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PERFECTLY HOMOGENEOUS BASES IN BANACH SPACES

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A bounded basis $\{x_n\}$ of a Banach space X is called perfectly homogeneous if every bounded block basic sequence $\{y_n\}$ of $\{x_n\}$ is equivalent to $\{x_n\}$. By a result of M. Zippin [4], a basis in a Banach space is perfectly homogeneous if and only if it is equivalent to the unit vector basis of c_0 or l_p, $1 \leq p < +\infty$. A basis $\{x_n\}$ of a Banach space X is called symmetric, if every permutation $\{x_{\sigma(n)}\}$ of $\{x_n\}$ is a basis of X, equivalent to the basis $\{x_n\}$. It is clear that every perfectly homogeneous basis is symmetric.

DEFINITION. A basis $\{x_n\}$ in a Banach space X is called lower (resp., upper) semi-homogeneous if it is bounded and every bounded block basic sequence $\{y_n\}$ of $\{x_n\}$ dominates (resp., is dominated by) $\{x_n\}$.

So a basis is perfectly homogeneous if and only if it is both lower and upper semi-homogeneous. In this note we shall construct Banach spaces with symmetric lower (resp., upper) semi-homogeneous bases which are not perfectly homogeneous.

We follow the notations and terminology of I. Singer's book [3]. If $\{x_n\}$ and $\{y_n\}$ are bases of Banach spaces X and Y, respectively, we say that $\{y_n\}$ dominates $\{x_n\}$ if for every sequence of scalars $\{\alpha_n\}$, $\sum_{n=1}^{\infty} \alpha_n y_n$ is convergent implies that $\sum_{n=1}^{\infty} \alpha_n x_n$ is convergent. We shall assume that every Banach space with symmetric basis is equipped with the equivalent symmetric norm [see e.g. p. 574; 3]. We shall also let $\mathcal{O}$ be the set of all strictly increasing subsequences of the natural numbers N.

It is easy to see that every lower (resp., upper) semi-homogeneous basis $\{x_n\}$ is unconditional and if $\sum_{n=1}^{\infty} \alpha_n x_n$ is convergent and

$$y_n = \sum_{i=p_n+1}^{p_{n+1}} \alpha_i x_i \Big/ \left\| \sum_{i=p_n+1}^{p_{n+1}} \alpha_i x_i \right\|, \; n = 1, 2, \ldots$$

where $\{p_n\} \in \mathcal{O}$ then $\sum_{n=1}^{\infty} \|\sum_{i=p_n+1}^{p_{n+1}} \alpha_i x_i\| \, y_n = \sum_{n=1}^{\infty} \alpha_n x_n$. Hence, if $\{x_n\}$ is lower semi-homogeneous and $\sum_{n=1}^{\infty} \alpha_n x_n$ converges then $\sum_{n=1}^{\infty} \|\sum_{i=p_n+1}^{p_{n+1}} \alpha_i x_i\| \, y_n$ converges and so $\sum_{n=1}^{\infty} \|\sum_{i=p_n+1}^{p_{n+1}} \alpha_i x_i\| \, x_n$ converges. Conversely, let $\{x_n\}$ be an unconditional basis of X with the property that $\sum_{n=1}^{\infty} \alpha_n x_n$ converges implies

$$\sum_{n=1}^{\infty} \left\| \sum_{i=p_n+1}^{p_{n+1}} \alpha_i x_i \right\| x_n$$

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converges for all $\{p_n\} \in \mathcal{O}$. Then for any normalized block basic sequence $y_n = \sum_{i=p_n+1}^{p_{n+1}} \alpha_i x_i$, $n=1, 2, \ldots$, of $\{x_n\}$, $\sum_{n=1}^{\infty} \beta_n y_n$ converges implies

$$\sum_{n=1}^{\infty} \|\beta_n y_n\| \, x_n = \sum_{n=1}^{\infty} |\beta_n| \, x_n$$

converges. Hence $\{x_n\}$ is lower semi-homogeneous. Using a similar argument for upper semi-homogeneous bases, we have the following proposition:

PROPOSITION 1. *Let $\{x_n\}$ be a basis in a Banach space X. Then*

(i) *$\{x_n\}$ is lower semi-homogeneous if and only if $\{x_n\}$ is unconditional and $\sum_{n=1}^{\infty} \|\sum_{i=p_n+1}^{p_{n+1}} \alpha_i x_i\| \, x_n$ is convergent in X for every $\sum_{n=1}^{\infty} \alpha_n x_n \in X$ and all $\{p_n\} \in \mathcal{O}$.*

(ii) *$\{x_n\}$ is upper semi-homogeneous if and only if $\{x_n\}$ is unconditional and for any sequence of scalars $\{\alpha_n\}$, $\sum_{n=}^{\infty} \|\sum_{i=p_n+1}^{p_{n+1}} \alpha_i x_i\|$ converges in X for some $\{p_n\} \in \mathcal{O}$ implies that $\sum_{n=1}^{\infty} \alpha_n x_n$ is convergent in X.*

REMARK. If $\{x_n\}$ is symmetric, then Proposition 1 holds when we replace $\sum_{i=p_n+1}^{p_{n+1}} \alpha_i x_i$ by $\sum_{i \in \mu_n} \alpha_i x_i$ where $\{\mu_n\}$ is a sequence of subsets (finite or infinite) of N such that $N = \bigcup_{n=1}^{\infty} \mu_n$ and $\mu_n \wedge \mu_m = \varnothing$ for all $n \neq m$.

PROPOSITION 2. *Let $\{x_n\}$ be a symmetric basis in a Banach space X and let $\{f_n\}$ be the sequence of biorthogonal functionals of $\{x_n\}$ in X^*. Then $\{x_n\}$ is upper semi-homogeneous if and only if $\{f_n\}$ is a lower semi-homogeneous basis of the closed linear subspace $[f_n]$ spanned by $\{f_n\}$.*

Proof. Suppose $\{x_n\}$ is upper semi-homogeneous. Let $g_n = \sum_{i=p_n+1}^{p_{n+1}} a_i f_i$, $\|g_n\| = 1$, $n=1, 2, \ldots$ be a block basic sequence of $\{f_n\}$. Since $\{x_n\}$ is symmetric, there exist $y_n = \sum_{i=p_n+1}^{p_{n+1}} b_i x_i$, $\|y_n\| = 1$ such that $g_n(y_n) = 1$, $n=1, 2, \ldots$. Suppose that $\sum_{n=1}^{\infty} \alpha_n g_n$ is convergent, then $\sum_{n=1}^{\infty} \alpha_n f_n$ is convergent if $(\sum_{n=1}^{\infty} \alpha_n f_n)(x)$ is convergent for all $x = \sum_{n=1}^{\infty} \beta_n x_n \in X$. Since $\{x_n\}$ is upper semi-homogeneous, we know that $\sum_{n=1}^{\infty} \beta_n y_n$ converges in X. Thus

$$\left(\sum_{n=1}^{\infty} \alpha_n g_n\right)\left(\sum_{n=1}^{\infty} \beta_n y_n\right) = \left(\sum_{n=1}^{\infty} \alpha_n f_n\right)(x)$$

is convergent. This completes the proof that $\{f_n\}$ is lower semi-homogeneous.

The proof of the converse is analogous. Q.E.D.

For any $a = (a_1, a_2, \ldots) \in c_0 \backslash l_1$, $a_1 \geq a_2 \geq \cdots \geq 0$ and $1 \leq p < +\infty$ let $d(a, p) = \{x = (\alpha_1, \alpha_2, \ldots) \in c_0 : \sup_{\sigma \in \pi} \sum_{n=1}^{\infty} |\alpha_{\sigma(n)}|^p a_n < +\infty\}$ where π is the set of all permutations of N. Then $d(a, p)$ with the norm $\|x\| = \sup_{\sigma \in \pi} (\sum_{n=1}^{\infty} |\alpha_{\sigma(n)}|^p a_n)^{1/p}$ for $x \in d(a, p)$ is a Banach space and the sequence $\{x_n\}$ of the unit vectors is a symmetric basis of the Lorentz sequence space $d(a, p)$. For $x = \sum_{n=1}^{\infty} \alpha_n x_n \in d(a, p)$, it is easy to see that $\|x\| = (\sum_{n=1}^{\infty} \hat{\alpha}_n^p a_n)^{1/p}$ where $\{\hat{\alpha}_n\}$ is a rearrangement of $\{|\alpha_n|\}$ in decreasing order. It is known [Theorem 4; 2] that $[f_n]$ has exactly two non-equivalent symmetric basic sequences if $\sup_{1 \leq n < +\infty} \sum_{i=1}^{n} d_i / \sum_{i=1}^{n} a_i < +\infty$ where

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$\{d_n\}$ is the rearrangement of the sequence $\{a_i a_j\}_{i,j=1,2,\ldots}$ in decreasing order. For example, if $a_1=a_2=1$, $a_n=1/\log n$, $n=3,4,\ldots$ then it can be proved that $\sup_{1\le n<+\infty} \sum_{i=1}^n d_i/\sum_{i=1}^n a_i<+\infty$ [Corollary 8; 2]. It is also known that every Lorentz sequence space has a unique symmetric basis up to equivalence and no $d(a,p)$ is isomorphic to c_0 or l_p, $1\le p<+\infty$. For the properties concerning Lorentz sequence spaces, we refer the reader to [1] and [2].

THEOREM. *Let $\{x_n, f_n\}$ be the unit vector basis of $d(a,p)$. Then $\{x_n\}$ is lower semi-homogeneous if and only if* $\sup_{1\le n<+\infty} \sum_{i=1}^n d_i/\sum_{i=1}^n a_i<+\infty$ *where $\{d_n\}$ is the rearrangement of $\{a_i a_j\}_{i,j=1,2,\ldots}$ in decreasing order.*

Proof. Suppose that $\{x_n\}$ is lower semi-homogeneous. Then there exists a constant $K>0$ such that $\|\sum_{n=1}^\infty \alpha_n x_n\|\le K\|\sum_{n=1}^\infty \alpha_n y_n\|$ for all normalized block basic sequences $\{y_n\}$ of $\{x_n\}$ and all scalars $\{\alpha_n\}$ such that $\sum_{n=1}^\infty \alpha_n y_n$ is convergent [see e.g. Proposition 24.2; 3]. Let $\{d_n\}$ be the rearrangement of $\{a_i a_j\}$ in decreasing order. For a fixed $n\in N$, let $n_1\ge n_2\cdots\ge n_k$ be such that $n=n_1+n_2+\cdots+n_k$ and $\sum_{i=1}^n d_i=\sum_{i=1}^k a_i s_{n_i}$ where $s_m=\sum_{i=1}^m a_i$, $m=1,2,\ldots$. Let

$$y_i=\frac{1}{s_{n_i}^{1/p}}\sum_{j=n_1+\cdots+n_{i-1}+1}^{n_1+\cdots+n_i} x_j$$

and $\alpha_i=s_{n_i}^{1/p}/s_n^{1/p}$, $i=1,2,\ldots,k$. Then for each $i=1,2,\ldots,k$, $\|y_i\|=1$, $\|\sum_{i=1}^k \alpha_i y_i\|=1$ and $\|\sum_{i=1}^k \alpha_i x_i\|^p=\sum_{i=1}^n d_i/\sum_{i=1}^n a_i$. Hence

$$\sup_{1\le n<+\infty}\sum_{i=1}^n d_i\Big/\sum_{i=1}^n a_i\le K^p<+\infty$$

Conversely, let $K>0$ be a constant such that $\sum_{i=1}^n d_i\le K\sum_{i=1}^n a_i$, $n=1,2,\ldots$. Then $\sum_{n=1}^\infty \alpha_n^p d_n\le K\sum_{n=1}^\infty \alpha_n^p a_n$ for all $\alpha_1\ge\alpha_2\ge\cdots\ge 0$. To show that $\{x_n\}$ is lower semi-homogeneous, by Proposition 1, it suffices to show that if $\sum_{n=1}^\infty \alpha_n x_n\in d(a,p)$ then $\sum_{n=1}^\infty \|\sum_{i\in\mu_n}\alpha_i x_i\|\,x_n$ is convergent for all $\{\mu_n\}$ such that $N=\bigcup_{n=1}^\infty \mu_n$, $\mu_n\wedge\mu_m=\varnothing$ if $n\ne m$. Let $\mu_n=\{(n,i)\}_{i=1,2,\ldots}$ and let $c_n=\|\sum_{i\in\mu_n}\alpha_i x_i\|$, $n=1,2,\ldots$. Since $\{x_n\}$ is symmetric, to show that $\sum_{n=1}^\infty c_n x_n$ is convergent we may assume that $c_1\ge c_2\ge\cdots\ge 0$. Now

$$\left\|\sum_{n=1}^\infty c_n x_n\right\|^p=\sum_{n=1}^\infty c_n^p a_n=\sum_{n=1}^\infty\sum_{i=1}^\infty \hat\alpha_{(n,i)}^p a_i a_n\le\sum_{n=1}^\infty \hat\alpha_n^p d_n\le K\sum_{n=1}^\infty \hat\alpha_n^p a_n=K\left\|\sum_{n=1}^\infty \alpha_n x_n\right\|^p$$

where $\{\hat\alpha_{(n,i)}\}$ (resp., $\{\hat\alpha_n\}$) is the rearrangement of $\{|\alpha_{(n,i)}|\}_{i=1,2,\ldots}$ (resp., $\{|\alpha_n|\}$) in decreasing order. Thus $\sum_{n=1}^\infty \|\sum_{i\in\mu_n}\alpha_i x_i\|\,x_n$ is convergent. This completes the proof that $\{x_n\}$ is lower semi-homogeneous. Q.E.D.

COROLLARY. *Let $\{x_n, f_n\}$ be the unit vector basis of $d(a,p)$. If*

$$\sup_{1\le n<+\infty}\sum_{i=1}^n d_i\Big/\sum_{i=1}^n a_i<+\infty$$

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where $\{d_n\}$ *is the rearrangement of* $\{a_i a_j\}_{i,j=1,2,\ldots}$ *in decreasing order then* $\{f_n\}$ *is a symmetric upper semi-homogeneous basis of* $[f_n]$ *which is not perfectly homogeneous.*

REMARK. Using the theorem, it is also easy to construct symmetric bases which are not lower (resp., upper) semi-homogeneous.

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