S. M. Frechette
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HECKE STRUCTURE OF SPACES OF HALF-INTEGRAL WEIGHT CUSP FORMS

SHARON M. FRECHETTE

Abstract. We investigate the connection between integral weight and half-integral weight modular forms. Building on results of Ueda [14], we obtain structure theorems for spaces of half-integral weight cusp forms $S_{k/2}(4N, \chi)$ where k and N are odd nonnegative integers with $k \geq 3$, and χ is an even quadratic Dirichlet character modulo $4N$. We give complete results in the case where N is a power of a single prime, and partial results in the more general case. Using these structure results, we give a classical reformulation of the representation-theoretic conditions given by Flicker [5] and Waldspurger [17] in results regarding the Shimura correspondence. Our version characterizes, in classical terms, the largest possible image of the Shimura lift given our restrictions on N and χ, by giving conditions under which a newform has an equivalent cusp form in $S_{k/2}(4N, \chi)$. We give examples (computed using tables of Cremona [4]) of newforms which have no equivalent half-integral weight cusp forms for any such N and χ. In addition, we compare our structure results to Ueda's [14] decompositions of the Kohnen subspace, illustrating more precisely how the Kohnen subspace sits inside the full space of cusp forms.

§1. Introduction

A vital part of the theory of integral weight modular forms is the study of simultaneous Hecke eigenforms, in particular newforms. The classical "multiplicity-one" result says that a newform is explicitly determined up to constant multiple by its eigenvalues for almost all the Hecke operators $T_k(p)$, p a prime, k a positive integer. If we attempt to define "half-integral weight newforms" using a definition analogous to that for integral weight, the theory breaks down rapidly, the crucial point being the lack of a multiplicity-one result. There are however significant connections between integral weight Hecke eigenforms and half-integral weight Hecke eigenforms, most notably the Shimura correspondence [12]. This correspondence maps Hecke eigenforms to Hecke eigenforms, which suggests that our knowledge

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of the integral weight structure can be "transported" to knowledge about half-integral weight forms. Through a representation-theoretic approach, Shintani [13] provides a mapping which is an adjoint to the Shimura lift and also preserves Hecke eigenforms. Unfortunately, the image of the Shintani map may be trivial, so it does not necessarily afford a practical method of transporting the Hecke structure back.

A promising alternative is to use trace identities to give decompositions of the spaces of cusp forms $S_{k/2}(4N,\chi)$ which illuminate their Hecke structure. These decompositions take the form of isomorphisms between $S_{k/2}(4N,\chi)$ and direct sums of spaces of integral weight newforms; the isomorphisms are as modules for the respective algebras generated by the Hecke operators acting on half-integral weight and integral weight cusp forms. Theorem 3.1 gives such decompositions when N is the power of a single odd prime and χ is even and quadratic. In Section 4 we compare these decompositions to Ueda's decompositions [14] of the Kohnen subspace, showing more precisely how this subspace sits inside the full space of cusp forms.

While Ueda's trace identity holds for levels $4N$ where N is any odd positive integer, transforming it into an isomorphism for $S_{k/2}(4N,\chi)$ becomes increasingly complex as the number of odd prime divisors of N increases. In the case of more general levels, partial Hecke structure results are sufficient to prove that subspaces of newforms satisfying certain conditions are missing from the decompositions of $S_{k/2}(4N,\chi)$ for all N and χ as above. Therefore all forms in these subspaces are not in the image of the Shimura lift [12] for any such N and χ. These results completely characterize the largest possible image of the Shimura lift from $S_{k/2}(4N,\chi)$ for N and χ as above, thus providing conditions under which this map will fail to be onto.

Specifically, Theorem 5.2 gives partial decompositions of $S_{k/2}(4\widehat{M},\chi)$ when $k \geq 5$ and the subspace $V_{3/2}(4\widehat{M},\chi) \subseteq S_{3/2}(4\widehat{M},\chi)$ when $k=3$, for odd positive integers $\widehat{M}$ satisfying certain restrictions. In Theorem 5.6 and Corollary 5.7, we show how introducing additional prime factors into the level affects the nature of the decompositions; essentially, shifting from the decomposition of $S_{k/2}(4N,\chi)$ to that of $S_{k/2}(4Nq,\chi)$ where $M|N$ and $q \nmid M$ does not result in the appearance of any additional newforms at levels dividing $2M$. Thus the nature of the decompositions with respect to any prime $p|M$ is unchanged.

For $t=0$ or 1 and M an odd positive integer, we consider the subspace $S_{k-1}^{n}(2^tM) \subseteq S_{k-1}^{0}(2^tM)$ defined in [15] which excludes all forms in $S_{k-1}^{0}(2^tM)$ that are twists of newforms of lower levels. As a consequence of Theorem 5.2 and Corollary 5.7, Corollary 5.1 characterizes those newforms $F \in S_{k-1}^{n}(2^tM)$ appearing in the image of the Shimura lift from $S_{k/2}(4N,\chi)$ for some N and χ as above. This characterization is given in terms of the congruence modulo 4 of the primes p dividing M, the exponents to which these primes occur, and the subspace of $S_{k-1}^{n}(2^tM)$ to which F belongs. Corollary 5.1 provides a classical reformulation of representation-theoretic conditions given by Flicker [5] and Waldspurger [17] in theorems determining when a newform $F \in S_{k-1}^{n}(2^tM)$ has equivalent half-integral weight cusp forms. These results are only given for $F \in S_{k-1}^{n}(2^tM)$; some discussion of how one may proceed in investigating the case $F \in S_{k-1}^{n\perp}(2^tM)$ is also given.

In Section 6 we provide examples of nonzero newforms which are not in the image of the Shimura lift for any $S_{k/2}(4N,\chi)$ as above. These examples are produced using tables of Cremona [4], turning to the case $k=3$. We compute lower bounds for the dimension of certain subspaces of $S_2(2^tM)$ for particular values of t and M.

Many interesting questions have been raised by these results, most notably questions about the role which the Atkin-Lehner involution W_p plays in determining whether a newform $F \in S_{k-1}^{n}(2^tM)$ is in the image of the Shimura lift at a given level. In the concluding remarks we discuss this and other related questions.

This paper contains results of my thesis work at Dartmouth College. I would like to express my deep gratitude to my thesis advisor, Thomas Shemanske, for all his guidance and encouragement. I would also like to thank John Rhodes for several insightful conversations.

§2. Preliminaries

2.1. Notation and terminology

Let $SL_2(\mathbf{Z}) = \left\{\left(\begin{smallmatrix} a & b \\ c & d\end{smallmatrix}\right) : a,b,c,d \in \mathbf{Z} \text{ and } ad-bc=1\right\}$, and for each positive integer N consider the congruence subgroup $\Gamma_0(N) = \left\{\left(\begin{smallmatrix} a & b \\ c & d\end{smallmatrix}\right) \in SL_2(\mathbf{Z}) : c \equiv 0 \pmod{N}\right\}$. Let χ be a Dirichlet character modulo N. Then $\chi = \prod_{p|N}\chi_p$ where χ_p is a Dirichlet character modulo $p^{\mathrm{ord}_p(N)}$. We will be concerned with $\chi_p = \left(\frac{*}{p}\right)$, the Legendre symbol modulo an odd prime p.

Let $k \geq 3$ be an odd positive integer. Denote the space of all cusp forms of weight $k-1$, level N and character χ by $S_{k-1}(N,\chi)$, or simply $S_{k-1}(N)$

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if the character is trivial. For each positive integer n relatively prime to N, we consider one Hecke operator $T_k(n)$ acting on $S_{k-1}(N,\chi)$. For each positive integer Q with $(Q,N/Q)=1$, let W_Q denote the Atkin-Lehner involution, and abbreviate $W_{q^{\mathrm{ord}_q(N)}}$ as W_q. Let R_χ denote the twisting operator with respect to the Dirichlet character χ, and write $R_\chi = R_p$ if $\chi = \left(\frac{*}{p}\right)$. Finally, let B_d denote the shift operator for a positive integer d. Definitions and details can be found in [8] or [15]. We will often need the commuting relationships between these operators, given in the following:

PROPOSITION 2.1. ([1], [2], [15]) *For N and n positive integers and $k>1$ an odd positive integer, let ψ be a quadratic character of conductor f_ψ, and let Q be a positive divisor of N with $(Q,N/Q)=1$. For any $F\in S_k(N)$, the following hold*:

(1) *If* $(n, Nf_\psi)=1$, *then* $F|R_\psi|T_k(n) = \psi(n)F|T_k(n)|R_\psi$.

(2) *If* $(n,N)=1$, *then* $F|T_k(n)|W_Q = F|W_Q|T_k(n)$.

(3) *If* $(Q,f_\psi)=1$, *then* $F|R_\psi|W_Q = \psi(n)F|W_Q|R_\psi$.

(4) *If Q' is another divisor of N such that* $(Q', QN/Q')=1$, *then* $F|W_{Q'}|W_Q = F|W_{Q'Q} = F|W_Q|W_{Q'}$.

Moreover, if $N=p^\nu M$, with p an odd prime, M a positive integer with $p\nmid M$, and $\nu=0$ or 1, then

(5) $F|R_p|W_{p^2} = \left(\frac{-1}{p}\right)F|R_p$.

We will also have need of several subspaces of $S_{k-1}(N,\chi)$: The subspace $S^-_{k-1}(N,\chi)$ generated by the oldforms, its orthogonal complement the subspace $S^0_{k-1}(N,\chi)$ generated by the newforms, and the image of this space under the action of R_p, denoted $S^0_{k-1}(N,\chi)|R_p$. For details, see [8] or [15]. We will also use the following corollary to the strong multiplicity-one result:

PROPOSITION 2.2. *Let N be a positive integer, let d be a positive divisor of N, and let $\delta(N/d)$ denote the number of positive divisors of N/d. Then we have the following isomorphism as modules for the Hecke algebra*:

$$S_k(N) \cong \bigoplus_{d|N} \delta(N/d) S^0_k(d).$$

Proof. This follows from Lemma 15 and Theorem 5 in [1]. □

In the half-integral weight setting, denote the space of all cusp forms of weight $k/2$, level $4N$ and character χ by $S_{k/2}(4N,\chi)$. For each positive integer n relatively prime to $2N$, we consider one Hecke operator $\widetilde{T}_{k/2}(n^2)$ acting on $S_{k/2}(4N,\chi)$. For details, see [15].

In the case $k = 3$, we must restrict attention to those half-integral weight cusp forms which correspond to integral weight cusp forms under the Shimura correspondence. We have the following construction: Let $U_{3/2}(4N,\chi)$ be the subspace of $S_{3/2}(4N,\chi)$ which is spanned by the functions $h_\psi(tz)$ where $h_\psi(z) = \sum_{m=1}^{\infty} \psi(m) m e^{2\pi i m^2 z}$, ψ is a primitive character modulo a positive integer r with $\psi(-1) = -1$, and t is an integer, such that the conditions $tr^2 | N$ and $\chi = \left(\frac{-t}{*}\right)\psi$ are satisfied. Let $V_{3/2}(4N,\chi)$ be the orthogonal complement of $U_{3/2}(4N,\chi)$ in $S_{3/2}(4N,\chi)$ with respect to the Petersson inner product. Under the Shimura correspondence, the forms in $U_{3/2}(4N,\chi)$ correspond to Eisenstein series and the forms in $V_{3/2}(4N,\chi)$ correspond to cusp forms (see [14]). Therefore when $k = 3$, we consider only $V_{3/2}(4N,\chi)$.

The decompositions given in this paper are obtained from trace identities. Let $\operatorname{tr}(T \mid V)$ denote the trace of an operator T on a vector space V. If we have subspaces $S_{half} \subseteq S_{k/2}(4N,\chi)$ and $S_{whole} \subseteq S_{k-1}(2N)$, where N is a positive integer and χ is a quadratic Dirichlet character modulo $4N$, it can be shown that

$$\begin{aligned}(2.1)\quad &\operatorname{tr}\big(\widetilde{T}_{k/2}(n^2) \mid S_{half}\big) = \operatorname{tr}(T_{k-1}(n) \mid S_{whole}) \text{ for all } n \text{ with } (n,2N) = 1\\ &\iff S_{half} \cong S_{whole} \text{ as modules for the algebra generated by all the Hecke operators.}\end{aligned}$$

See [6] for details. By explicitly calculating $\operatorname{tr}\big(\widetilde{T}_{k/2}(n^2) \mid [S_{k/2}(4N,\chi)]\big)$ when $(n,2N) = 1$ and χ is even and quadratic, Ueda [14] proved an identity relating this trace to certain traces on spaces of integral weight forms. We use this identity in proving Theorem 3.1 and Theorem 5.6 and its corollaries. To do so, we must break apart the newform spaces, isolating a subspace $S_{k-1}^n(N) \subseteq S_{k-1}^0(N)$ which is closed under not only the appropriate W and T operators but also under the twisting operators R_p for any odd prime p with $p^2 | N$. This space was defined by Ueda, and is denoted by $S_{k-1}^n(N)$ in [14] and by $S_{k-1}^*(N)$ in [15]. We give definitions of $S_{k-1}^n(N)$ and its relevant subspaces, as well as several properties which will be needed throughout.

2.2. The subspace $S^n_{k-1}(N)$

In general, the space $S^0_{k-1}(N)$ need not be closed under quadratic twists. In fact, for an odd prime p with $\mathrm{ord}_p(N) = 2$, taking a newform of level p^2M with $p \nmid M$ and twisting by R_p gives us a newform of level pM or M by Theorem 6 of [1]. Moreover, $S^0_{k-1}(p^\nu M)|R_p \subseteq S^0_{k-1}(p^2M)$ for $\nu = 0, 1$. To obtain a subspace of $S^0_{k-1}(N)$ which is closed under quadratic twists, we must "split off" all forms in $S^0_{k-1}(N)$ which are quadratic twists of newforms of lower levels.

Let Ω denote the set of all odd primes dividing N, and for each $p \in \Omega$ let $\nu_p = \mathrm{ord}_p(N)$. Let $\widetilde{\Omega} = \Omega \cup \{2\}$, and write $N = 2^tM = \prod_{p\in\widetilde{\Omega}} p^{\nu_p}$. Put $\Omega_2 = \{p \in \Omega : \nu_p = 2\}$, and let $R_A = \prod_{p\in A} R_p$ for any subset A of Ω_2. For any partition $\Omega_2 = A + B + C$ of the set of primes occurring with exponent 2, define an integer

$$N(B, C) = \prod_{p\in\widetilde{\Omega}-(B+C)} p^{\nu_p} \prod_{p\in B} p.$$

Then $S^0_{k-1}(N(B, C))|R_{B+C} \subseteq S^0_{k-1}(N)$ by repeated application of Theorem 6 of [1]. If we take the sum of the subspaces $S^0_{k-1}(N(B, C))|R_{B+C}$ over all partitions $\Omega_2 = A + B + C$ where $\Omega_2 \neq A$, this will include all forms in $S^0_{k-1}(N)$ which are quadratic twists of newforms of lower levels. We therefore put

$$S^2_{k-1}(N) = \sum_{\substack{\Omega_2=A+B+C\\ \Omega_2\neq A}} S^0_{k-1}(N(B, C))|R_{B+C},$$

and define $S^n_{k-1}(N)$ to be the orthogonal complement of $S^2_{k-1}(N)$ in $S^0_{k-1}(N)$ with respect to the Petersson inner product. For example, if $N = 2p^2$, we have

$$S^2_{k-1}(2p^2) = S^0_{k-1}(2p)|R_p + S^0_{k-1}(2)|R_p.$$

Proposition 2.3. (Ueda [15]) *Let the notation and terminology be as above. Then*

$$S^0_{k-1}(N) = \bigoplus_{\Omega_2=A+B+C} S^n_{k-1}(N(B, C))|R_{B+C}.$$

Note that the $\Omega_2 = A$ summand is $S^n_{k-1}(N)$. We refer to the direct sum of the remaining terms as $S^{n\perp}_{k-1}(N)$.

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With respect to each odd prime p dividing N with $\mathrm{ord}_p(N) \geq 2$, we define four subspaces of $S^n_{k-1}(N)$ as follows: for each choice of $\alpha_p, \beta_p = \pm 1$ put

$$S^{p\alpha_p\beta_p}_{k-1}(N) = \{F \in S^n_{k-1}(N) : F|W_p = \alpha_p F \text{ and } F|R_p|W_p = \beta_p F|R_p\}.$$

These subspaces appeared in [11], denoted by S_I, S_{II}, S_{II_ψ}, and S_{III}. It is easy to show that $S^n_{k-1}(N) = S^{p++}_{k-1}(N) \oplus S^{p+-}_{k-1}(N) \oplus S^{p-+}_{k-1}(N) \oplus S^{p--}_{k-1}(N)$. In general, we will need to split the space $S^n_{k-1}(N)$ into subspaces depending on each odd prime $p|N$ with $\mathrm{ord}_p(N) \geq 2$. Let Ω_{2+} denote the set of odd prime divisors of N with $\mathrm{ord}_p(N) \geq 2$. For any choice of $\alpha_p, \beta_p = \pm 1$ for each $p \in \Omega_{2+}$, let $S^{(p\alpha_p\beta_p)_{p\in\Omega_{2+}}}_{k-1}(N)$ denote the subspace of $S^n_{k-1}(N)$ consisting of forms F which satisfy the relations $F|W_p = \alpha_p F$, and $F|R_p|W_p = \beta_p F|R_p$. We have

$$S^n_{k-1}(N) = \bigoplus S^{(p\alpha_p\beta_p)_{p\in\Omega_{2+}}}_{k-1}(N)$$

where the direct sum is taken over all possible choices for the tuple $(p\alpha_p\beta_p)_{p\in\Omega_{2+}}$. Using these definitions and Proposition 2.1, we can easily prove the following facts about the behavior of these subspaces $S^{p\alpha_p\beta_p}_{k-1}(N)$ under the action of various operators:

PROPOSITION 2.4. *Let N be a positive integer with $\mathrm{ord}_p(N) \geq 2$ for some odd prime p. The subspaces $S^{p\alpha_p\beta_p}_{k-1}(N)$ as defined above behave under the action of the Hecke operators $T_{k-1}(n)$, involution W_p, and twisting operator R_p in the following way*:

(1) *$S^{p\alpha_p\beta_p}_{k-1}(N)$ is closed under the action of $T_{k-1}(n)$ for $\alpha_p, \beta_p = \pm 1$ and $(n, N) = 1$.*

(2) *$S^{p\alpha_p\beta_p}_{k-1}(N)$ is closed under the action of W_p for $\alpha_p, \beta_p = \pm 1$.*

(3) *$S^{p++}_{k-1}(N)$ and $S^{p--}_{k-1}(N)$ are both closed under the action of R_p, while $S^{p+-}_{k-1}(N)|R_p = S^{p-+}_{k-1}(N)$ and $S^{p-+}_{k-1}(N)|R_p = S^{p+-}_{k-1}(N)$.*

An important consequence of Proposition 2.4 is that $S^n_{k-1}(N)$ is closed under the action of the appropriate Hecke operators, involutions and twists. Moreover, we have the following:

PROPOSITION 2.5. *Let the notation and terminology be as above. Then each summand $S^n_{k-1}(N(B,C))|R_{B+C}$ of $S^{n\perp}_{k-1}(N)$ is closed under the action of W_p for each odd prime $p|N$.*

Proof. Let $F \in S^n_{k-1}(N(B,C))|R_{B+C}$, so that $F = G|R_{B+C}$ for some $G \in S^n_{k-1}(N(B,C))$. First suppose $p \notin B+C$. Then $p^{\nu_p}\|N(B,C)$, and we have $F|W_p = G|R_{B+C}|W_p = G|W_p|R_{B+C}$ by (3) of Proposition 2.1. Since $G \in S^n_{k-1}(N(B,C))$, we can write G as a linear combination of some $G_{p\alpha_p\beta_p} \in S^{p\alpha_p\beta_p}_{k-1}(N(B,C))$. Each subspace $S^{p\alpha_p\beta_p}_{k-1}(N(B,C))$ is closed under the action of W_p by Proposition 2.4, hence $G_{p\alpha_p\beta_p}|W_p \in S^{p\alpha_p\beta_p}_{k-1}(N(B,C)) \subseteq S^n_{k-1}(N(B,C))$. By linearity, $G|W_p \subseteq S^n_{k-1}(N(B,C))$ as well, and therefore $F|W_p \in S^n_{k-1}(N(B,C))|R_{B+C}$.

If $p \in B+C$, then $\mathrm{ord}_p(N(B,C)) = 0$ if $p \in C$ or 1 if $p \in B$. Put $R_{B+C} = R_p R_{B+C-\{p\}}$. Thus by Proposition 2.1,

$$\begin{aligned} F|W_p = F|W_{p^2} &= G|R_p|R_{B+C-\{p\}}|W_{p^2} = G|R_p|W_{p^2}|R_{B+C-\{p\}} \\ &= \left(\frac{-1}{p}\right)G|R_p|R_{B+C-\{p\}} = \left(\frac{-1}{p}\right)G|R_{B+C} \\ &= \pm G|R_{B+C} = \pm F \in S^n_{k-1}(N(B,C))|R_{B+C}. \end{aligned}$$

□

2.3. Equivalent forms

The results in Sections 5 and 6 pertain to half-integral weight cusp forms which are equivalent to a given integral weight newform F. We say that a newform $F \in S^0_{k-1}(2N,\chi^2)$ is *equivalent* to a cusp form $f \in S_{k/2}(4N,\chi)$ if f and F are both Hecke eigenforms with corresponding eigenvalues equal for almost all primes p. That is, $f|\widetilde{T}_{k/2}(p^2) = \lambda_p f$ and $F|T_{k-1}(p) = \lambda_p F$ for almost all p, where $\lambda_p \in \mathbf{C}$. Let $S_{k/2}(4N,\chi,F)$ denote the subspace of $S_{k/2}(4N,\chi)$ consisting of all forms equivalent to F. We have the following direct sum,

$$S_{k/2}(4N,\chi) = \bigoplus_F S_{k/2}(4N,\chi,F)$$

taken over all newforms F of levels dividing $2N$.

§3. The decompositions at level $4p^m$

In this section we give explicit means of constructing decompositions for the spaces $S_{k/2}(4p^m,\chi)$ for $k \geq 5$ (resp. $V_{3/2}(4p^m,\chi)$ for $k = 3$). Decompositions were also computed explicitly for levels with two distinct odd prime divisors, and the regular structure of these decompositions illustrates quite well what should happen in the case of general level $4N$. We discuss these more general levels in the remark at the end of this section as well as in Section 5.

THEOREM 3.1. *Let the notation and terminology be as above, and let* $a \geq -1$, t, *and* u *be integers. For any specified* m *and* χ, *we explicitly construct an isomorphism between* $S_{k/2}(4p^m, \chi)$ *for* $k \geq 5$ (*resp.* $V_{3/2}(4p^m, \chi)$ *for* $k = 3$) *and a direct sum of integral weight forms which depends on* m *and* χ. *This is an isomorphism as modules for the Hecke algebra. The summands are* (*subspaces of*) $S^0_{k-1}(2^t p^u)|R_A$ *for* $0 \leq t \leq 1$ *and* $0 \leq u \leq m$, *where* $A = 1$ *or* p (*with* R_1 *trivial*). *For each such level and twist, consult the table below to determine the subspace and coefficient with respect to* p. *Then multiply each coefficient by* $2 - t$, *and take the direct sum over all possible choices of* $0 \leq t \leq 1$, $0 \leq u \leq m$, *and* $A = 1$ *or* p.

Contribution at level $2^t p^u$ twisted by R_A						
Cases				Sum over:	Coefficient:	Subspace:
m	A	u	χ			
0	1	0	1	—	1	S^0
$2a{+}3$	1	>0, even	either	$\alpha_p,\beta_p{=}\pm1$	$\left[2{+}\alpha_p{+}\beta_p\left(\frac{-1}{p}\right)\right](a{+}2{-}\frac{u}{2})$	$S^{p\alpha_p\beta_p}$
”	”	≥3, odd	either	—	$2a{+}4{-}u$	S^0
”	”	0 or 1	either	—	$4a{+}6{-}u(a{+}2)$	S^0
”	p	0 or 1	either	—	$\left(2{+}(2{-}u)\left(\frac{-1}{p}\right)\right)(a{+}1)$	S^0
$2a{+}4$	1	>0, even, $u{=}m$	1	$\beta_p{=}\pm1$	2	$S^{p+\beta_p}$
”	”	”	$\left(\frac{p}{*}\right)$	$\alpha_p,\beta_p{=}\pm1$	$1{+}\beta_p\left(\frac{-1}{p}\right)$	$S^{p\alpha_p\beta_p}$
”	”	>0, even, $u{<}m$	1	$\alpha_p,\beta_p{=}\pm1$	$(a{+}2{-}u)\left[2{+}\alpha_p{+}\beta_p\left(\frac{-1}{p}\right)\right]{+}(1{+}\alpha_p)$	$S^{p\alpha_p\beta_p}$
”	”	”	$\left(\frac{p}{*}\right)$	$\alpha_p,\beta_p{=}\pm1$	$(a{+}2{-}u)\left[2{+}\alpha_p{+}\beta_p\left(\frac{-1}{p}\right)\right]{+}\left(1{+}\beta_p\left(\frac{-1}{p}\right)\right)$	$S^{p\alpha_p\beta_p}$
”	”	≥3, odd	either	—	$2a{+}5{-}u$	S^0
”	”	0 or 1	1	—	$4a{+}8{-}u(a{+}3)$	S^0
”	”	”	$\left(\frac{p}{*}\right)$	—	$4a{+}8{-}u(a{+}2)$	S^0
$2a{+}4$; $a{\neq}{-}1$	p	0 or 1	1	—	$(2a{+}3{-}u(a{+}1))\left(1{+}\left(\frac{-1}{p}\right)\right){+}u(a{+}1)$	$S^0\|R_p$
”	”	”	$\left(\frac{p}{*}\right)$	—	$(2a{+}3{-}u(a{+}2))\left(1{+}\left(\frac{-1}{p}\right)\right){+}u(a{+}2)$	$S^0\|R_p$
2	”	0 or 1	1	—	$1{+}\left(\frac{-1}{p}\right)$	$S^0\|R_p$
”	”	”	$\left(\frac{p}{*}\right)$	—	$1{+}(1{-}u)\left(\frac{-1}{p}\right)$	$S^0\|R_p$

EXAMPLE 3.2. By Theorem 3.1, for $k \geq 5$ and $\chi = 1$ or $\left(\frac{p}{*}\right)$, we have:

$$S_{k/2}(4p^3, \chi) \cong \bigoplus_{t=0}^{1}(2-t)\Big\{ S^0_{k-1}(2^t p^3) \oplus \left(3 + \left(\frac{-1}{p}\right)\right) S^{p++}_{k-1}(2^t p^2)$$

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$$\oplus\Big(3-\Big(\frac{-1}{p}\Big)\Big)S_{k-1}^{p+-}(2^tp^2)\oplus\Big(1+\Big(\frac{-1}{p}\Big)\Big)S_{k-1}^{p-+}(2^tp^2)$$
$$\oplus\Big(1-\Big(\frac{-1}{p}\Big)\Big)S_{k-1}^{p--}(2^tp^2)\oplus 4S_{k-1}^0(2^tp)\oplus 6S_{k-1}^0(2^t)$$
$$\oplus\Big(2+\Big(\frac{-1}{p}\Big)\Big)S_{k-1}^0(2^tp)|R_p\oplus 2\Big(1+\Big(\frac{-1}{p}\Big)\Big)S_{k-1}^0(2^t)|R_p\Big\}.$$

Notice that exactly one of $\big(1\pm\big(\frac{-1}{p}\big)\big)$ is zero according to the congruence of p modulo 4. Hence exactly one of the spaces $S_{k-1}^{p-\pm}(2^tp^2)$ will be "missing" from this isomorphism. Consequently, newforms in that space are not in the image of the Shimura lift from $S_{k/2}(4p^3,\chi)$. In Section 5, we characterize those newforms in the S^n-spaces which *are* in the image of the Shimura lift from $S_{k/2}(4N,\chi)$ for some odd positive integer N and some even quadratic Dirichlet character χ modulo $4N$. Moreover, we give conditions under which a newform in an S^n-space has no such preimage.

Proof of Theorem 3.1. The isomorphisms are obtained by manipulating the following trace identities of Ueda's:

THEOREM 3.3. (Ueda [14]) *Let N be a positive integer such that $2\le \mathrm{ord}_2(N)=\mu\le 4$ and put $M=2^{-\mu}N$. Let χ be an even quadratic Dirichlet character modulo N and suppose that the conductor of χ is divisible by 8 if $\mu=4$. Then for $k\ge 5$ and for all positive integers n with $(n,N)=1$ we have the following relation*:

$$\mathrm{tr}\big(\widetilde{T}_{k/2}(n^2)\,|\,[S_{k/2}(N,\chi)]\big)=\mathrm{tr}\big(T_{k-1}(n)\,|\,[S_{k-1}(N/2)]\big)$$
$$+\sum_{L_0}\Lambda(n,L_0)\,\mathrm{tr}\big(W_{L_0}T_{k-1}(n)\,|\,[S_{k-1}(2^{\mu-1}L_0L_1)]\big)$$

and for $k=3$ we have the following relation:

$$\mathrm{tr}\big(\widetilde{T}_{3/2}(n^2)\,|\,[V_{3/2}(N,\chi)]\big)=\mathrm{tr}\big(T_2(n)\,|\,[S_2(N/2)]\big)$$
$$+\sum_{L_0}\Lambda(n,L_0)\,\mathrm{tr}\big(W_{L_0}T_2(n)\,|\,[S_2(2^{\mu-1}L_0L_1)]\big)$$

where

(1) $\sum_{L_0}$ *runs over all square divisors L_0 of M with $L_0>1$,*

(2) *to each L_0 the corresponding L_1 is given by $L_1=M\prod_{p|L_0}p^{-\mathrm{ord}_p(M)}$,*

(3) *and the constant* $\Lambda(n, L_0)$ *is defined as follows*:

$$\Lambda(n, L_0) = \prod_{p|M} \lambda(p, n; \mathrm{ord}_p(L_0)/2) \text{ with}$$

$$\lambda(p, n; a) = \begin{cases} 1 & \text{if } a = 0, \\ 1 + \left(\frac{-n}{p}\right) & \text{if } 1 \leq a \leq \left[\frac{\mathrm{ord}_p(N)-1}{2}\right], \\ \chi_p(-n) & \text{if } \mathrm{ord}_p(N) \text{ is even and } a = \frac{\mathrm{ord}_p(N)}{2}. \end{cases}$$

We handle the case $k \geq 5$. All computations when $k = 3$ are completely analogous. We must consider $\chi = 1$ or $\left(\frac{p}{*}\right)$, however if the exponent m is odd, the trace identity has no dependence on χ.

Case 1: $m = 0$ or 1. In this case, N is square-free and the trace identity in Theorem 3.3 becomes

$$\mathrm{tr}\big(\widetilde{T}_{k/2}(n^2) \mid [S_{k/2}(4N, \chi)]\big) = \mathrm{tr}\big(T_{k-1}(n) \mid [S_{k-1}(2N)]\big)$$

yielding the isomorphism $S_{k/2}(4N, \chi) \cong S_{k-1}(2N)$ by application of (2.1). We then break $S_{k-1}(2N)$ into a direct sum of newform spaces according to Proposition 2.2 in order to obtain the decomposition.

Case 2: $m \geq 2$. The decompositions for non-square-free levels are obtained by bootstrapping up from decompositions for lower levels. For $c \geq 1$ and $a \geq 0$, define the following expressions:

$$\begin{aligned} D_{2c} &= \mathrm{tr}\big(\widetilde{T}_{k/2}(n^2) \mid [S_{k/2}(4p^{2c}, 1)]\big) - \mathrm{tr}\big(\widetilde{T}_{k/2}(n^2) \mid [S_{k/2}(4p^{2c-1}, \chi)]\big), \\ E_{2c} &= \mathrm{tr}\big(\widetilde{T}_{k/2}(n^2) \mid [S_{k/2}(4p^{2c}, (\tfrac{p}{*}))]\big) - \mathrm{tr}\big(\widetilde{T}_{k/2}(n^2) \mid [S_{k/2}(4p^{2c-1}, \chi)]\big), \\ F_a &= \mathrm{tr}\big(\widetilde{T}_{k/2}(n^2) \mid [S_{k/2}(4p^{2a+3}, \chi)]\big) - \mathrm{tr}\big(\widetilde{T}_{k/2}(n^2) \mid [S_{k/2}(4p^{2a+2}, 1)]\big) \\ &\quad - \mathrm{tr}\big(\widetilde{T}_{k/2}(n^2) \mid [S_{k/2}(4p^{2a+2}, (\tfrac{p}{*}))]\big) + \mathrm{tr}\big(\widetilde{T}_{k/2}(n^2) \mid [S_{k/2}(4p^{2a+1}, \chi)]\big). \end{aligned}$$

Substituting appropriately for each term using the trace identity in Theorem 3.3 yields significant cancellation:

$$\begin{aligned} D_{2c} &= \mathrm{tr}\big(T_{k-1}(n) \mid [S_{k-1}(2p^{2c})]\big) - \mathrm{tr}\big(T_{k-1}(n) \mid [S_{k-1}(2p^{2c-1})]\big) \\ &\quad + \mathrm{tr}\big(W_{p^{2c}} T_{k-1}(n) \mid [S_{k-1}(2p^{2c})]\big), \end{aligned}$$

$$\begin{aligned} E_{2c} &= \mathrm{tr}\big(T_{k-1}(n) \mid [S_{k-1}(2p^{2c})]\big) - \mathrm{tr}\big(T_{k-1}(n) \mid [S_{k-1}(2p^{2c-1})]\big) \\ &\quad + \left(\frac{-n}{p}\right) \mathrm{tr}\big(W_{p^{2c}} T_{k-1}(n) \mid [S_{k-1}(2p^{2c})]\big), \end{aligned}$$

$$\begin{aligned} F_a &= \mathrm{tr}\big(T_{k-1}(n) \mid [S_{k-1}(2p^{2a+3})]\big) - 2\,\mathrm{tr}\big(T_{k-1}(n) \mid [S_{k-1}(2p^{2a+2})]\big) \\ &\quad + \mathrm{tr}\big(T_{k-1}(n) \mid [S_{k-1}(2p^{2a+1})]\big). \end{aligned}$$

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We then reduce these three expressions to ones involving traces of only Hecke operators acting on spaces of newforms. For each $T_{k-1}(n)$-term, this is done via Proposition 2.2. For each term involving W-operators, we first apply the following:

PROPOSITION 3.4. (Ueda [15]) *Let A, B be finite sets consisting of prime numbers such that $A \cap B = \emptyset$ and also let a_p for $p \in A$, and b_q for $q \in B$ be any non-negative integers. Then for a positive integer n prime to $\prod_{p\in A} p^{a_p} \prod_{q\in B} q^{b_q}$, we have the following identity:*

$$\begin{aligned}&\operatorname{tr}\big(W_A T_{k-1}(n) \mid \big[S_{k-1}(\textstyle\prod_{p\in A} p^{a_p} \prod_{q\in B} q^{b_q})\big]\big)\\ &\quad = \sum_{\substack{(t_p)_{p\in A}\\ 0\le t_p\le [a_p/2]}} \sum_{\substack{(u_q)_{q\in B}\\ 0\le u_q\le b_q}} \prod_{q\in B} (b_q - u_q + 1)\\ &\qquad\quad \times \operatorname{tr}\big(W_A T_{k-1}(n) \mid \big[S^0_{k-1}(\textstyle\prod_{p\in A} p^{a_p-2t_p} \prod_{q\in B} q^{u_q})\big]\big).\end{aligned}$$

We then eliminate all W-operators via Propositions 2.3 and 2.1, and incorporate any coefficients of $\left(\frac{n}{p}\right)$ into the trace terms via the following:

LEMMA 3.5. ([6], [9]) *Let k, N, M be positive integers with $k \ge 3$, and let ψ be a primitive Dirichlet character modulo M. Let $N' = \operatorname{lcm}(N, M^2)$. Then for $(n, N') = 1$,*

$$\psi(n) \operatorname{tr}\big(T_{k-1}(n) \mid \big[S^0_{k-1}(N)\big]\big) = \operatorname{tr}\big(T_{k-1}(n) \mid \big[S^0_{k-1}(N)|R_\psi\big]\big)$$

where we consider $S^0_{k-1}(N)|R_\psi$ as a submodule of $S_{k-1}(N', \psi^2)$.

After collecting terms according to level and subspace with respect to p, we have the following reduced expressions:

$$\begin{aligned}D_{2c} = \sum_{t=0}^{1} (2-t)\Bigg[&2\sum_{u=1}^{c} \operatorname{tr}\big(T_{k-1}(n) \mid \big[S^{p++}_{k-1}(2^t p^{2u}) + S^{p+-}_{k-1}(2^t p^{2u})\big]\big)\\ &+ \sum_{u=1}^{c} \operatorname{tr}\big(T_{k-1}(n) \mid \big[S^{n}_{k-1}(2^t p^{2u-1})\big]\big) + 2\operatorname{tr}\big(T_{k-1}(n) \mid \big[S^0_{k-1}(2^t)\big]\big)\\ &+ \left(1 + \left(\frac{-1}{p}\right)\right) \operatorname{tr}\big(T_{k-1}(n) \mid \big[S^0_{k-1}(2^t p)|R_p \oplus S^0_{k-1}(2^t)|R_p\big]\big)\Bigg],\end{aligned}$$

$$E_{2c} = \sum_{t=0}^{1} (2-t)\Bigg[\left(1 + \left(\frac{-1}{p}\right)\right) \sum_{u=1}^{c} \operatorname{tr}\big(T_{k-1}(n) \mid \big[S^{p++}_{k-1}(2^t p^{2u}) \oplus S^{p-+}_{k-1}(2^t p^{2u})\big]\big)$$

$$+\Big(1-\Big(\frac{-1}{p}\Big)\Big)\sum_{u=1}^{c}\operatorname{tr}\big(T_{k-1}(n)\,|\,\big[S_{k-1}^{p+-}(2^tp^{2u})\oplus S_{k-1}^{p--}(2^tp^{2u})\big]\big)$$
$$+\sum_{u=1}^{c}\operatorname{tr}\big(T_{k-1}(n)\,|\,\big[S_{k-1}^{n}(2^tp^{2u-1})\big]\big)+\operatorname{tr}\big(T_{k-1}(n)\,|\,\big[S_{k-1}^{0}(2^tp)\big]\big)$$
$$+2\operatorname{tr}\big(T_{k-1}(n)\,|\,\big[S_{k-1}^{0}(2^t)\big]\big)+\operatorname{tr}\big(T_{k-1}(n)\,|\,\big[S_{k-1}^{0}(2^tp)|R_p\big]\big)$$
$$+\Big(1+\Big(\frac{-1}{p}\Big)\Big)\operatorname{tr}\big(T_{k-1}(n)\,|\,\big[S_{k-1}^{0}(2^t)|R_p\big]\big)\Big],$$
$$F_a=\sum_{t=0}^{1}(2-t)\operatorname{tr}\big(T_{k-1}(n)\,|\,\big[S_{k-1}^{0}(2^tp^{2a+3})\big]\big).$$

When $m=2$, if $\chi=1$, rearrange the definition of D_2 to obtain

$$\operatorname{tr}\big(\widetilde{T}_{k/2}(n^2)\,|\,\big[S_{k/2}(4p^2,1)\big]\big)=\operatorname{tr}\big(\widetilde{T}_{k/2}(n^2)\,|\,\big[S_{k/2}(4p,\chi)\big]\big)+D_2.$$

Substitute the expression for $\operatorname{tr}\big(\widetilde{T}_{k/2}(n^2)\,|\,\big[S_{k/2}(4p,1)\big]\big)$ from Case 1, along with the reduced expression for D_2. Collect terms according to level and subspace with respect to p to obtain a reduced expression for $\operatorname{tr}\big(\widetilde{T}_{k/2}(n^2)\,|\,\big[S_{k/2}(4p^2,1)\big]\big)$, and hence the decomposition by (2.1). Similarly, use E_2 when $\chi=\big(\frac{p}{*}\big)$.

When $m=2a+3$ for $a\geq 0$, rearranging the definition of F_a gives:

$$\begin{aligned}\operatorname{tr}\big(\widetilde{T}_{k/2}(n^2)\,|\,\big[S_{k/2}(4p^{2a+3},\chi)\big]\big)&=F_a+\operatorname{tr}\big(\widetilde{T}_{k/2}(n^2)\,|\,\big[S_{k/2}(4p^{2a+2},1)\big]\big)\\&\quad+\operatorname{tr}\big(\widetilde{T}_{k/2}(n^2)\,|\,\big[S_{k/2}(4p^{2a+2},(\tfrac{p}{*}))\big]\big)\\&\quad-\operatorname{tr}\big(\widetilde{T}_{k/2}(n^2)\,|\,\big[S_{k/2}(4p^{2a+1},\chi)\big]\big).\end{aligned}$$

Substituting for the level $4p^{2a+2}$ terms and then inducting on $a\geq 0$, we have

$$\begin{aligned}\operatorname{tr}\big(\widetilde{T}_{k/2}(n^2)\,|\,\big[S_{k/2}(4p^{2a+3},\chi)\big]\big)&=\sum_{c=0}^{a}\big[F_a+D_{2a+2}+E_{2a+2}\big]\\&\quad+\operatorname{tr}\big(\widetilde{T}_{k/2}(n^2)\,|\,\big[S_{k/2}(4p,\chi)\big]\big).\end{aligned}$$

Substitute for $\operatorname{tr}\big(\widetilde{T}_{k/2}(n^2)\,|\,\big[S_{k/2}(4p,\chi)\big]\big)$ from Case 1 along with the summands F_a, D_{2a+2}, E_{2a+2}, and collect terms according to level and subspace with respect to p. This gives the reduced expression for $\operatorname{tr}\big(\widetilde{T}_{k/2}(n^2)\,|\,\big[S_{k/2}(4p^{2a+3},\chi)\big]\big)$. The decomposition follows.

When $m = 2a+4$ for $a \geq 0$, the results again depend on the choice of χ. If $\chi = 1$, rearrange the expression for D_{2a+4} to obtain:

$$\mathrm{tr}\big(\widetilde{T}_{k/2}(n^2)\,|\,\big[S_{k/2}(4p^{2a+4},1)\big]\big) = \mathrm{tr}\big(\widetilde{T}_{k/2}(n^2)\,|\,\big[S_{k/2}(4p^{2a+3},\chi)\big]\big) + D_{2a+4}.$$

Then substitute for $\mathrm{tr}\big(\widetilde{T}_{k/2}(n^2)\,|\,\big[S_{k/2}(4p^{2a+3},\chi)\big]\big)$ and D_{2a+4}, and combine terms as usual. Similarly, use E_{2a+4} if $\chi = \left(\frac{p}{*}\right)$. □

Remark. If p and q are distinct odd primes, the nature of the decomposition at level $4p^m q^s$ was found to be the same *with respect to* p as it was at level $4p^m$. This suggests that this decomposition theorem may generalize to level $4N$, N any odd positive integer; one would use the given table for each distinct $p|N$ and then combine the information for all such primes. Defining terms like those used in Case 2, and using them to build up to level $4N$ one prime at a time, may lead to such a generalization.

§4. Comparison to decompositions of the Kohnen subspace

The decompositions given in Theorem 3.1 have a strong relationship to the decompositions given by Ueda [14] for the Kohnen subspace $S_{k/2}(4p^m,\chi)_K$ when $k \geq 5$. Comparing our decompositions to Ueda's, we see precisely how the Kohnen subspace sits inside the full space of cusp forms. For example, with the notation and definitions as above, by Theorem 3.1 we have:

$$\begin{aligned} S_{k/2}(4p^2,\left(\tfrac{p}{*}\right)) \cong \bigoplus_{t=0}^{1}(2-t)\Big\{&\Big(1+\Big(\frac{-1}{p}\Big)\Big)\big\{S_{k-1}^{p++}(2^tp^2) \oplus S_{k-1}^{p+-}(2^tp^2)\big\} \\ &\oplus \Big(1-\Big(\frac{-1}{p}\Big)\Big)\big\{S_{k-1}^{p-+}(2^tp^2) \oplus S_{k-1}^{p--}(2^tp^2)\big\} \\ &\oplus 3S_{k-1}^{0}(2^tp) \oplus S_{k-1}^{0}(2^tp)|R_p \\ &\oplus 4S_{k-1}^{0}(2^t) \oplus \Big(1+\Big(\frac{-1}{p}\Big)\Big)S_{k-1}^{0}(2^t)|R_p\Big\} \end{aligned}$$

while Ueda [14] gives the following decomposition for the Kohnen subspace:

$$\begin{aligned} S_{k/2}(4p^2,\left(\tfrac{p}{*}\right))_K \cong &\Big(1+\Big(\frac{-1}{p}\Big)\Big)\big\{S_{k-1}^{p++}(p^2) \oplus S_{k-1}^{p+-}(p^2)\big\} \\ &\oplus \Big(1-\Big(\frac{-1}{p}\Big)\Big)\big\{S_{k-1}^{p-+}(p^2) \oplus S_{k-1}^{p--}(p^2)\big\} \\ &\oplus 3S_{k-1}^{0}(p) \oplus S_{k-1}^{0}(p)|R_p \\ &\oplus 4S_{k-1}^{0}(1) \oplus \Big(1+\Big(\frac{-1}{p}\Big)\Big)S_{k-1}^{0}(1)|R_p. \end{aligned}$$

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For levels dividing p^2, the forms occurring in the isomorphism for the full space are precisely those which occurred in Ueda's isomorphism for the Kohnen space, with their multiplicities doubled. In addition, the structure of the forms occurring at level $2p^u$ is parallel to the structure occurring at level p^u — the same subspaces appear with the same coefficients depending on p. This parallelism exists between our decompisitions and Ueda's at any level $4p^m$, and indicates a beautiful structure in the relationship between the Kohnen subspace and the full space of cusp forms. The regular structure of the decompositions suggests that this parallelism should extend to all levels.

§5. When $S_{k/2}(4N, \chi, F)$ is nonzero for newforms $F \in S^n_{k-1}(2^t M)$

Throughout this section, $t = 0$ or 1, M and N are odd positive integers, $\widehat{M}$ is an odd positive integer with the same prime factors as M, each occurring to odd exponent at least 3, and χ and χ' are even quadratic Dirichlet characters modulo the relevant levels. The discussion focuses on $S_{k/2}(4N, \chi)$ for $k \geq 5$. All proofs are given for this case, and analogous arguments hold for $V_{3/2}(4N, \chi)$ when $k = 3$. We give a series of partial results regarding the structure of the decompositions for $S_{k/2}(4N, \chi)$ and $V_{3/2}(4N, \chi)$ which provide significant information about the image of the Shimura lift. For all newforms $F \in S^n_{k-1}(2^t M)$, these results lead to the following characterization, in terms of classical conditions, of whether F has equivalent half-integral weight cusp forms of level $4N$ and character χ:

Corollary 5.1. *With the notation and terminology as above, let $F \in S^n_{k-1}(2^t M)$. Then $S_{k/2}(4N, \chi, F) = \{0\}$ for all odd positive integers N and all even quadratic Dirichlet characters χ modulo $4N$ if and only if the following hold*:

(1) *There is at least one prime $p|M$ such that* $\mathrm{ord}_p(M)$ *is even.*

(2) *For any such prime p, either $p \equiv 1 \pmod 4$ and $F \in S^{p--}_{k-1}(2^t M)$, or $p \equiv 3 \pmod 4$ and $F \in S^{p-+}_{k-1}(2^t M)$.*

In case $S_{k/2}(4N, \chi, F) \neq \{0\}$, the minimal level for which this occurs is $4N = 4M$.

This corollary depends on several results given below: First, Theorem 5.2 takes the space $S^n_{k-1}(2^t M)$ and tracks its "appearance" in the decomposition of $S_{k/2}(4\widehat{M}, \chi)$. This piece of the decomposition of $S_{k/2}(4\widehat{M}, \chi)$

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leads to Corollary 5.5 which explicitly gives the dimension of $S_{k/2}(4\widehat{M},\chi,F)$ for all $F\in S^n_{k-1}(2^tM)$; in particular, it gives conditions under which this dimension is zero. Theorem 5.6 and Corollary 5.7 then indicate the effect of introducing additional prime factors into the level.

5.1. Subspaces of $S^n_{k-1}(2^tM)$ appearing in the decomposition of $S_{k/2}(4\widehat{M},\chi)$

For any newform $F\in S^n_{k-1}(2^tM)$, we can determine precisely the number of copies of F appearing in the decomposition of $S_{k/2}(4\widehat{M},\chi)$ (i.e., the dimension of $S_{k/2}(4\widehat{M},\chi,F)$):

THEOREM 5.2. *Let $t\in\{0,1\}$ and let $M=\prod_p p^{b_p}$ be the product of distinct odd primes p to positive integer exponents b_p. Split the primes dividing M into the following three sets: $\mathcal{U}=\{p|M : b_p=1\}$, $\mathcal{E}=\{p|M : b_p\geq 2 \text{ is even}\}$, and $\mathcal{O}=\{p|M : b_p\geq 3 \text{ is odd}\}$. Consider any $\widehat{M}=\prod_{p|M}p^{a_p}$ with a_p odd integers such that $a_p\geq\max\{3,b_p\}$, and any even quadratic Dirichlet character χ modulo $4\widehat{M}$.*

(1) *If $\mathcal{E}=\emptyset$, then for $k\geq 5$ the total contribution of summands in the decomposition of $S_{k/2}(4\widehat{M},\chi)$ which are subspaces of $S^n_{k-1}(2^tM)$ is:*

$$(2-t)\prod_{p\in\mathcal{U}}\Big(3\Big[\frac{a_p}{2}\Big]+1\Big)\prod_{p\in\mathcal{O}}(a_p+1-b_p)S^n_{k-1}(2^tM).$$

(2) *If $\mathcal{E}\neq\emptyset$, then for $k\geq 5$ the total contribution of summands in the decomposition of $S_{k/2}(4\widehat{M},\chi)$ which are subspaces of $S^n_{k-1}(2^tM)$ is:*

$$(2-t)\prod_{p\in\mathcal{U}}\Big(3\Big[\frac{a_p}{2}\Big]+1\Big)\prod_{p\in\mathcal{E}}\Big(\Big[\frac{a_p}{2}\Big]+1-\frac{b_p}{2}\Big)\prod_{p\in\mathcal{O}}(a_p+1-b_p)$$
$$\times\bigoplus_{\substack{p\in\mathcal{E}\\ \alpha_p,\beta_p=\pm1}}\prod_{p\in\mathcal{E}}c_{p\alpha_p\beta_p}S^{(p\alpha_p\beta_p)_{p\in\mathcal{E}}}_{k-1}(2^tM),$$

where $c_{p\alpha_p\beta_p}=(2+\alpha_p)+\beta_p\big(\frac{-1}{p}\big)$.

(3) *Statements (1) and (2) hold for $k=3$, with $S_{k/2}(4\widehat{M},\chi)$ replaced by $V_{3/2}(4\widehat{M},\chi)$.*

Proof. We first isolate and simplify all terms in the trace identity for $S_{k/2}(4\widehat{M},\chi)$ which give contributions to $S^n_{k-1}(2^tM)$. We then show by induction that these expressions reduce to the structure of coefficients and subspaces as given in the theorem.

Since each a_p is odd, the trace identity for $S_{k/2}(4\widehat{M},\chi)$ is independent of the choice of character χ. Thus $\Lambda(n,L_0)=\prod_{p|L_0}(1+\left(\frac{-n}{p}\right))$ for each square $L_0|\widehat{M}$ with $L_0>1$. Moreover, since each $a_p\geq 3$, each prime dividing $\widehat{M}$ (by construction, the primes dividing M) will occur in some L_0. For convenience, we group terms in the L_0-sum of the trace identity by the set of prime divisors of L_0. For subsets $P_{\mathcal{U}}\subseteq\mathcal{U}$ and $P_{\mathcal{E}}\subseteq\mathcal{E}$, we will be concerned with terms for which the set of prime divisors of L_0 is $P_{\mathcal{U}}\cup P_{\mathcal{E}}$. We refer to these as the "$P_{\mathcal{U}}P_{\mathcal{E}}$-sums" in the trace identity. In the following proposition, we determine the contribution to $S^n_{k-1}(2^tM)$ from these sums together with the $T_{k-1}(n)$-term.

PROPOSITION 5.3. *Let the notation and terminology be as above, and let A denote the set of prime divisors of M. For any subsets $P_{\mathcal{E}}\subseteq\mathcal{E}$ and $P_{\mathcal{U}}\subseteq\mathcal{U}$, we get a contribution to $S^n_{k-1}(2^tM)$ from the $P_{\mathcal{U}}P_{\mathcal{E}}$-sum. The total contribution to $S^n_{k-1}(2^tM)$ from these sums is given by*:

$$(5.1)\qquad (2-t)\Bigg[\sum_{P_{\mathcal{U}}\subseteq\mathcal{U}}\sum_{P_{\mathcal{E}}\subseteq\mathcal{E}}K_{P_{\mathcal{U}},P_{\mathcal{E}}}\operatorname{tr}\big(W_{P_{\mathcal{E}}}T_{k-1}(n)\,|\,\big[S^n_{k-1}(2^tM)\big]\big)\Bigg],$$

where

$$K_{P_{\mathcal{U}},P_{\mathcal{E}}}=\prod_{p\in P_{\mathcal{E}}}\left[\left(1+\left(\frac{-n}{p}\right)\right)\left(\left[\frac{a_p}{2}\right]+1-\frac{b_p}{2}\right)\right]\prod_{p\in P_{\mathcal{U}}}\left[\frac{a_p}{2}\right]\prod_{\substack{p\in A\\ p\notin(P_{\mathcal{E}}\cup P_{\mathcal{U}})}}(a_p+1-b_p).$$

Proof. The case $P_{\mathcal{E}}=P_{\mathcal{U}}=\emptyset$ corresponds to the $T_{k-1}(n)$-term in the trace identity. Reducing this by Proposition 2.2 and isolating the level 2^tM term yields $(2-t)\prod_{p\in A}(a_p+1-b_p)\operatorname{tr}\big(T_{k-1}(n)\,|\,\big[S^0_{k-1}(2^tM)\big]\big)$ as desired.

If $P_{\mathcal{U}}=\emptyset$ but $P_{\mathcal{E}}\neq\emptyset$, by reducing each $P_{\mathcal{E}}$-sum by Proposition 3.4 and then simplifying, we can isolate the level 2^tM term which is:

$$(2-t)\Bigg[\prod_{p\in P_{\mathcal{E}}}\left[\left(1+\left(\frac{-n}{p}\right)\right)\left(\left[\frac{a_p}{2}\right]+1-\frac{b_p}{2}\right)\right]\prod_{p\in A-P_{\mathcal{E}}}(a_p+1-b_p)\Bigg]$$
$$\times\operatorname{tr}\big(W_{P_{\mathcal{E}}}T_{k-1}(n)\,|\,\big[S^0_{k-1}(2^tM)\big]\big).$$

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Since $S^0_{k-1}(2^tM) = S^n_{k-1}(2^tM) \oplus S^{n\perp}_{k-1}(2^tM)$ and since both pieces are closed under the action of $W_{P_\mathcal{E}}$, applying Proposition 2.5 and discarding all $S^{n\perp}_{k-1}(2^tM)$ terms yields the desired expression. Moreover, we get no contributions to $S^n_{k-1}(2^tM)$ from terms at levels *other than* 2^tM in the $P_\mathcal{E}$-sums: For a fixed $P_\mathcal{E}$-sum, any contributions to $S^n_{k-1}(2^tM)$ from a level other than 2^tM would necessarily come from a summand $S^n_{k-1}(2^tM)|R_C \subseteq S^{n\perp}_{k-1}(2^t\widetilde{M})$ for some $\widetilde{M} > M$, but only if coefficients $\left(\frac{n}{p}\right)$ twisted in to "undo" the twist R_C. The method of construction in Proposition 2.3 prescribes $\widetilde{M} = M\prod_{p\in P_\mathcal{U}} p$ for some nonempty $P_\mathcal{U} \subseteq \mathcal{U}$, with corresponding $C = \prod_{p\in P_\mathcal{U}} p$. The Legendre symbol coefficients appearing in our terms are all with respect to primes in $P_\mathcal{E}$ however, so they will not affect the twist R_C.

Now if $P_\mathcal{U} \neq \emptyset$, write $\widetilde{P}_\mathcal{U} = \prod_{p\in P_\mathcal{U}} p$. In this case, contributions to $S^n_{k-1}(2^tM)$ from the $P_\mathcal{U}P_\mathcal{E}$-sums arise only from summands $S^n_{k-1}(2^tM)|R_{\widetilde{P}_\mathcal{U}} \subseteq S^{n\perp}_{k-1}(2^tM\widetilde{P}_\mathcal{U})$: For each term in a $P_\mathcal{U}P_\mathcal{E}$-sum, all primes $p \in P_\mathcal{U}$ will occur to even exponents in the level. Therefore we cannot obtain level 2^tM directly. Reducing each $P_\mathcal{U}P_\mathcal{E}$-sum by Proposition 3.4 and simplifying, we can isolate the level $2^tM\widetilde{P}_\mathcal{U}$ term. By Propositions 2.5 and 2.3, we can then isolate the $S^n_{k-1}(2^tM)|R_{\widetilde{P}_\mathcal{U}}$ summand which we reduce by repeated application of Proposition 2.1 part (5) to

$$\prod_{p\in P_\mathcal{U}}\left(\left(\frac{-1}{p}\right)+\left(\frac{n}{p}\right)\right)L_{P_\mathcal{U},P_\mathcal{E}}\operatorname{tr}\left(W_{P_\mathcal{E}}T_{k-1}(n)\,\middle|\,\left[S^n_{k-1}(2^tM)|R_{\widetilde{P}_\mathcal{U}}\right]\right)$$

where

$$L_{P_\mathcal{U},P_\mathcal{E}} = \prod_{p\in P_\mathcal{E}}\left[\left(1+\left(\frac{-n}{p}\right)\right)\left(\left[\frac{a_p}{2}\right]+1-\frac{b_p}{2}\right)\right]\prod_{p\in P_\mathcal{U}}\left[\frac{a_p}{2}\right]\prod_{\substack{p\in A\\ p\notin(P_\mathcal{E}\cup P_\mathcal{U})}}(a_p+1-b_p).$$

By Lemma 5.4 given below, the only term in $\prod_{p\in P_\mathcal{U}}\left(\left(\frac{-1}{p}\right)+\left(\frac{n}{p}\right)\right)$ which will "undo" the twist $R_{\widetilde{P}_\mathcal{U}}$ is $\prod_{p\in P_\mathcal{U}}\left(\frac{n}{p}\right)$. Thus the expression above becomes

$$(2-t)\left[\sum_{P_\mathcal{U}}\sum_{P_\mathcal{E}}K_{P_\mathcal{U},P_\mathcal{E}}\operatorname{tr}\left(W_{P_\mathcal{E}}T_{k-1}(n)\,\middle|\,\left[S^n_{k-1}(2^tM)\right]\right)\right]$$
$$+\ (\text{twists of terms of level } 2^tM).$$

We then discard the twists of terms of level 2^tM and take the sum over all subsets $P_\mathcal{E}$ and $P_\mathcal{U}$ to obtain the result. ∎

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The following generalization of Lemma 3.5 allows $\left(\frac{n}{p}\right)$ to be "twisted in" when the trace expression still involves W_Q operators (as above), provided $p \nmid Q$:

LEMMA 5.4. *Let k, N, M, Q be positive integers with $k \geq 3$ and $(Q, f_\psi) = 1$, where f_ψ is the conductor of ψ, a primitive Dirichlet character modulo M. Let $N' = \mathrm{lcm}(N, M^2)$. Then for $(n, N') = 1$ we have:*

(1) $\psi(n)\,\mathrm{tr}\big(W_Q T_{k-1}(n) \mid \big[S^0_{k-1}(N)\big]\big) = \mathrm{tr}\big(W_Q T_{k-1}(n) \mid \big[S^0_{k-1}(N)|R_\psi\big]\big).$

(2) $\psi(n)\,\mathrm{tr}\big(W_Q T_{k-1}(n) \mid \big[S^0_{k-1}(N)|R_\psi\big]\big) = \mathrm{tr}\big(W_Q T_{k-1}(n) \mid \big[S^0_{k-1}(N)\big]\big).$

Proof. Let $[T]^{\mathcal{B}}_{\mathcal{B}}$ denote the matrix of an operator T in terms of a basis $\mathcal{B}$. Choose a basis $\mathcal{B} = \{F_1, \ldots, F_d\}$ of $S^0_{k-1}(N)$ consisting of normalized newforms. One can show that $\mathcal{C} = \{F_1|R_\psi, \ldots, F_d|R_\psi\}$ is then a basis of $S^0_{k-1}(N)|R_\psi$. Using Proposition 2.1 and multiplicity-one, we explicitly compute $\mathrm{trace}([W_Q T_{k-1}(n)]^{\mathcal{C}}_{\mathcal{C}}) = \mathrm{tr}\big(W_Q T_{k-1}(n) \mid \big[S^0_{k-1}(N)|R_\psi\big]\big)$ and $\mathrm{trace}([W_Q T_{k-1}(n)]^{\mathcal{B}}_{\mathcal{B}}) = \psi(n)\,\mathrm{tr}\big(W_Q T_{k-1}(n) \mid \big[S^0_{k-1}(N)\big]\big)$, showing them to be equal. This proves (1), and (2) follows since $F|R_p|R_p = F$ for all newforms $F \in S^0_{k-1}(N)$ and all $p \in \Omega_2$ (see [11]). ∎

Proposition 5.3 handles contributions to $S^n_{k-1}(2^t M)$ from the $P_\mathcal{U} P_\mathcal{E}$-sums. In fact, no other terms in the trace identity contribute at this level: Using Proposition 3.4, we replace each L_0-sum with a sum of terms in which each $p|L_0$ occurs to an even exponent in the levels. When looking for contributions at level $2^t M$ we can therefore disregard L_0-terms where any prime $p \in \mathcal{O}$ divides L_0. Thus the expression in Proposition 5.3 gives *all* contributions to $S^n_{k-1}(2^t M)$ in the trace identity of $S_{k/2}(4\widehat{M}, \chi)$. To complete the proof of Theorem 5.2, we will now use induction to show that eliminating the W-operators in (5.1) gives the structure of coefficients and subspaces as stated in the theorem.

Case 1: $\mathcal{E} = \emptyset$. By Proposition 5.3, the contributions to $S^n_{k-1}(2^t M)$ in the trace identity for $S_{k/2}(4\widehat{M}, \chi)$ are in this case given by

$$(2-t)\prod_{p\in\mathcal{O}}(a_p + 1 - b_p)\left[\prod_{p\in\mathcal{U}} a_p + \sum_{P_\mathcal{U}}\left(\prod_{p\in P_\mathcal{U}}\left[\frac{a_p}{2}\right]\prod_{p\in\mathcal{U}-P_\mathcal{U}} a_p\right)\right] \times \mathrm{tr}\big(T_{k-1}(n) \mid \big[S^n_{k-1}(2^t M)\big]\big).$$

where $\sum_{P_\mathcal{U}}$ denotes the sum over all nonempty $P_\mathcal{U} \subseteq \mathcal{U}$.

To obtain the desired structure, we need only show that

$$\left[\prod_{p\in\mathcal{U}} a_p + \sum_{P_{\mathcal{U}}}\left(\prod_{p\in P_{\mathcal{U}}}\left[\frac{a_p}{2}\right]\prod_{p\in\mathcal{U}-P_{\mathcal{U}}} a_p\right)\right] = \prod_{p\in\mathcal{U}}\left(3\left[\frac{a_p}{2}\right]+1\right).$$

This is done by inducting on $|\mathcal{U}|$, noting that $a_p = 2\left[\frac{a_p}{2}\right]+1$ since a_p is odd.

Case 2: $\mathcal{E} \neq \emptyset$. Due to technicalities in the induction, we prove a more general result. For any positive integers R and Q with $(R, M) = 1$ and W_Q defined on $S^n_{k-1}(2^t MR)$, we show that for all nonnegative integral choices of $e = |\mathcal{E}|$, $o = |\mathcal{O}|$, and $u = |\mathcal{U}|$, the following equality holds:

$$(5.2)\quad (2-t)\Bigg[\sum_{P_{\mathcal{E}}} K_{P_{\mathcal{E}}}\operatorname{tr}\big(W_Q W_{P_{\mathcal{E}}} T_{k-1}(n) \mid [S^n_{k-1}(2^t MR)]\big) + \sum_{P_{\mathcal{U}}}\sum_{P_{\mathcal{E}}} K_{P_{\mathcal{U}},P_{\mathcal{E}}}\operatorname{tr}\big(W_Q W_{P_{\mathcal{E}}} T_{k-1}(n) \mid [S^n_{k-1}(2^t MR)]\big)\Bigg]$$
$$= (2-t)\prod_{p\in\mathcal{U}}\left(3\left[\frac{a_p}{2}\right]+1\right)\prod_{p\in\mathcal{E}}\left(\left[\frac{a_p}{2}\right]+1-\frac{b_p}{2}\right)\prod_{p\in\mathcal{O}}(a_p+1-b_p)$$
$$\times \sum_{\substack{p\in\mathcal{E}\\ \alpha_p,\beta_p=\pm 1}}\prod_{p\in\mathcal{E}} c_{p\alpha_p\beta_p}\operatorname{tr}\big(W_Q T_{k-1}(n) \mid [S^{(p\alpha_p\beta_p)_{p\in\mathcal{E}}}_{k-1}(2^t MR)]\big).$$

The theorem then follows by setting $R = Q = 1$. We prove (5.2) by inducting on both e and u. When inducting on e, the base case $e = 0$ for all $u \geq 0$ and $o \geq 0$ is proved using an induction on u analogous to the one used in Case 1. Now assume (5.2) holds for all $0 \leq \ell < e$, $u \geq 0$, and $o \geq 0$. Separate off one prime $q \in \mathcal{E}$ and write $\mathcal{E} = \mathcal{E}' \cup \{q\}$. Split the subsets $P_{\mathcal{E}}$ into two types: (1) $P_{\mathcal{E}} = P_{\mathcal{E}'} \subseteq \mathcal{E}'$ (including $\emptyset$), and (2) $P_{\mathcal{E}} = P_{\mathcal{E}'} \cup \{q\}$ for

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some $P_{\mathcal{E}'}$ as in (1). Rewrite the left-hand side of (5.2) in terms of $P_{\mathcal{E}'}$:

$$(2-t)\Bigg[(a_q+1-b_q)\Bigg\{\Bigg(\sum_{P_{\mathcal{E}'}} K_{P_{\mathcal{E}'}}+\sum_{P_{\mathcal{U}}}\sum_{P_{\mathcal{E}'}} K_{P_{\mathcal{U}},P_{\mathcal{E}'}}\Bigg) \times \operatorname{tr}\big(W_Q W_{P_{\mathcal{E}'}} T_{k-1}(n) \mid \big[S^{n}_{k-1}(2^t MR)\big]\big)\Bigg\}$$
$$+\Big(\Big[\frac{a_q}{2}\Big]+1-\frac{b_q}{2}\Big)\Bigg\{\Bigg(\sum_{P_{\mathcal{E}'}} K_{P_{\mathcal{E}'}}+\sum_{P_{\mathcal{U}}}\sum_{P_{\mathcal{E}'}} K_{P_{\mathcal{U}},P_{\mathcal{E}'}}\Bigg)\Big(1+\Big(\frac{-n}{q}\Big)\Big) \times \operatorname{tr}\big(W_Q W_q W_{P_{\mathcal{E}'}} T_{k-1}(n) \mid \big[S^{n}_{k-1}(2^t MR)\big]\big)\Bigg\}\Bigg].$$

Write $MR = M'R'$ where $R' = q^{b_q}R$ and $M' = Mq^{-b_q}$, and consider the operators W_Q and W_q together as $W_{Q'}$. M' has $\mathcal{E}'$ as its set of prime divisors occurring to even exponents, with $|\mathcal{E}'| = e-1 < e$. Thus by induction the expression above equals

$$(2-t)\prod_{p\in\mathcal{U}}\Big(3\Big[\frac{a_p}{2}\Big]+1\Big)\prod_{p\in\mathcal{E}'}\Big(\Big[\frac{a_p}{2}\Big]+1-\frac{b_p}{2}\Big)\prod_{p\in\mathcal{O}}(a_p+1-b_p)$$
$$\times\sum_{\substack{p\in\mathcal{E}'\\ \alpha_p,\beta_p=\pm1}}\prod_{p\in\mathcal{E}'} c_{p\alpha_p\beta_p}\Big[(a_q+1-b_q)\operatorname{tr}\big(W_Q T_{k-1}(n)\mid\big[S^{(p\alpha_p\beta_p)_{p\in\mathcal{E}'}}_{k-1}(2^tMR)\big]\big)$$
$$+\Big(\Big[\frac{a_q}{2}\Big]+1-\frac{b_q}{2}\Big)\Big(1+\Big(\frac{-n}{q}\Big)\Big)\operatorname{tr}\big(W_Q W_q T_{k-1}(n)\mid\big[S^{(p\alpha_p\beta_p)_{p\in\mathcal{E}'}}_{k-1}(2^tMR)\big]\big)\Big].$$

All that remains is to eliminate the W_q-operator and combine the terms in brackets. Decompose each subspace $S^{(p\alpha_p\beta_p)_{p\in\mathcal{E}'}}_{k-1}(2^tMR)$ into a direct sum of four subspaces $S^{(p\alpha_p\beta_p)_{p\in\mathcal{E}'},q\alpha_q\beta_q}_{k-1}(2^tMR)$, with $\alpha_q,\beta_q=\pm1$. Simplify the expression using Propositions 3.5 and 2.1, and then notice that the subspaces $S^{(p\alpha_p\beta_p)_{p\in\mathcal{E}'},q\alpha_q\beta_q}_{k-1}(2^tMR)$ are precisely $S^{(p\alpha_p\beta_p)_{p\in\mathcal{E}}}_{k-1}(2^tMR)$, and the two sums combine to sum over all $p\in\mathcal{E}$ with the appropriate coefficients and subspaces. We therefore obtain the right-hand side of (5.2). This completes the proof of Theorem 5.2. ∎

5.2. The dimension of $S_{k/2}(4\widehat{M},\chi,F)$

Corollary 5.5. *Let the notation and terminology be as in Theorem* 5.2 *and let* $F \in S^n_{k-1}(2^t M)$ *be a newform. The dimension of the space* $S_{k/2}(4\widehat{M}, \chi, F)$ *is given in the following expressions*:

Case 1: *If* $\mathcal{E} = \emptyset$, *then*

$$\dim(S_{k/2}(4\widehat{M}, \chi, F)) = (2-t) \prod_{p \in \mathcal{U}} \Big(3\Big[\frac{a_p}{2}\Big] + 1\Big) \prod_{p \in \mathcal{O}} (a_p + 1 - b_p).$$

Case 2: *If* $\mathcal{E} \neq \emptyset$, *then accordingly as* $F \in S^{(p\alpha_p\beta_p)_{p\in\mathcal{E}}}_{k/2}(2^t M)$,

$$\dim(S_{k/2}(4\widehat{M}, \chi, F)) = (2-t) \prod_{p \in \mathcal{U}} \Big(3\Big[\frac{a_p}{2}\Big] + 1\Big) \prod_{p \in \mathcal{E}} \Big(\Big[\frac{a_p}{2}\Big] + 1 - \frac{b_p}{2}\Big) \times \prod_{p \in \mathcal{O}} (a_p + 1 - b_p) \prod_{p \in \mathcal{E}} c_{p\alpha_p\beta_p}.$$

Proof. Case 1 follows immediately from Theorem 5.2. Once it is established that any newform $F \in S^n_{k-1}(2^t M)$ belongs to one of the subspaces $S^{(p\alpha_p\beta_p)_{p\in\mathcal{E}}}_{k-1}(2^t M)$, Case 2 will also follow from the theorem. A priori, F may be a linear combination of forms in these subspaces. However, since F is a newform, by Theorem 3 of [1] we have $F|W_p = \alpha_p F$, for some choice of $\alpha_p = \pm 1$ for each $p \in \mathcal{E}$. Since $F|R_p$ is also a newform by Theorem 6 of [1], applying Theorem 4 of [1] to $F|R_p$ shows that $F|R_p|W_p = \beta_p F|R_p$ for some choice of $\beta_p = \pm 1$ for each $p \in \mathcal{E}$. Thus we have $F \in S^{(p\alpha_p\beta_p)_{p\in\mathcal{E}}}_{k-1}(2^t M)$ for some tuple $(p\alpha_p\beta_p)_{p\in\mathcal{E}}$. ∎

5.3. The minimal level for which $S_{k/2}(4N, \chi, F)$ may be nontrivial

For a newform $F \in S^0_{k-1}(2^t M)$, we say "F appears (resp. does not appear) in the decomposition of $S_{k/2}(4N, \chi)$" if F is (resp. is not) an element of a summand in the isomorphism for $S_{k/2}(4N, \chi)$.

Theorem 5.6. *Let* k, M *and* N *be odd positive integers such that* $M|N$ *and* $k \geq 3$. *Let* $t \in \{0, 1\}$ *and consider a newform* $F \in S^0_{k-1}(2^t M)$. *Let* q *be an odd prime such that* $q \nmid M$, *and require that* $\nu_p = \mathrm{ord}_p(N) \geq 3$ *be odd for all* $p|N$ *with* $p \neq q$. *Finally, let* χ (*resp.* χ') *be any even quadratic Dirichlet character modulo* $4N$ (*resp.* $4Nq$). *For* $k \geq 5$ (*resp.* $k = 3$), *if* F *does not appear in the decomposition of* $S_{k/2}(4N, \chi)$ (*resp.* $V_{3/2}(4N, \chi)$), *then* F *does not appear in the decomposition of* $S_{k/2}(4Nq, \chi')$ (*resp.* $V_{3/2}(4Nq, \chi')$).

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Before giving the proof of Theorem 5.6, we state and prove the following:

Corollary 5.7. *Let M and t be as above, and consider a newform $F \in S^0_{k-1}(2^t M)$. For $k \geq 5$ (resp. $k = 3$), if F does not appear in the decomposition of $S_{k/2}(4\widehat{M}, \chi)$ (resp. $V_{3/2}(4\widehat{M}, \chi)$) for any positive integer $\widehat{M} = \prod_{p|M} p^{\alpha_p}$ with odd integers $\alpha_p \geq 3$, and for any even quadratic Dirichlet character modulo $4\widehat{M}$, then F does not appear in the decomposition of $S_{k/2}(4N, \chi')$ (resp. $V_{3/2}(4N, \chi')$) for any odd positive integer N and any even quadratic Dirichlet character χ' modulo $4N$.*

Proof. For convenience, we will abbreviate "F appears (resp. does not appear) in the decomposition of $S_{k/2}(4N, \chi)$" as "F appears (resp. does not appear) in $S_{k/2}(4N, \chi)$". Let $F \in S^0_{k-1}(2^t M)$. If $M \nmid N$, then F clearly cannot appear in $S_{k/2}(4N, \chi)$, so suppose $M|N$. Write $N = M' q_1^{\beta_1} \cdots q_r^{\beta_r}$, splitting off all primes q_i not dividing M, where $\beta_i = \mathrm{ord}_{q_i}(N)$ for $i = 1, \ldots, r$. There is no character-dependence in the trace identity when the prime exponents are odd, so put $\widehat{N} = \widehat{M} q_1^{\beta_1} \cdots q_r^{\beta_r}$, with $\widehat{M} = \prod_{p|M'} p^{\alpha_p}$ where $\alpha_p \geq \max\{3, \mathrm{ord}_p(M')\}$ and odd.

By hypothesis, F does not appear in $S_{k/2}(4\widehat{M}, \chi)$ for any even quadratic Dirichlet character χ modulo $4\widehat{M}$, so by repeated application of Theorem 5.6 on the primes q_i, F does not appear in $S_{k/2}(4\widehat{N}, \psi)$ for any even quadratic Dirichlet character ψ modulo $4\widehat{N}$. Since $N|\widehat{N}$ and they have the same prime factors, any even quadratic Dirichlet character χ' modulo $4N$ can be obtained from a choice of ψ, viewed as a character modulo $4N$. The usual containment relations among spaces of modular forms then show that F does not appear in $S_{k/2}(4N, \chi')$. ∎

Proof of Theorem 5.6. Suppose a newform $F \in S^0_{k-1}(2^t M)$ does not appear in $S_{k/2}(4N, \chi)$, for some odd positive integer N and some even quadratic Dirichlet character modulo $4N$. Let q be an odd prime such that $q \nmid M$, and let $\nu_q = \mathrm{ord}_q(N)$.

Case 1: $\nu_q = 0$. Using Theorem 3.3 and methods previously discussed, one computes that

$$\mathrm{tr}\big(\widetilde{T}_{k/2}(n^2) \,|\, [S_{k/2}(4N, \chi)]\big) = \sum_{d|2N} \delta(2N/d)\, \mathrm{tr}\big(T_{k-1}(n) \,|\, [S^0_{k-1}(d)]\big) + {\sum_{L_0}}' C_{L_0, u_p}\, \mathrm{tr}\big(W_{L_0} T_{k-1}(n) \,|\, [S^0_{k-1}(2^t N_{t_p, u_p})]\big)$$

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and

$$\begin{aligned}
&\operatorname{tr}\big(\widetilde{T}_{k/2}(n^2)\,|\,[S_{k/2}(4Nq,\chi')]\big)\\
&\quad=\sum_{v=0}^{1}(2-v)\Bigg[\sum_{d|2N}\delta(2N/d)\operatorname{tr}\big(T_{k-1}(n)\,|\,[S^0_{k-1}(dq^v)]\big)\\
&\qquad\qquad+\sum\nolimits' C_{L_0,u_p}\operatorname{tr}\big(W_{L_0}T_{k-1}(n)\,|\,[S^0_{k-1}(2^tq^vN_{t_p,u_p})]\big)\Bigg]
\end{aligned}$$

where

(1) $\sum'_{L_0}$ denotes the following multiple sum:

$$\sum_{\substack{\text{squares } L_0\\ 1<L_0|N}}\ \sum_{\substack{p|L_0\\ 0\le t_p\le[\frac{a_p}{2}]}}\ \sum_{\substack{p|2L_1\\ 0\le u_p\le b_p}}.$$

(2) C_{L_o,u_p} and N_{t_p,u_p} are defined as follows: For each L_0, put $a_p=\operatorname{ord}_p(K_0)$ for each prime dividing $p|L_0$ and put $b_p=\operatorname{ord}_p(2K_1)$ for each prime $p|2L_1$. For integers t_p and u_p with $0\le u_p\le b_p$ and $0\le t_p\le\left[\frac{a_p}{2}\right]$, put

$$\begin{aligned}
C_{L_0,u_p}&=\prod_{p|L_0}\left(1+\left(\frac{-n}{p}\right)\right)\prod_{p|2L_1}(b_p-u_p+1),\quad\text{and}\\
N_{t_p,u_p}&=\prod_{p|L_0}p^{2t_p}\prod_{p|2L_1}p^{u_p}.
\end{aligned}$$

These two trace identities are almost identical; the main distinction is the appearance of q^v in the levels in the second. The decomposition $S^0_{k-1}(2^tq^vN_{t_p,u_p})=S^n_{k-1}(2^tq^vN_{t_p,u_p})\oplus S^{n\perp}_{k-1}(2^tq^vN_{t_p,u_p})$ is blind to q^v, since $v=0$ or 1 and only prime divisors with exponents equal to 2 affect the nature of S^n. Therefore, the reduction of the respective trace terms to eliminate all W-operators will be parallel, with the factor of q^v "along for the ride" in the levels of the second identity. Therefore, since F does not appear in $S_{k/2}(4N,\chi)$, when we introduce the additional prime q into the level and look at the $v=0$ terms in the trace expression for $S_{k/2}(4Nq,\chi')$, we see that F does not appear in $S_{k/2}(4Nq,\chi')$.

Case 2: $\nu_q=2b+1$ for some nonnegative integer b. We prove that if F does not appear in $S_{k/2}(4N,\chi)$, then F does not appear in $S_{k/2}(4Nq^2,\chi'')$

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for any even quadratic Dirichlet character χ'' modulo $4Nq^2$, in order to avoid the issue of character-dependence in the trace identity. Then by the usual containment relations, F does not appear in $S_{k/2}(4Nq, \chi')$.

In transforming the trace identity for $\operatorname{tr}\big(\widetilde{T}_{k/2}(n^2) \mid [S_{k/2}(4N, \chi)]\big)$ into an isomorphism for $S_{k/2}(4N, \chi)$, we must combine certain terms to determine the summands at any specified level. The way in which these terms combine produces Legendre symbol factors in the coefficients which depend on the level, and gives us some constant multiple of the basic "building block" of forms occurring at that level. Put

$$P_{N,q^2} = \operatorname{tr}\big(\widetilde{T}_{k/2}(n^2) \mid [S_{k/2}(4Nq^2, \chi'')]\big) - \operatorname{tr}\big(\widetilde{T}_{k/2}(n^2) \mid [S_{k/2}(4N, \chi)]\big).$$

Then $\operatorname{tr}\big(\widetilde{T}_{k/2}(n^2) \mid [S_{k/2}(4Nq^2, \chi'')]\big) = P_{N,q^2} + \operatorname{tr}\big(\widetilde{T}_{k/2}(n^2) \mid [S_{k/2}(4N, \chi)]\big)$. To relate the decompositions of $S_{k/2}(4Nq^2, \chi'')$ and $S_{k/2}(4N, \chi)$, we must compare the structure of the expressions for $\operatorname{tr}\big(\widetilde{T}_{k/2}(n^2) \mid [S_{k/2}(4N, \chi)]\big)$ and P_{N,q^2} in terms of newforms. In particular, we must show that in adding P_{N,q^2} to $\operatorname{tr}\big(\widetilde{T}_{k/2}(n^2) \mid [S_{k/2}(4N, \chi)]\big)$, we add a constant multiple of the entire "building block" at any specified level in order to prove that we introduce no additional forms.

We introduce abbreviated notation which condenses the trace expressions, yet still illustrates the behavior at each level. Write $N = N'q^{2b+1}$ (so that $q \nmid N'$) and put $c_p = \operatorname{ord}_p(2N')$ for each prime $p|2N'$. For integers v with $0 \leq v \leq 2b+3$, define

$$\begin{aligned} X_{N',v} = \sum_{\substack{p|2N' \\ 0 \leq u_p \leq b_p}} \prod_{p|2N'} (c_p - u_p + 1) \operatorname{tr}\big(T_{k-1}(n) \mid [S^0_{k-1}(q^v \textstyle\prod_{p|2N'} p^{u_p})]\big) \\ + {\sum_{L_0}}' C_{L_0,u_p} \operatorname{tr}\big(W_{L_0} T_{k-1}(n) \mid [S^0_{k-1}(q^v N_{t_p,u_p})]\big) \end{aligned}$$

with ${\sum}'_{L_0}$, C_{L_0,u_p} and N_{t_p,u_p} as in Case 1. Since our expressions will often differ only by an additional W-operator, also define

$$\begin{aligned} W_Q[X_{N',v}] = \sum_{\substack{p|2N' \\ 0 \leq u_p \leq b_p}} \prod_{p|2N'} (c_p - u_p + 1) \operatorname{tr}\big(W_Q T_{k-1}(n) \mid [S^0_{k-1}(q^v \textstyle\prod_{p|2N'} p^{u_p})]\big) \\ + {\sum_{L_0}}' C_{L_0,u_p} \operatorname{tr}\big(W_Q W_{L_0} T_{k-1}(n) \mid [S^0_{k-1}(q^v N_{t_p,u_p})]\big). \end{aligned}$$

LEMMA 5.8. *With the abbreviated notation and terminology as above, we have the following expressions in terms of newforms*:

$$\operatorname{tr}\big(\widetilde{T}_{k/2}(n^2)\,|\,[S_{k/2}(4N,\chi)]\big) = \sum_{v=0}^{2b+1}(2b-v+2)X_{N',v} + \left(1+\left(\frac{-n}{q}\right)\right)\left[b\,X_{N',0}+\sum_{v=1}^{b}(b-v+1)W_{q^{2v}}\left[X_{N',2v}\right]\right]$$

and

$$P_{N,q^2} = X_{N',2b+3} + 2\sum_{v=0}^{2b+2}X_{N',v} + \left(1+\left(\frac{-n}{q}\right)\right)\left[X_{N',0}+\sum_{v=1}^{b+1}W_{q^{2v}}\left[X_{N',2v}\right]\right].$$

Proof. By Theorem 3.3,

$$\operatorname{tr}\big(\widetilde{T}_{k/2}(n^2)\,|\,[S_{k/2}(4N,\chi)]\big) = \operatorname{tr}\big(T_{k-1}(n)\,|\,[S_{k-1}(2N)]\big) + \sum_{L_0'}\prod_{p|L_0'}\left(1+\left(\frac{-n}{p}\right)\right)\operatorname{tr}\big(W_{L_0'}T_{k-1}(n)\,|\,[S_{k-1}(2L_0'L_1')]\big)$$

where L_0' runs over all squares $1 < L_0'|N$ with corresponding $L_1' = N\prod_{p|L_0'}p^{-\operatorname{ord}_p(N)}$. Let L_0 denote a square divisor of N' with $L_0 \neq 1$, and put $L_1 = N'\prod_{p|L_0'}p^{-\operatorname{ord}_p(N)}$. We then have $L_0' = L_0$, $q^{2\ell}$, or $L_0q^{2\ell}$ for some L_0 and some $\ell = 1, 2, \ldots, b$, with corresponding $L_1' = L_1q^{2b+1}$, N', or L_1 respectively. After rewriting the trace expression above in terms of L_0, straightforward computation using Proposition 2.2 yields the desired expression. The proof for P_{N,q^2} is analogous. ∎

Notice that the terms $\big(1+\big(\frac{-n}{q}\big)\big)W_{q^{2b+2}}\left[X_{N',2b+2}\right]+2X_{N',2b+2}+X_{N',2b+3}$ appear in P_{N,q^2} but not in $\operatorname{tr}\big(\widetilde{T}_{k/2}(n^2)\,|\,[S_{k/2}(4N,\chi)]\big)$. Aside from these terms, we are adding constant multiples of existing terms, with the constants independent of $2N'$. It remains to show the following claims:

(1) The terms $\big(1+\big(\frac{-n}{q}\big)\big)W_{q^{2b+2}}\left[X_{N',2b+2}\right]+2X_{N',2b+2}+X_{N',2b+3}$ give no contribution at levels dividing $2N'$.

(2) For terms in $\operatorname{tr}\big(\widetilde{T}_{k/2}(n^2)\,|\,[S_{k/2}(4N,\chi)]\big)$ giving contribution at any particular level dividing $2N'$, we have added the *same* constant multiple of each term, thus preserving the structure of the decomposition at that level.

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Let d be a positive divisor of $2N'$. Since $S^0_{k-1}(d)|R_q \subseteq S^0_{k-1}(dq^2)$ and one piece of the coefficient $\big(1+\big(\frac{-n}{q}\big)\big)$ "twists in" to eliminate the R_q twist, we will get contributions at level d from the term $\big(1+\big(\frac{-n}{q}\big)\big)W_{q^2}\big[X_{N',2}\big]$. Clearly we also get contributions at level d from the term $X_{N',0}$. However, $X_{N',2v}$ for $v \neq 0$ and $\big(1+\big(\frac{-n}{q}\big)\big)W_{q^{2v}}\big[X_{N',2v}\big]$ for $v \neq 1$ involve only terms whose levels are divisible by q. Thus they give no contributions at levels dividing $2N'$. This proves the first claim.

To prove the second claim, we need the following

Lemma 5.9. *With the notation and terminology as above, we have*

$$\left(1+\left(\frac{-n}{q}\right)\right)W_{q^2}\left[X_{N',2}\right] = X_{N',0} + E(q,R_q)$$

where $E(q,R_q)$ is a sum of terms of levels divisible by q and/or twisted by R_q, hence no terms in $E(q,R_q)$ contribute at levels dividing $2N'$.

Proof. We have

$$S^0_{k-1}(q^2\textstyle\prod_{p|2N'}p^{u_p}) = \displaystyle\bigoplus_{\Omega_2=A+B+C} S^n_{k-1}(N(B,C))|R_{B+C}$$

with Ω_2 and $N(B,C)$ as in Proposition 2.3. Our interest is limited to those summands where $q \nmid N(B,C)$, for only these will give contributions at levels dividing $2N'$. By construction, $q \nmid N(B,C)$ occurs only when $q \in C$. After the usual manipulations to eliminate the W_{q^2}-operators and twist in coefficients of $\big(\frac{n}{q}\big)$, and after separating out terms involving subspaces $S^n_{k-1}(N(B,C))|R_{B+C}$ where $q|N(B,C)$, we have:

$$\begin{aligned}\left(1+\left(\frac{-n}{q}\right)\right)W_{q^2}\left[X_{N',2}\right] &= \sum_{\substack{p|2N'\\0\le u_p\le b_p}} \prod_{p|2N'}(b_p-u_p+1)\\ &\quad\times \operatorname{tr}\Big(T_{k-1}(n)\,\Big|\Big[\bigoplus_{\substack{\Omega_2=A+B+C\\q\in C}} S^n_{k-1}(N(B,C))|R_{B+C-\{q\}}\Big]\Big)\\ &\quad+\sum_{L_0}{}' C_{L_0,u_p}\operatorname{tr}\Big(W_{L_0}T_{k-1}(n)\,\Big|\Big[\bigoplus_{\substack{\Omega_2'=A'+B'+C'\\q\in C'}} S^n_{k-1}(N(B',C'))|R_{B'+C'-\{q\}}\Big]\Big)\\ &\quad+E(q,R_q)\end{aligned}$$

with $E(q, R_q)$ as above. The set of partitions $\Omega_2 = A + B + C$ with $q \in C$ is precisely the set of partitions of $\Omega_2 - \{q\}$. Thus

$$\bigoplus_{\substack{\Omega_2=A+B+C\\ q\in C}} S^n_{k-1}(N(B,C))|R_{B+C-\{q\}} = S^0_{k-1}(\textstyle\prod_{p|2N'} p^{u_p}).$$

The direct sum over partitions of Ω_2' is handled similarly, and by the definition of $X_{N',0}$, we obtain the result. ∎

By Lemmas 5.8 and 5.9, we see that adding the expression in terms of newforms for P_{N,q^2} to that for $\mathrm{tr}\big(T_{k-1}(n) \mid [S^n_{k-1}(4N)\chi]\big)$, we add $3b+5$ copies of $X_{N',0}$ to the existing $b+1$ copies. Therefore we have added the same constant multiple of each term at any level dividing $2N'$, with the constants being independent of $2N'$ (note that $X_{N',0}$ is the aforementioned "building block").

Case 3: $\nu_q = 2b+2$ for some nonnegative integer b. Write $N = N'q^{2b+2}$. F does not appear in $S_{k/2}(4N, \chi'') = S_{k/2}(4N'q^{2b+2}, \chi'')$ for any even quadratic χ'', hence by containment relations, F does not appear in $S_{k/2}(4N'q^{2b+1}, \chi')$ for any even quadratic χ'. Then F does not appear in $S_{k/2}(4N'q^{2b+3}, \chi) = S_{k/2}(4Nq, \chi)$ for any even quadratic χ by Case 2. This completes the proof of Theorem 5.6. ∎

5.4. Connections between these results and Flicker's theorem

Restricting our attention to $S^n(2^tM)$, Theorem 5.2 allows us to identify certain forms which are "missing" from decompositions for $S_{k/2}(4\widehat{M}, \chi)$. For example, when $p \equiv 1 \pmod 4$, any subspaces with a coefficient of $1 - \left(\frac{-1}{p}\right)$ will not appear. These will also be missing from $S_{k/2}(4N, \chi)$ for any odd N and any even quadratic Dirichlet character χ modulo $4N$, by Corollary 5.7. As a consequence, Corollary 5.1 characterizes in terms of classical invariants those integral weight newforms $F \in S^n_{k-1}(2^tM)$ which have equivalent half-integral weight cusp forms at some level $4N$ with N odd. This is a partial reformulation of a representation-theoretic result of Flicker's [5] regarding $S_{k/2}(4N, \chi, F)$:

THEOREM. (Flicker [5]) *Let $(H1)$ denote the following condition on the p^{th} components ρ_p of the automorphic representation ρ associated to F: For all primes p such that ρ_p is of the principal series, $\mu_{1,p}(-1) = \mu_{2,p}(-1) = 1$, where $\mu_{1,p}$ and $\mu_{2,p}$ denote the characters of $\mathbf{Q}_p^\times$ such that $\rho_p \sim \pi(\mu_{1,p}, \mu_{2,p})$. Then there exists N with $S_{k/2}(4N, \chi, F) \neq \{0\}$ if and only if $(H1)$ is satisfied.*

We have rephrased these representation-theoretic conditions in terms of the prime-powers in the level of the form F and in terms of the W_q-eigenspace for F with respect to those primes occurring with even exponent. At present, we have reformulated Flicker's theorem only in the case $F \in S^n_{k-1}(2^t M)$, since Theorem 5.2 and Corollary 5.5 only held for newforms in $S^n_{k-1}(2^t M)$. However, careful examination of the full decompositions when M has one or two distinct prime factors revealed certain patterns in the appearance of the twist terms. Evidence suggests that the appearance of subspaces of $S^{n\perp}_{k-1}(2^t M)$ in the decomposition of $S_{k/2}(4\widehat{M}, \chi)$ may follow a pattern similar to the one observed at level $4p^m$:

Recall that for any choice of nonempty subset $B \subseteq \Omega_2$, the space $S^{n\perp}_{k-1}(2^t M)$ contains summands $S^n_{k-1}(2^t \prod_{q\in B} q^{\nu_q} \prod_{p\in A-B} p^{b_p})|R_B$ where A is the set of prime divisors of M, $\nu_q = 0$ or 1, and $R_B = R_{\prod_{q\in B} q}$. The evidence discussed above suggests the following:

(1) When $\mathcal{E} = \{p|M : b_p \geq 2 \text{ is even}\} = B$, the entire space $S^n_{k-1}(2^t \prod_{q\in B} q^{\nu_q} \prod_{p\in A-B} p^{b_p})|R_{\mathcal{E}} \subseteq S^{n\perp}_{k-1}(2^t M)$ should appear in the decomposition of $S_{k/2}(4\widehat{M}, \chi)$. Its coefficient structure with respect to $\mathcal{U} = \{p|M : b_p = 1\}$ and $\mathcal{O} = \{p|M : b_p \geq 3 \text{ is odd}\}$ should parallel that of Case (1) in Theorem 5.2. Additionally, we expect some Legendre symbol coefficients with respect to $\mathcal{E}$ to occur.

(2) When $\mathcal{E} \neq B$, subspaces $S^{(p\alpha_p\beta_p)_{p\in\mathcal{E}-B}}_{k-1}(2^t \prod_{q\in\mathcal{E}} q^{\nu_q} \prod_{p\in A-B} p^{b_p})|R_{\mathcal{E}} \subseteq S^{n\perp}_{k-1}(2^t M)$ should appear in the decomposition of $S_{k/2}(4\widehat{M}, \chi)$. Their coefficient structure with respect to $\mathcal{U}$ and $\mathcal{O}$ should parallel that of Case (2) in Theorem 5.2. Additionally, we expect different Legendre symbol coefficients with respect to B and $\mathcal{E} - B$.

The theory contained in this section also has a connection to an important result of Waldspurger's: When $(H1)$ is satisfied, Waldspurger [17] gives a means of constructing $S_{k/2}(4N, \chi, F)$ explicitly, by first identifying $\widetilde{N}$, the minimal N for which this space is nonzero, and then analyzing cases depending on $\widetilde{N}$ and the level of F. There are many cases to consider, and the conditions given in some cases involve a great deal of complexity. Alternatively, we determine $\widetilde{N}$ in terms of the subspace $S^{(p\alpha_p\beta_p)_{p\in\mathcal{E}}}_{k-1}(2^t M)$ to which F belongs.

§6. Examples: non-zero newforms F with $S_{k-1}(4N, \chi, F) = 0$

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Our results raise the following question: Do the "missing subspaces" have positive dimension? That is, are there *nonzero* forms which are not in the image of the Shimura lift from $S_{k/2}(4N,\chi)$ for any odd positive integer N and any even quadratic Dirichlet character χ? We give examples when $k=3$ which show that the answer is yes. These are computed using Cremona's tables [4] which list the following identifying information for rational newforms $F \in S^0_{k-1}(M)$:

(1) The Hecke eigenvalues λ_p of F for $T_2(p)$ when $p \nmid M$ and $p \leq 100$.

(2) The eigenvalues, either $+1$ or -1, of F for W_q when $q|M$ and $q \leq 100$.

To utilize this information, suppose $F(z) = \sum_{n=1}^{\infty} a(n)e^{2\pi inz}$ is a normalized newform in $S^0_{k-1}(M)$. For a prime $p|M$, we then have $F|R_p = \sum_{n=1}^{\infty} b(n)e^{2\pi inz}$ where $b(n) = \left(\frac{n}{p}\right)a(n)$. If $p|M$, $b(n)=0$. Otherwise, corresponding $a(n)$ and $b(n)$ can differ only in sign, and will differ precisely when n is a quadratic non-residue modulo p. Since $a(n)=\lambda_n$, the eigenvalue of F for the Hecke operator $T_2(n)$, we have a relationship between the Hecke eigenvalues of F and those of $F|R_p$. We can therefore make use of the eigenvalue information in the tables to determine whether $F \in S^n_2(M)$ or F is the twist of some newform of lower level. Moreover, if $F \in S^n_2(M)$, we can then determine the subspace of $S^n_2(M)$ to which F belongs with respect to a prime $p|M$ (i.e., whether $F \in S^{p++}_2(M)$, etc.).

Example 6.1. Using this method, we have $\dim(S^{13--}_2(338)) \geq 2$:

Cremona lists 6 distinct rational newforms of level 338, called $338A$ through $338F$, so $\dim(S_2(338)) \geq 6$. In comparing the Hecke eigenvalues of $338A$ through $338F$ to the corresponding eigenvalues of the forms listed at level $26 = 2\cdot 13$, we see that $338C$ and $338F$ are both twists by R_{13} of newforms of level 26. Investigation of the Hecke eigenvalues as above shows that $338A|R_{13} = 338B$ and $338D|R_{13} = 338E$, so these four are in $S^n_2(338)$. Checking the sign of the W_{13}-eigenvalue for each form, we find that $338A$ and $338B$ are in $S^{13++}_2(338)$, while $338D$ and $338E$ are in $S^{13--}_2(338)$. Thus $\dim(S^{13--}_2(338)) \geq 2$.

Example 6.2. Similarly, we have $\dim(S^{19-+}_2(722)) \geq 1$.

We have given examples of missing subspaces with positive dimension, of both the $p--$ and $p-+$ types. In computing with Cremona's tables for forms in $S^0_2(p^2)$ with p an odd prime less than 100, it was not possible

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to show that any of the missing spaces had positive dimension. However, extended tables for levels $1001 \leq N \leq 5000$ given on Cremona's webpage led to the computations that $\dim(S_2^{37--}(37^2)) \geq 2$ and $\dim(S_2^{43-+}(43^2)) \geq 1$. Cremona's tables deal only with rational newforms, and with more complete information it may be possible to obtain examples at levels p^2 for smaller primes.

§7. Conclusion

In this paper, we have examined the Hecke structure of spaces of half-integral weight cusp forms by "looking backwards" through the Shimura correspondence. Our partial decompositions for $S_{k/2}(4N,\chi)$ and $V_{3/2}(4N,\chi)$ gave important information about the image of the Shimura lift. Decompositions in certain cases illustrated the relationship between the Kohnen subspace and the full space of cusp forms. Certain results were restricted to newforms in $S_{k-1}^n(2^tM)$ for $t = 0, 1$ and M an odd positive integer, although possible methods for obtaining analogous results for newforms $F \in S_{k-1}^{n\perp}(2^tM)$ were discussed.

Several interesting question are raised by these results. First, are all the missing spaces of newforms in these decompositions positive-dimensional? (we have seen some examples in Section 6). Since the trace of the Hecke operator T_1 on any space S of integral weight cusp forms is equal to the dimension of S, computing the trace of T_1 on $S_{k-1}^{p--}(2^tM)$ and $S_{k-1}^{p-+}(2^tM)$ can provide the answer. This computation is currently being pursued, using the formulas for traces of T_n and of the composition W_pT_n on $S_{k-1}^0(2^tM)$ given in Ross [10] and Yamauchi [18] respectively.

Second, is the Shimura lift always non-surjective? Determining dimensions of the missing spaces of newforms in general may provide additional examples of non-surjectivity for other values of k when N is odd. Additionally, extending our decompositions to the case of arbitrary N would be a step towards answering this question. This would require additional trace identities, as Theorem 3.3 is only equipped to handle levels where $\mathrm{ord}_2(N)$ is at most two. Trace relationships handling almost all cases of $\mathrm{ord}_2(N)$ are given by Ueda in [16], and could be used to obtain such decompositions via similar methods.

Perhaps most interestingly, what exactly is the significance of the -1-eigenspace of the W_p operator? Our results indicate that forms in this eigenspace for some p dividing their level may not have equivalent half-integral weight forms. To gain insight into this question we appeal to the theory of L-series: The sign in the functional equation for the Dirichlet L-series $L(F,s)$ associated to an integral weight form $F \in S^0_{k-1}(N)$ is determined by the *Fricke involution* H_N, a composition of W_p operators for all $p|N$. When this sign is -1, $L(F,s)$ vanishes for the value of s in the center of the critical strip. If F has weight 2 and integral coefficients, it can be shown that $L(F,s) = L(E,s)$ for some elliptic curve E over $\mathbf{Q}$, and the Birch–Swinnerton-Dyer [3] conjecture states that the rank of the associated elliptic curve E is equal to the order of vanishing in the functional equation for $L(E,s)$. The -1-eigenspace of W_p plays a role in these vanishings, and we hope to develop a greater understanding of this role.

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Department of Mathematics
Wellesley College
106 Central Street
Wellesley, MA 02481-8203
U.S.A.
sfrechette@wellesley.edu