 + \|v_R\|^2 \leq \langle v, v_D \rangle + \langle v, v_R \rangle = \langle v, v_D + v_R \rangle \leq \|v\| \|v_D + v_R\|$. Hence, $\|v_D + v_R\| \leq \|v\|$. Altogether, $\|v\| = \|v_D + v_R\|$. In view of (11) and Fact 5.5, we learn that $v = v_D + v_R$. (ii): Combine (i) and Proposition 5.4(vii). ■

Proposition 5.7. Suppose that U and V are nonempty closed convex subsets of X . Set $(A, B) = (N_U, N_V)$. Then the following hold:

- (i) $v_R = 0$.
- (ii) $v_D = v$.
- (iii) $v = v_D + v_R$.

Proof. (i): It follows from [23, Theorem 3.1] that $(\overline{\text{ran}} A, \overline{\text{ran}} B) = ((\text{rec } U)^\ominus, (\text{rec } V)^\ominus)$. Therefore, $0 \in (\text{rec } U)^\ominus + (\text{rec } V)^\ominus \subseteq \overline{R}$. Hence, $v_R = 0$.

(ii): This is [5, Proposition 3.5].

(iii): Combine (i) and (ii). ■

Corollary 5.8. Suppose that $\overline{\text{ran}} (\text{Id} - T) = \overline{D \cap R}$ (see Theorem 2.5 for sufficient conditions). Suppose additionally that one of the following holds:

- (i) $\overline{D \cap R} = \overline{D} \cap \overline{R}$.
- (ii) X is finite-dimensional.
- (iii) $(\exists C \in \{A, B\})$ such that $\text{dom } C$ and $\text{ran } C$ are affine.
- (iv) $(\exists C \in \{A, B\})$ such that $\text{dom } C = X$ or $\text{ran } C = X$.
- (v) $(\exists C \in \{A, B\})$ $\text{dom } C$ is bounded or $\text{ran } C$ is bounded.
- (vi) A and B are 3^* monotone, $\text{int } D \neq \emptyset$ and $\text{int } R \neq \emptyset$.
- (vii) $(A, B) = (N_U, N_V)$, U and V are nonempty closed subsets of X .

Then $v = v_D + v_R$.

Proof. (i): This is Lemma 5.6(ii). (ii): This follows from combining Remark 4.1(i) and (i). (iii)–(v): Combine Theorem 4.2(i)–(iv) and Lemma 5.6(ii). (vi): Combine (i) and Theorem 4.4. (vii): This is Proposition 5.7(iii). ■

Remark 5.9. The minimal displacement vector in $\overline{\text{ran}}(\text{Id} - T)$ can be found via (see [1], [10] and [17])

$$(\forall x \in X) \quad v = - \lim_{n \rightarrow \infty} \frac{T^n x}{n} = \lim_{n \rightarrow \infty} T^n x - T^{n+1} x. \quad (64)$$

Working in $X \times X$ and recalling (49) we observe that (see [4, Proposition 29.4])

$$(v_R, v_D) = P_{\overline{R} \times \overline{D}}(0). \quad (65)$$

Theorem 5.10 (computing v_D and v_R). Suppose that one of the following holds:

- (i) $\overline{D \cap R} = \overline{D} \cap \overline{R}$.
- (ii) X is finite-dimensional.
- (iii) $(\exists C \in \{A, B\})$ such that $\text{dom } C$ and $\text{ran } C$ are affine.
- (iv) $(\exists C \in \{A, B\})$ such that $\text{dom } C = X$ or $\text{ran } C = X$.
- (v) $(\exists C \in \{A, B\})$ $\text{dom } C$ is bounded or $\text{ran } C$ is bounded.
- (vi) A and B are 3^* monotone, $\text{int } D \neq \emptyset$ and $\text{int } R \neq \emptyset$.
- (vii) $(A, B) = (N_U, N_V)$, U and V are nonempty closed subsets of X .

Then $v = v_D + v_R$. Let $x \in X$. Then we have :

$$v_R = - \lim_{n \rightarrow \infty} \frac{J_A T^n x}{n} = \lim_{n \rightarrow \infty} (J_A T^n x - J_A T^{n+1} x) \quad (66a)$$

$$v_D = - \lim_{n \rightarrow \infty} \frac{J_{A^{-1}} T^n x}{n} = \lim_{n \rightarrow \infty} (J_{A^{-1}} T^n x - J_{A^{-1}} T^{n+1} x). \quad (66b)$$

Proof. The identity $v = v_D + v_R$ follows from Corollary 5.8. We now turn to (66). First we verify that

$$\text{ran}(J_A - J_A T) \subseteq \overline{R} \quad \text{and} \quad \text{ran}(J_{A^{-1}} - J_{A^{-1}} T) \subseteq \overline{D}. \quad (67)$$

Indeed, observe that $J_A T + J_{A^{-1}} T = T = \text{Id} - J_A + J_B R_A = J_A - R_A + J_B R_A = J_A - J_{B^{-1}} R_A$. Hence, $J_A - J_A T = J_{A^{-1}} T + J_{B^{-1}} R_A$. Consequently, $\text{ran}(J_A - J_A T) \subseteq \text{ran } A + \text{ran } B \subseteq \overline{R}$. Similarly we show that $\text{ran}(J_{A^{-1}} - J_{A^{-1}} T) \subseteq \overline{D}$. This verifies (67). Now let $x \in X$, let $n \geq 1$ and observe that (67) and the convexity of $\overline{D}$ and $\overline{R}$ (see (48)) imply

$$\{(J_A T^n x - J_A T^{n+1} x, J_{A^{-1}} T^n x - J_{A^{-1}} T^{n+1} x), \frac{1}{n}(J_A x - J_A T^n x, J_{A^{-1}} x - J_{A^{-1}} T^n x)\} \subseteq \overline{R} \times \overline{D}. \quad (68)$$

Now, [Proposition 5.4\(v\)](#), [\(65\)](#), [\(68\)](#), the firm nonexpansiveness of J_A and [\(64\)](#) yield

$$\|v_R + v_D\|^2 = \|v_R\|^2 + \|v_D\|^2 \quad (69a)$$

$$= \|(v_R, v_D)\|^2 \leq \|\frac{1}{n}(J_A x - J_A T^n x, J_{A^{-1}} x - J_{A^{-1}} T^n x)\|^2 \quad (69b)$$

$$= \frac{1}{n^2} \|J_A x - J_A T^n x\|^2 + \frac{1}{n^2} \|J_{A^{-1}} x - J_{A^{-1}} T^n x\|^2 \leq \frac{1}{n^2} \|(x - T^n x)\|^2 \quad (69c)$$

$$= \|\frac{1}{n}(x - T^n x)\|^2 \rightarrow \|v\|^2 = \|v_R + v_D\|^2, \quad (69d)$$

and

$$\|v_R + v_D\|^2 = \|v_R\|^2 + \|v_D\|^2 \quad (70a)$$

$$= \|(v_R, v_D)\|^2 \leq \|(J_A T^n x - J_A T^{n+1} x, J_{A^{-1}} T^n x - J_{A^{-1}} T^{n+1} x)\|^2 \quad (70b)$$

$$= \|J_A T^n x - J_A T^{n+1} x\|^2 + \|J_{A^{-1}} T^n x - J_{A^{-1}} T^{n+1} x\|^2 \quad (70c)$$

$$\leq \|T^n x - T^{n+1} x\|^2 \rightarrow \|v\|^2 = \|v_R + v_D\|^2. \quad (70d)$$

Therefore we learn from [\(69\)](#) and [\(70\)](#) respectively that

$$\frac{1}{n} \|(J_A x - J_A T^n x, J_{A^{-1}} x - J_{A^{-1}} T^n x)\| \rightarrow \|(v_R, v_D)\| \quad (71a)$$

$$\|(J_A T^n x - J_A T^{n+1} x, J_{A^{-1}} T^n x - J_{A^{-1}} T^{n+1} x)\| \rightarrow \|(v_R, v_D)\|. \quad (71b)$$

Now combine [\(71a\)](#) (respectively [\(71b\)](#)), [\(68\)](#), [Fact 5.5](#) applied with X replaced by $X \times X$, C replaced by $\bar{R} \times \bar{D}$ and w replaced by (v_R, v_D) in view of [\(65\)](#) to verify [\(66a\)](#) (respectively [\(66b\)](#)). The proof is complete. ■

We conclude this section with an application of our results which employs Pierra's technique to the product space.

Proposition 5.11. *Suppose that $m \in \{2, 3, \dots\}$. For every $i \in \{1, 2, \dots, m\}$, suppose that $A_i: X \rightrightarrows X$ is maximally monotone and 3^* monotone. Set $\Delta := \{(x, \dots, x) \in X^m \mid x \in X\}$, set $\mathbf{A} = \times_{i=1}^m A_i$, set $\mathbf{B} = N_\Delta$, and set $\mathbf{T} = T_{(\mathbf{A}, \mathbf{B})}$. Then the following hold:*

$$(i) \quad \Delta^\perp = \{(u_1, \dots, u_m) \in X^m \mid \sum_{i=1}^m u_i = 0\}.$$

$$(ii) \quad \overline{\text{ran}}(\text{Id} - \mathbf{T}) = \overline{\times_{i=1}^m \text{dom } A_i - \Delta} \cap \overline{\times_{i=1}^m \text{ran } A_i + \Delta^\perp}.$$

$$(iii) \quad \overline{\text{ran}} \mathbf{T} = \overline{\times_{i=1}^m \text{dom } A_i - \Delta^\perp} \cap \overline{\times_{i=1}^m \text{ran } A_i + \Delta}.$$

Proof. It is straightforward to verify that $\mathbf{A}$ is 3^* monotone, that $\text{dom } \mathbf{A} = \times_{i=1}^m \text{dom } A_i$ and that $\text{ran } \mathbf{A} = \times_{i=1}^m \text{ran } A_i$. (i): This is [\[4, Proposition 26.4\(i\)\]](#). (ii): Combine [Theorem 2.5\(i\)](#) applied with (A, B) replaced by $(\mathbf{A}, \mathbf{B})$ and [Theorem 4.2\(i\)](#) applied with B replaced by $\mathbf{B}$. (iii): Apply [Theorem 2.6\(i\)](#) with (A, B) replaced by $(\mathbf{A}, \mathbf{B})$ and [Theorem 4.2\(i\)](#) applied with B replaced by $\mathbf{B}^{-1}$. ■

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