PERFECT CODES IN THE GRAPHS O_k AND $L(O_k)$

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In [6] the question of the existence of perfect e-codes in the infinite family of distance-transitive graphs O_k was considered. It was pointed out that it is difficult to rule out completely any particular value of e because of the difficulty of working with the sphere packing condition. For $e=1,2,3$ it can be seen from the results of [6] that the condition given by the generalisation of Lloyd's theorem is satisfied for infinitely many values of k. We shall show that this is not the case for $e=4$ and we shall prove that there are no perfect 4-codes in O_k.

Hammond [5] has constructed perfect 1-codes in the line graphs $L(O_k)$. In fact $L(O_k)$ contains $2k-1$ perfect 1-codes which form a partition of the vertex set of $L(O_k)$. We show that the codes described by Hammond are unique.

Definition. The graph O_k $(k\geqslant 2)$ has $\binom{2k-1}{k-1}$ vertices indexed by the $(k-1)$-subsets of the set $\{1,2,\ldots,2k-1\}$. Two vertices are joined by an edge if and only if their indexing sets are disjoint.

Definition. The line graph $L(O_k)$ has vertices which correspond to the edges of O_k, with two vertices of $L(O_k)$ being adjacent whenever the corresponding edges of O_k are incident.

Definition. A *perfect e-code* in a graph Γ is a subset C of the vertices of Γ with minimum distance $2e+1$ such that any vertex of Γ is at distance at most e from some vertex of C. We consider only nontrivial codes ($|C|>2$).

Definition. Define the sequence of polynomials $\{v_i(\lambda)\}$ by $v_0(\lambda)=1$, $v_1(\lambda)=\lambda$, $c_{i+1}v_{i+1}(\lambda)-\lambda v_i(\lambda)+b_{i-1}v_{i-1}(\lambda)=0$ where $c_i=[\frac{1}{2}(i+1)]$ and $b_i=k-[\frac{1}{2}(i+1)]$ $(i=1,2,\ldots,d-1)$. Let

$$x_j(\lambda)=\sum_{i=0}^{j} v_i(\lambda).$$

The following lemma is the generalisation of Lloyd's theorem.

Lemma 1. [4], [6]. *If O_k contains a perfect e-code, then the roots of $x_e(\lambda)$ are members of the set $\{-(k-1), (k-2), -(k-3), \ldots, (-1)^{k+1}\}$.*

Lemma 2 [6]. *If O_k contains a nontrivial perfect e-code, then $k\geqslant(e^2+4e+2)/2$ (e even) and $k\geqslant(e^2+4e+3)/2$ (e odd).*

Lemma 3 [6]. *If $\alpha\neq-1$ is a root of $x_e(\lambda)$, then so is $-\alpha-1$. If e is odd, -1 is a root of $x_e(\lambda)$.*

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Now let $\bar{x}_e(\lambda) = cx_e(\lambda)$ (c constant) be a monic polynomial and consider $\bar{x}_e(-\frac{1}{2})$ which is a polynomial in k of degree $[e/2]$. (The use of the polynomial $\bar{x}_e(-\frac{1}{2})$ was suggested by E. Bannai.)

Consider the case $e=4$. Suppose the roots of $x_4(\lambda)$ are α_1, α_2, $-\alpha_1-1$, $-\alpha_2-1$.

$$\bar{x}_4(-\tfrac{1}{2}) = (-\tfrac{1}{2}-\alpha_1)(-\tfrac{1}{2}-\alpha_2)(-\tfrac{1}{2}+\alpha_1+1)(-\tfrac{1}{2}+\alpha_2+1)$$

and so we have

$$16\bar{x}_4(-\tfrac{1}{2}) = (2\alpha_1+1)^2(2\alpha_2+1)^2 = w^2. \tag{1}$$

Straightforward calculation reveals that

$$x_4(\lambda) = [\lambda^4+2\lambda^3+\lambda^2(7-4k)+\lambda(6-4k)+2(k-1)(k-2)]/4$$

and so the equation (1) becomes

$$32k^2-80k+41 = y^2. \tag{2}$$

Also since

$$\begin{aligned} 7-4k &= \alpha_1(-\alpha_1-1)+\alpha_1\alpha_2+\alpha_1(-\alpha_2-1)+(-\alpha_1-1)\alpha_2 \\ &\quad +(-\alpha_1-1)(-\alpha_2-1)+\alpha_2(-\alpha_2-1) \\ &= -\alpha_1(\alpha_1+1)-\alpha_2(\alpha_2+1)+1, \end{aligned}$$

we have

$$4k-6 = \alpha_1(\alpha_1+1)+\alpha_2(\alpha_2+1)$$

and

$$2(k-1)(k-2) = \alpha_1(\alpha_1+1)\alpha_2(\alpha_2+1).$$

Hence $\alpha_1(\alpha_1+1) = 2k-3\pm\sqrt{(2k^2-6k+5)}$. Since α_1 is an integer, $2k^2-6k+5$ is a perfect square and so

$$32k^2-96k+80 = z^2. \tag{3}$$

Hence if a perfect 4-code exists in O_k, equations (2) and (3) have a simultaneous integer solution and from Lemma 2, $k \geq 17$. From (2) and (3) we have $16k-39 = y^2-z^2$. Write $z=\gamma k>0$. Then $(\gamma k)^2 = 32k^2-96k+80$ gives $\gamma<4\sqrt{2}<6$ and $k\geq 17$ gives $(\gamma k)^2>25k^2$ so $\gamma>5$. If we write $y=\gamma k+i$ (where i is a positive integer) we have $16k-39 = 2\gamma ki+i^2$, so

$$10ki+i^2 < 16k-39 < 12ki+i^2.$$

The first inequality gives $i<2$ and the second excludes $i=1$. Hence we have:

THEOREM 1. *There is no nontrivial perfect* 4-*code in* O_k.

NOTE. It seems possible that a similar method would work for $e=5$. The equations replacing (2) and (3) can be written

$$6(8k-11)^2-114 = p^2, \qquad 3(k-2)^2+1 = q^2.$$

It is possible that the method of Baker and Davenport **[1]** will extend to this case, but the calculation is formidable.

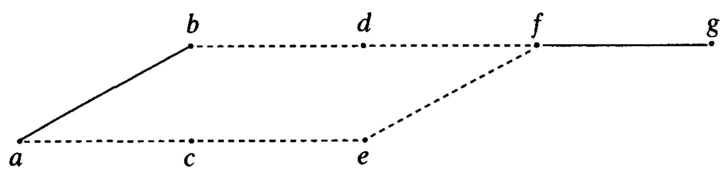
Now consider the case of perfect 1-codes in $L(O_k)$. Let $X=\{1,2,\ldots,2k-1\}$ and let e–f be the edge of O_k joining vertices e and f. For any $x\in X$ let $\bar{C}_x=\{c-d\mid\{c\cup d\}=X\backslash x\}$ and C_x be the corresponding set of vertices in $L(O_k)$. Hammond **[5]** has shown that for each $x\in X$ the code C_x is a perfect 1-code.

THEOREM 2. *The codes $C_x(x\in X)$ are the only perfect 1-codes in $L(O_k)$.*

Proof. The case $k=3$ is easily dealt with directly. Suppose $k>3$. Let D be a code in $L(O_k)$ not isomorphic to any C_x and let $\bar{D}$ be the corresponding set of edges in O_k. D contains vertices of C_x and C_y for some $x, y\in X$, $x\neq y$. Choose $x, y, p\in C_x$, $q\in C_y$ with $p, q\in D$ in such a way that p and q are as close as possible with $x\neq y$. Let C'_x consist of those vertices in $L(O_k)$ adjacent to vertices of C_x. Let $p, a_1, a_2, \ldots, a_n, q$ be a path of minimum length joining p and q. Clearly $a_1\in C'_x$, $a_n\in C'_y$ but all possibilities for a_2 contradict the choice of p and q unless $n=2$.

Since O_k has girth 6 ($k>3$) we have two cases:

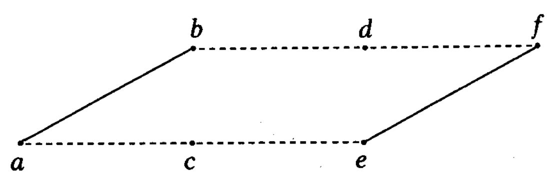
Case 1



($a-b\in\bar{D}$ corresponds to p, $f-g\in\bar{D}$ corresponds to q).

Either $c-e$ is in $\bar{D}$ or $c-e$ is adjacent to an edge of $\bar{D}$. In either case the minimum distance of D would be 2. This is a contradiction.

Case 2



($a-b\in\bar{D}$ corresponds to p, $e-f\in\bar{D}$ corresponds to q).

Then, rearranging X if necessary, we can write without loss of generality

$$a=(1\,2\ldots k-1)\qquad b=(k\;k+1\ldots 2k-2)$$
$$d=(2\,3\ldots k-1;2k-1)\qquad c=(k+1\;k+2\ldots 2k-1).$$

Then it is easy to see that $f=(1;k+1\;k+2\ldots 2k-2)$, $e=(2\,3\ldots k)$. Then $e\cup f=\{X\backslash(2k-1)\}=a\cup b$ contradicting the fact that $a-b$ corresponds to p and $e-f$ corresponds to q.

NOTE. Theorem 1 together with Theorem 2 of [**6**] show that there are no nontrivial perfect 4-codes in the graphs $2 \,.\, O_k$ [**6**]. The modifications required to the proof of Theorem 2 for the case of perfect 1-codes in the graphs $L(2 \,.\, O_k)$ are straightforward.

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