*J. Aust. Math. Soc.* **100** (2016), 241–251
doi:10.1017/S1446788715000464

# RELATIVE ELEMENTARY ABELIAN GROUPS AND A CLASS OF EDGE-TRANSITIVE CAYLEY GRAPHS

CAI HENG LI and LEI WANG[✉]

(Received 10 December 2014; accepted 4 August 2015; first published online 20 November 2015)

Communicated by B. Alspach

**Abstract**

Motivated by a problem of characterising a family of Cayley graphs, we study a class of finite groups $G$ which behave similarly to elementary abelian $p$-groups with $p$ prime, that is, there exists a subgroup $N$ such that all elements of $G\backslash N$ are conjugate or inverse-conjugate under $\mathsf{Aut}(G)$. It is shown that such groups correspond to complete multipartite graphs which are normal edge-transitive Cayley graphs.

2010 *Mathematics subject classification*: primary 20B25; secondary 05C25, 05E18.

*Keywords and phrases*: Cayley graph, complete graph, REA group, Frobenius group.

## 1. Introduction

Two elements $a, b$ of a group $G$ are called *fused* or *inverse-fused* if $a$ is conjugate under $\mathsf{Aut}(G)$ to $b$ or $b^{-1}$, respectively. A finite group is an elementary abelian $p$-group if and only if any two nonidentity elements are fused or inverse-fused because all nonidentity elements of this group have equal order, and the center is nonidentity and is a characteristic subgroup.

Definition 1.1. A group $G$ is called a *relative elementary abelian group*, or simply called an *REA group* for short, if there exists a subgroup $N < G$ such that any two elements of $G\backslash N$ are fused or inverse-fused. To emphasise the subgroup $N$, we sometimes call it an *REA group relative to $N$*.

A group $G$ is called a *Camina group* if all elements of $gG'$ with $g \notin G'$ are conjugate to $g$ (refer to [2, 3, 10, 11, 13]). The concept of REA group is in some sense a generalisation of Camina group.

Theorem 1.2. *Let $G$ be an REA group relative to $N$. Then the following statements hold:*

This work was partially supported by the NNSF of China (No. 11301468, 11231008) and the NSF of Yunnan Province (No. 2013FB001).

(i) *all elements in $G\backslash N$ are of order $p^e$, where $p$ is a prime, and centralise no element of $G$ of order coprime to $p$;*
(ii) *$N$ is a normal subgroup of $G$, but not necessarily a characteristic subgroup;*
(iii) *the quotient group $G/N \cong \mathrm{C}_p^d$ is elementary abelian.*

Our principal motivation of studying REA groups is to study a problem regarding a type of edge-transitive Cayley graph, defined below, and covered in more detail in Section 3.

Definition 1.3. A graph $\Gamma$ is called a *normal edge-transitive Cayley graph* if $\Gamma$ is a Cayley graph of some group $G$ and the normaliser $\mathbf{N}_{\mathsf{Aut}\,\Gamma}(G)$ is transitive on the set of edges of $\Gamma$.

Edge-transitive Cayley graphs are not necessarily normal edge-transitive. Praeger [14] proposed to characterise normal edge-transitive Cayley graphs (also refer to [15]).

For positive integers $m$ and $b$, we denote by $\mathbf{K}_{m[b]}$ a complete multipartite graph which has $m$ parts of equal cardinality $b$. Then $\mathbf{K}_{m[b]}$ is an edge-transitive Cayley graph, and we are interested in the question under what condition it is normal edge-transitive.

Problem A. Determine the pairs of integers $m, b$ such that $\mathbf{K}_{m[b]}$ is a normal edge-transitive Cayley graph.

Recently, this problem was solved in [5] for the so-called *normal 2-geodesic-transitive Cayley graphs*, which form a special subclass of normal edge-transitive Cayley graphs. The following theorem reduces Problem A to the problem of studying finite REA groups.

Theorem 1.4. *If a complete multipartite graph $\mathbf{K}_{m[b]}$ is a normal edge-transitive Cayley graph of a group $G$, then $G$ is an REA group relative to a normal subgroup $N$ of order $b$ such that $|G/N| = m$.*

We can determine the values for a single parameter $m$ or $b$, although we cannot determine the pairs $(m, b)$ (see Corollaries 4.3 and 3.7).

Corollary 1.5.

(1) *For any prime power $m$, there exists an integer $b$ such that $\mathbf{K}_{m[b]}$ is a normal edge-transitive Cayley graph.*
(2) *For each positive integer $b$, there exists an integer $m$ such that $\mathbf{K}_{m[b]}$ is a normal edge-transitive Cayley graph.*

From the known examples, we are inclined to guess that $m \le b$ in general; the only known counterexamples to this are $(m, b) = (m, 1)$ or $(p^2, p)$.

Theorem 1.4 leads us to addressing the following classification problem.

Problem B. Classify finite REA groups.

The examples of REA groups include abelian $p$-groups, the groups of order $p^3$, certain Camina groups and some Frobenius groups.

Theorem 1.6. *Let $G = V : \mathrm{Q}_8$ be a Frobenius group with a Frobenius complement $\mathrm{Q}_8$. Then:*

(i) $V = \mathrm{C}^2_{p_1^{e_1}} \times \cdots \times \mathrm{C}^2_{p_t^{e_t}}$, *where $p_i$ are primes and $e_i \geq 1$; and*

(ii) *each $\mathrm{C}^2_{p_i^{e_i}} : \mathrm{Q}_8$ is a Frobenius group; and*

(iii) *$G$ is an REA group relative to $V : \mathrm{C}_2$ and $V : \mathrm{C}_4$.*

In subsequent work, a classification will be given of Frobenius REA groups.

## 2. Proof of Theorem 1.2

Let $G$ be a finite REA group relative to a subgroup $N$.

Lemma 2.1. *There exists an integer $p^e$, where $p$ is a prime and $e \geq 1$, such that all elements of $G\backslash N$ are of order $p^e$. Each element of $G\backslash N$ centralises no $p'$-element of $G$ (if any).*

Proof. Let $x$ be an element of $G\backslash N$ of order $n$. For any prime divisor $r$ of $n$, the element $x^r$ has order $n/r$ and thus $x$ and $x^r$ are not conjugate under $\mathsf{Aut}(G)$. Hence, $x^r \in N$. If $n = n_1n_2$ is such that $n_1$ and $n_2$ are relatively prime, then $x^{n_1}, x^{n_2} \in N$. Thus, $\langle x^{n_1}, x^{n_2}\rangle \leq N$ and it implies that $x \in N$, which is a contradiction. So, $n$ is a power of a prime, namely, $n = p^e$, where $p$ is a prime. □

The next lemma shows that the class of REA groups is closed under taking quotients with respect to characteristic subgroups.

Lemma 2.2. *If $M$ is a characteristic subgroup of $G$ which is contained in $N$, then the factor group $G/M$ is an REA group relative to $N/M$.*

Proof. Let $\bar{x}, \bar{y}$ be two elements of $(G/M)\backslash(N/M)$. Let $x$ and $y$ be the preimages of $\bar{x}$ and $\bar{y}$, respectively, under $G \to G/M$. Then $x, y \in G\backslash N$. Since $G$ is an REA group relative to $N$, there exists an automorphism $\sigma \in \mathsf{Aut}(G)$ such that $x^\sigma = y$ or $y^{-1}$. Since $M$ is characteristic, we have $M^\sigma = M$ and hence $(xM)^\sigma = yM$ or $y^{-1}M$. Therefore, $G/M$ is an REA group relative to $N/M$. □

The next example shows that a group may be an REA group relative to different subgroups.

Example 2.3. Let $G = \mathrm{Q}_8$, the quaternion group, and let $G = \langle x, y\rangle$ and $z = xy$. Then $\mathsf{Aut}(G) \cong \mathrm{S}_4$ and hence there exist $\sigma, \tau \in \mathsf{Aut}(G)$ with $\langle \sigma, \tau \mid \sigma^\tau = \sigma^{-1}\rangle \cong \mathrm{S}_3$ such that $x^\sigma = y$, $y^\sigma = z$ and $z^\sigma = x$, and $\tau$ fixes $\langle z\rangle$ and interchanges $x$ and $y$.

Let $N_1 = \langle x^2\rangle \cong \mathrm{C}_2$. Then $G\backslash N_1$ consists of all elements of order 4, and $\mathsf{Aut}(G)$ is transitive on $G\backslash N_1$. So, $G = \mathrm{Q}_8$ is an REA group relative to $N_1$.

Let $N_2 = \langle z\rangle \cong \mathrm{C}_4$. Then $N_2$ is normal but not characteristic in $G$, and $G\backslash N_2 = \{x, x^{-1}, y, y^{-1}\}$. Now $\tau$ maps $x$ to $y$ and $x^{-1}$ to $y^{-1}$. So, $G = \mathrm{Q}_8$ is an REA group relative to $N_2$. □

https://doi.org/10.1017/S1446788715000464 Published online by Cambridge University Press

Lemma 2.4. *The subgroup $N$ is a normal subgroup of $G$. However, $N$ is not necessarily a characteristic subgroup.*

Proof. Suppose that $G$ is a minimal counterexample to the statement. Then, in particular, $N$ is not a normal subgroup of $G$.

By Lemma 2.1, the elements of $G\backslash N$ are of order $p^e$, where $p$ is a prime. Let $T$ be the set of all elements of $G$ of order divisible by a prime $r \neq p$ (if any). Then $T \subset N$ and $T$ generates a characteristic subgroup of $G$. Let $M = \langle T \rangle$. Since $N$ is a group, $M$ is a subgroup of $N$. Since $T$ contains all $p'$-elements of $G$, the factor group $G/M$ is a $p$-group, and is an REA group relative to $N/M$ by Lemma 2.2.

If $G/M$ is abelian, then $N/M$ is a normal subgroup of $G/M$ and so $N$ is a normal subgroup of $G$, which is a contradiction. Thus, $G/M$ is nonabelian.

If $(G\backslash N) \cap \mathbf{Z}(G) \neq \emptyset$, then $G\backslash N \subset \mathbf{Z}(G)$ as $\mathbf{Z}(G)$ is a characteristic subgroup of $G$ and all elements of $G\backslash N$ are fused or inverse-fused. Noticing that $G = N \cup (G\backslash N)$, we have $G = N \cup \mathbf{Z}(G)$, which contradicts the fact that a group is not equal to the union of two proper subgroups. Hence, $(G\backslash N) \cap \mathbf{Z}(G) = \emptyset$.

Since $G = N \cup (G\backslash N)$, we have $\mathbf{Z}(G) \leq N$. By Lemma 2.2, the factor group $G/\mathbf{Z}(G)$ is an REA group relative to $N/\mathbf{Z}(G)$. By the minimality of $G$, $N/\mathbf{Z}(G)$ is normal in $G/\mathbf{Z}(G)$. It implies that $N$ is normal in $G$, which is again a contradiction.

We therefore conclude that $N$ is a normal subgroup of $G$. By Example 2.3, $N$ is not necessarily a characteristic subgroup of $G$. This completes the proof of the lemma. □

Lemma 2.5. *The factor group $G/N \cong \mathrm{C}_p^d$, where $p$ is a prime.*

Proof. By Lemma 2.1, the elements of $G\backslash N$ have order $p^e$, where $p$ is a prime. Since all elements of $G\backslash N$ are fused or inverse-fused, each nonidentity element of $G/N$ is of order $p$.

Assume first that $G$ is a $p$-group. Then the commutator subgroup $G'$ is nontrivial, and $G/G'$ is abelian. We claim that $G' \leq N$. Suppose, to the contrary, that $G' \nleq N$. Then $(G\backslash N) \cap G' \neq \emptyset$. Since $G'$ is a characteristic subgroup of $G$ and all elements of $G\backslash N$ are fused or inverse-fused, every element of $G\backslash N$ lies in $G'$, namely, $G\backslash N \subset G'$. Thus, $G = N \cup (G\backslash N) = N \cup G'$, which contradicts the fact that a group is not equal to the union of two proper subgroups. Hence, $G' \leq N$, and $G/N$ is abelian. Therefore, $G/N \cong \mathrm{C}_p^d$ for some positive integer $d$.

Suppose now that $G$ is not a $p$-group. Since the elements in $G\backslash N$ have the same order $p^e$, each element of $G$ of order not equal to $p^e$ lies in the normal subgroup $N$. Thus, the set

$$T = \{g \in G \mid o(g) \neq p^e\}$$

is a subset of the subgroup $N$, and $M := \langle T \rangle \leq N$. Clearly, any automorphism $\sigma \in \mathsf{Aut}(G)$ fixes $T$ setwise, namely, $T^\sigma = T$. It implies that $T$ generates a characteristic subgroup of $G$. By Lemma 2.2, $G/M$ is an REA group relative to $N/M$. Since $T$ contains all $p'$-elements of $G$, so does $N \geq M$. So, the factor group $G/N$ is a $p$-group. Therefore, the factor group $(G/M)/(N/M)$ is an elementary abelian $p$-group by the previous paragraph. So is $G/N$, because $G/N \cong (G/M)/(N/M)$ is elementary abelian, completing the proof. □

https://doi.org/10.1017/S1446788715000464 Published online by Cambridge University Press

Proof of Theorem 1.2. Let $G$ be a finite REA group relative to a subgroup $N$. By Lemma 2.1, there exists a prime $p$ such that all elements of $G\backslash N$ are of order $p^e$, as in part (i) of Theorem 1.2. Then Lemma 2.4 shows that the subgroup $N$ is normal but not necessarily characteristic in $G$, as in part (ii). Finally, by Lemma 2.5, the factor group $G/N$ is an elementary abelian $p$-group, as in part (iii). So, Theorem 1.2 holds. □

## 3. Normal edge-transitive Cayley graphs

For a group $G$ and a self-inverse subset $S$ of $G$ (namely, an element $x \in S$ if and only if the inverse $x^{-1} \in S$), a *Cayley graph* $\mathsf{Cay}(G, S)$ is the graph with vertex set $G$ such that two vertices $x, y \in G$ are adjacent if and only if $yx^{-1} \in S$.

For a Cayley graph $\Gamma = \mathsf{Cay}(G, S)$, the right multiplication of elements of $G$ on $G$ forms a subgroup $\hat{G}$ of $\mathsf{Aut}\,\Gamma$, which is regular on the vertex set $G$. There is a criterion to decide whether a graph $\Gamma = (V, E)$ is a Cayley graph.

Lemma 3.1 (See [1, Proposition 16.3]). *A graph $\Gamma$ is a Cayley graph if and only if the automorphism group $\mathsf{Aut}\,\Gamma$ has a subgroup which is vertex-regular.*

Let $\Gamma = \mathsf{Cay}(G, S)$, and let

$$\mathsf{Aut}(G, S) = \{\sigma \in \mathsf{Aut}(G) \mid S^{\sigma} = S\},$$

which is a subgroup of the automorphism group $\mathsf{Aut}(G)$ and fixes the subset $S$ setwise. Each element of $\mathsf{Aut}(G, S)$ induces an automorphism of the Cayley graph $\Gamma$ and fixes the vertex corresponding to the identity of $G$. An important property (by [9, Lemma 2.1]) for this subgroup is

$$\mathbf{N}_{\mathsf{Aut}\,\Gamma}(\hat{G}) = \hat{G} : \mathsf{Aut}(G, S),$$

the normaliser of the regular subgroup $\hat{G}$ in the full automorphism group $\mathsf{Aut}\,\Gamma$. In general, the subgroup $\mathbf{N}_{\mathsf{Aut}\,\Gamma}(\hat{G})$ is not necessarily edge-transitive on $\Gamma$.

Let $\Gamma = \mathsf{Cay}(G, S)$ be a Cayley graph. If $\Gamma$ is disconnected, then the component that contains the identity is a subgroup of $G$, and other components are the cosets of this subgroup. Suppose that, for any elements $s, t \in S$, there exists $\sigma \in \mathsf{Aut}(G, S)$ such that $s^{\tau} = t$ or $t^{-1}$. Then the edges $\{x, sx\}$ and $\{y, ty\}$ are equivalent under $\hat{G} : \mathsf{Aut}(G, S)$ for, if $s^{\tau} = t$, then $\{x, sx\}^{\hat{x}^{-1}\tau\hat{y}} = \{1, s\}^{\tau\hat{y}} = \{1, t\}^{\hat{y}} = \{y, ty\}$ and, if $s^{\tau} = t^{-1}$, then $\{x, sx\}^{\hat{x}^{-1}\tau\widehat{ty}} = \{1, s\}^{\tau\widehat{ty}} = \{1, t^{-1}\}^{\widehat{ty}} = \{ty, y\}$. We thus have the following simple conclusion.

Lemma 3.2. *A Cayley graph $\Gamma = \mathsf{Cay}(G, S)$ is a normal edge-transitive Cayley graph if and only if any two elements of $S$ are conjugate or inverse-conjugate under $\mathsf{Aut}(G, S)$.*

A graph may be not a Cayley graph even if it is vertex-transitive, for example, the Petersen graph. A Cayley graph may be expressed as a Cayley graph of different groups.

Example 3.3. The complete graph $\Gamma = \mathbf{K}_8$ is a Cayley graph of any group $G$ of order 8. However, it is not a normal edge-transitive Cayley graph of $G$ unless $G$ is elementary abelian.

This example tells us that the normal edge-transitivity of a Cayley graph is an algebraic property, but not a combinatorial property.

It is clear that any two vertices of $\mathbf{K}_{m[b]}$ are equivalent under automorphisms, and so are any two edges. Hence, $\mathbf{K}_{m[b]}$ is vertex-transitive and edge-transitive. The automorphism group of $\mathbf{K}_{m[b]}$ is $\mathrm{S}_b \wr \mathrm{S}_m$, the *wreath product* of $\mathrm{S}_b$ by $\mathrm{S}_m$. A natural question is to describe the edge-transitive subgroups.

Problem C. Determine the subgroups of $\mathrm{S}_b \wr \mathrm{S}_m$ which are edge-transitive on $\mathbf{K}_{m[b]}$.

We remark that edge-transitive automorphism groups of complete multipartite graphs include some important classes of groups, such as imprimitive permutation groups of rank 3 (refer to [4]). See [7, 8] for the study of Problem C for some special cases.

Next we prove Theorem 1.4, beginning with treating complete graphs.

Lemma 3.4. *A complete graph $\mathbf{K}_n$ is a normal edge-transitive Cayley graph if and only if n is a prime power.*

Proof. Let $\Gamma = \mathbf{K}_n$, where $n \geq 2$. Then, for any group $G$ of order $n$, we have $\Gamma = \mathsf{Cay}(G, S)$, where $S = G \backslash \{1\}$, the set of nonidentity elements.

Assume that $\Gamma$ is a normal edge-transitive Cayley graph of $G$. Then all nonidentity elements of $G$ are of the same order, and it follows that all nonidentity elements of $G$ are of order $p$ for some prime $p$, namely, $G$ is a $p$-group. Thus, the center $\mathbf{Z}(G) \neq 1$. Since any two nonidentity elements of $G$ are fused or inverse-fused, $\mathbf{Z}(G) = G$, and $G = \mathrm{C}_p^d$ is an elementary abelian $p$-group. In particular, the order $n$ is a power of a prime.

Conversely, let $n = p^d$ with $p$ prime, and let $G = \mathrm{C}_p^d$. Let $N = \{1\}$ and $S = G \backslash \{1\}$. Then $\mathsf{Aut}(G) = \mathsf{Aut}(G, S) \cong \mathrm{GL}(d, p)$ is transitive on $S$, and so $\Gamma$ is a normal edge-transitive Cayley graph of $G = \mathrm{C}_p^d$. □

The next lemma deals with the general case, which proves Theorem 1.4.

Lemma 3.5. *If $\mathbf{K}_{m[b]}$ is a normal edge-transitive Cayley graph of a group G, then G is an REA group relative to a subgroup N of order b.*

Proof. Let $\Gamma = (V, E) = \mathbf{K}_{m[b]}$ be a normal edge-transitive Cayley graph of $G$. Then $\Gamma = \mathsf{Cay}(G, S)$ is such that any two elements of $S$ are fused or inverse-fused under $\mathsf{Aut}(G, S)$.

Let $\Delta_1, \Delta_2, \ldots, \Delta_m$ be the $m$ parts of $\Gamma$. Then $V = \Delta_1 \cup \Delta_2 \cup \cdots \cup \Delta_m$ and $|\Delta_1| = \cdots = |\Delta_m| = b$. Let $\Delta_1$ contain the vertex $\alpha$ corresponding to the identity 1 of $G$. The complement of $\Gamma$ is disconnected and all components are isomorphic to the complete graph $\mathbf{K}_b$. By Lemma 3.1, the component on $\Delta_1$ is a subgroup of $G$. Let $N$ be this subgroup. Then $|N| = b$, and $S = G \backslash N$. Since $\Gamma$ is a normal edge-transitive Cayley graph, all elements of $G \backslash N$ are fused or inverse-fused by Lemma 3.2, and so $G$ is an REA group relative to $N$. □

We end this section with treating complete bipartite graphs.

https://doi.org/10.1017/S1446788715000464 Published online by Cambridge University Press

LEMMA 3.6. *A complete bipartite graph* $\mathbf{K}_{n,n}$ *with* $n$ *odd is a normal edge-transitive Cayley graph of a group* $G$ *if and only if* $G = N : \mathrm{C}_2$ *is a Frobenius group and* $N$ *is abelian.*

PROOF. Let $\Gamma = \mathbf{K}_{n,n}$ be a Cayley graph $\mathsf{Cay}(G, S)$. Then the order $|G| = 2n$ and $S = G \backslash N$, where $N$ is a subgroup of $G$ of index 2. In particular, $N \lhd G$, and $G = \langle N, g\rangle$ is such that $g^2 \in N$. For any odd integer $m$, we have $g^m \in S$.

Suppose that $\Gamma = \mathsf{Cay}(G, S)$ is a normal edge-transitive Cayley graph. Then any two elements of $S = G\backslash N$ are fused or inverse-fused under $\mathsf{Aut}(G, S) \le \mathsf{Aut}(G)$. Hence, $G$ is an REA group relative to the normal subgroup $N$, and all elements of $S$ are involutions.

Since $n$ is odd, the order $|N|$ is odd. Thus, $\mathbf{C}_N(g) = 1$, that is, $g$ acts on $N$ by conjugation and is fixed-point-free. Let $g \in S$. Then, for any $h \in N$, the product $hg \notin N$, and so $hg \in S$ is of order 2. Hence, $g^{-1}hg = ghg = h^{-1}$, namely, $g$ inverts all nonidentity elements of $N$. For any two elements $h_1, h_2 \in N$,

$$h_2^{-1}h_1^{-1} = (h_1h_2)^{-1} = (h_1h_2)^g = h_1^g h_2^g = h_1^{-1}h_2^{-1},$$

and thus $N$ is abelian. It implies that $G$ is a Frobenius group. □

COROLLARY 3.7. *For any integer* $n \ge 2$*, the complete bipartite graph* $\mathbf{K}_{n,n}$ *is a normal edge-transitive Cayley graph.*

PROOF. Let $n = 2^e m$ with $m$ odd, and let $M$ be a cyclic group of order $m$. Let

$$G = M : \langle z\rangle = \mathrm{C}_m : \mathrm{C}_{2^{e+1}}$$

be such that $z$ inverts every nonidentity element of $M$, namely, for any $x \in M$,

$$x^z = x^{-1}.$$

For any odd integer $\lambda$, there is an automorphism $\sigma \in \mathsf{Aut}(G)$ such that

$$x^\sigma = x, \quad z^\sigma = z^\lambda, \quad \text{where } x \in M.$$

It implies that all elements of $G$ of order $2^{e+1}$ are conjugate under $\mathsf{Aut}(G)$. Thus, $G$ is an REA group relative to $\mathrm{C}_{2^e m}$, and $\mathbf{K}_{n,n}$ is a normal edge-transitive Cayley graph of the group $G$. □

## 4. Several families of REA groups

In this section, we present some examples of REA groups.

**4.1. Nilpotent groups.** We first consider nilpotent REA groups.

LEMMA 4.1. *A nilpotent REA group is a* $p$*-group, where* $p$ *is a prime.*

PROOF. Let $G$ be a nilpotent REA group relative to a subgroup $N$. By Lemma 2.1, all elements of $G\backslash N$ are of order $p^e$, where $p$ is a prime. Let $g$ be an element of $G\backslash N$. If $G$ contains an element $x$ of order $q$, where $q \ne p$ is a prime, then $x \in N$ and $xg$ is of order $p^e q$, which is a contradiction, for $g$ and $xg$ should be fused or inverse-fused. Thus, $G$ is a $p$-group. □

Moreover, for abelian groups, we have the following result.

https://doi.org/10.1017/S1446788715000464 Published online by Cambridge University Press

Proposition 4.2. *An abelian group is an REA group if and only if it is a $p$-group, where $p$ is a prime.*

Proof. By Lemma 4.1, we only need to prove that each abelian $p$-group is an REA group.

Let $G$ be an abelian $p$-group. Let $p^e$ be the exponent of $G$. Let $N$ be the subgroup of $G$ generated by elements of $G$ of order at most $p^{e-1}$. Then $G\backslash N$ consists of all elements of $G$ of order $p^e$. Since $G$ is abelian, it is easily shown that any two elements of order $p^e$ are conjugate under $\mathsf{Aut}(G)$. So, $G$ is an REA group relative to $N$. □

For an abelian REA group $G$, if $|G| = p^k$ and $|N| = p^\ell$, then either $\ell = 0$ and $G$ is elementary abelian, or $\ell \geq k/2$. Thus, $b = p^\ell$, and $m = p^{k-\ell} \leq p^\ell$.

Corollary 4.3. *For any prime $p$ and integers $n \leqslant \ell$, the complete multipartite graph $\mathbf{K}_{p^n[p^\ell]}$ is a normal edge-transitive Cayley graph.*

Next we consider nonabelian $p$-groups which are REA groups.

Proposition 4.4. *All groups of order $p^3$ are REA groups, where $p$ is a prime.*

Proof. Let $G$ be a group of order $p^3$. If $G$ is abelian, then, by Proposition 4.2, we are done.

Assume that $G = \langle a \rangle : \langle b \rangle = \mathrm{C}_{p^2} : \mathrm{C}_p$, where $a^b = a^{1+p}$. For any element $x \in G$ of order $p^2$, we have $x = a^i b^j$, where $\gcd(i, p) = 1$, and $G = \langle x \rangle : \langle b \rangle$ such that $x^b = x^{1+p}$. It follows that there exists $\sigma \in \mathsf{Aut}(G)$ such that $a^\sigma = x$. Therefore, all elements of $G$ of order $p^2$ are conjugate in $\mathsf{Aut}(G)$. Let $S$ consist of elements of $G$ of order $p^2$, and $N$ be the subgroup of $G$ generated by elements of order $p$. Then $N = \mathrm{C}_p^2$, $S = G\backslash N$ has cardinality $p^3 - p^2 = p^2(p-1)$ and $G$ is an REA group relative to $N$. In this case, $\mathsf{Cay}(G, S) \cong \mathbf{K}_{p[p^2]}$ is normal edge-transitive, and $\mathsf{Aut}(G, S) = \mathsf{Aut}(G)$.

Suppose now that $G$ is nonabelian and of exponent $p$. Let $N = \mathbf{Z}(G) = \mathrm{C}_p$. Then $S = G\backslash N$ consists of $p^3 - p = p(p^2 - 1)$ elements of order $p$. The automorphism group $\mathsf{Aut}(G)$ is isomorphic to $\mathrm{C}_p^2 : \mathrm{GL}(2, p)$ (refer to [12, Lemma 2.3]), and is transitive on the set $S$. Thus, $G$ is an REA group relative to $N$, and $\mathsf{Cay}(G, S) = \mathbf{K}_{p^2[p]}$ is a normal edge-transitive Cayley graph. □

**4.2. Frobenius groups.** We study a family of Frobenius groups which are REA groups. A *Frobenius group* $G$ has the form $G = F : H$ such that each nonidentity element of $H$ centralises no nonidentity element of $F$, that is, $xy \neq yx$ for any $x \in F\backslash\{1\}$ and $y \in H\backslash\{1\}$ (refer to [6]). In this case, the normal subgroup $F$ is called the *Frobenius kernel*, and the subgroup $H$ is called a *Frobenius complement* of $G$.

We consider Frobenius groups with a Frobenius complement $\mathrm{Q}_8$.

Lemma 4.5. *Let $p$ be an odd prime, and let $G = \mathrm{C}_{p^e}^2 : \mathrm{Q}_8$ be a Frobenius group. Then the following hold:*

(i) $\mathsf{Aut}(G) = \mathrm{C}_{p^e}^2 \cdot (\mathrm{C}_{p^{e-1}(p-1)} \circ \mathrm{GL}(2, 3))$;

(ii) *$G$ is an REA group relative to $\mathrm{C}_{p^e}^2 : \mathrm{C}_2$ and $\mathrm{C}_{p^e}^2 : \mathrm{C}_4$.*

https://doi.org/10.1017/S1446788715000464 Published online by Cambridge University Press

Proof. Let $V = \mathrm{C}_{p^e} \times \mathrm{C}_{p^e}$, and let $G_2$ be a Sylow 2-subgroup of $G$. Then $G_2 = \langle x_1, x_2\rangle \cong \mathrm{Q}_8$, and $\langle x_1\rangle$, $\langle x_2\rangle$ and $\langle x_1x_2\rangle$ are the subgroups of $G_2$ of order 4. Let $x_3 = x_1x_2$.

It is known that $\mathsf{Aut}(V) = P \cdot \mathrm{GL}(2, p)$, where $P$ is a $p$-group (refer to [7]). Noting that $V$ is a characteristic subgroup of $G$, it follows that each automorphism of $G$ induces a nontrivial automorphism of $V$. Since $G = V : H$ is a Frobenius group, we have $V \,\mathrm{char}\, G \cong \mathsf{Inn}(G) \lhd \mathsf{Aut}(G)$. Thus,

$$\mathrm{Q}_8 \cong G/V \lhd \mathsf{Aut}(G)/V \leqslant \mathsf{Aut}(V).$$

It implies $\mathsf{Aut}(G)/V \cong \mathbf{N}_{\mathsf{Aut}(V)}(\mathrm{Q}_8)$, and

$$\mathsf{Aut}(G) = V \cdot \mathbf{N}_{\mathsf{Aut}(V)}(\mathrm{Q}_8) = V \cdot (\mathrm{C}_{p^{e-1}(p-1)} \circ \mathrm{GL}(2, 3)).$$

Noticing that $\mathrm{GL}(2, 3) \cong \mathrm{Q}_8 : \mathrm{S}_3$, there exist automorphisms $\sigma, \tau \in \mathsf{Aut}(G)$ such that $\langle \sigma, \tau \mid \sigma^\tau = \sigma^{-1}\rangle \cong \mathrm{S}_3$, and

$$\langle x_1\rangle^\sigma = \langle x_2\rangle, \quad \langle x_2\rangle^\sigma = \langle x_3\rangle \quad \text{and} \quad \langle x_3\rangle^\sigma = \langle x_1\rangle,$$
$$\langle x_1\rangle^\tau = \langle x_2\rangle, \quad \langle x_2\rangle^\tau = \langle x_1\rangle \quad \text{and} \quad \langle x_3\rangle^\tau = \langle x_3\rangle.$$

Let $N_1 = \mathrm{C}_{p^e}^2 : \langle x_1^2\rangle = \mathrm{C}_{p^e}^2 : \mathrm{C}_2 \lhd G$. Then $G\backslash N_1$ consists of all elements of $G$ of order 4. Let $a, b \in G\backslash N_1$ be such that $b \neq a$ or $a^{-1}$. By Sylow's theorem, we may assume that $a, b \in G_2$. Without loss of generality, let $a = x_1$. Then $\langle b\rangle = \langle x_2\rangle$ or $\langle x_3\rangle$. For the former, $\langle a\rangle^\sigma = \langle x_1\rangle^\sigma = \langle x_2\rangle = \langle b\rangle$, and hence $a^\sigma = b$ or $b^{-1}$, and, for the latter, $\langle a\rangle^{\sigma^{-1}} = \langle x_1\rangle^{\sigma^{-1}} = \langle x_3\rangle = \langle b\rangle$, and so it follows that $a^{\sigma^{-1}} = b$ or $b^{-1}$. So, $G$ is an REA group relative to $N_1$.

Let $N_2 = \mathrm{C}_{p^e}^2 : \langle x_3\rangle = \mathrm{C}_{p^e}^2 : \mathrm{C}_4 \lhd G$. Then $x_1, x_2 \notin N_2$. Let $a, b$ be distinct elements of $G\backslash N_2$ such that $b \neq a^{-1}$. By Sylow's theorem, we may suppose that $a, b \in G_2$. Without loss of generality, we may assume that $\langle a\rangle = \langle x_1\rangle$. Then $\langle b\rangle = \langle x_2\rangle$. Thus, $\langle a\rangle^\tau = \langle x_1\rangle^\tau = \langle x_2\rangle = \langle b\rangle$, and $a^\tau = b$ or $b^{-1}$. Therefore, $G$ is an REA group relative to $N_2$. □

We remark that in the above proof the full automorphism group $\mathsf{Aut}(G)$ acts on $G\backslash N_1$. However, since $\sigma$ does not normalise $N_2$, the subgroup $\langle\sigma\rangle \cong \mathrm{C}_3$ does not act on $G\backslash N_2$. It implies that $N_2$ is not a characteristic subgroup of $G$.

Finally, we verify that all Frobenius groups with Frobenius complements $\mathrm{Q}_8$ are indeed REA groups.

Proof of Theorem 1.6. Let $G = V : \mathrm{Q}_8$ be a Frobenius group. Let $H = \mathrm{Q}_8$ be a Frobenius complement of $G$. Then the involution $g$ of $H$ fixes no nonidentity element of the Frobenius kernel $V$. It implies that $g$ inverts every nonidentity element of $V$, and then $V$ is abelian. By Maschke's theorem, $V$ can be decomposed as

$$V = V_1 \times V_2 \times \cdots \times V_t$$

such that $H$ normalises each $V_i$, and $V_i$ is indecomposable with respect to the action of $H$. Since $G$ is a Frobenius group, $H$ acts faithfully on $V_i$, and thus $H$ acts on $V_i/\Phi(V_i)$ irreducibly and faithfully, where $\Phi(V_i)$ is the Frattini subgroup of $V_i$. It is known that

https://doi.org/10.1017/S1446788715000464 Published online by Cambridge University Press

a faithful irreducible representation of $Q_8$ is of dimension 2. Thus, $V_i = C_{p_i^{e_i}} \times C_{p_i^{e_i}}$ for some prime power $p_i^{e_i}$, as in Theorem 1.6(i).

Let $W_i$ be the factor group of $G$ modulo $\prod_{j\neq i} V_j$, where $1 \le i \le t$. Then $W_i = V_i : H_i$, where $H_i \cong H = Q_8$, and it implies that $W_i = C^2_{p_i^{e_i}} : Q_8$ is a Frobenius group, as in Theorem 1.6(ii).

Let $H_i = \langle x_i, y_i\rangle$, and $z_i = x_i y_i$, where $1 \le i \le t$. By Lemma 4.5, there are automorphisms $\sigma_i, \tau_i \in \mathsf{Aut}(W_i)$ such that $\langle\sigma_i, \tau_i\rangle \cong S_3$, where $o(\sigma_i) = 3$ and $o(\tau_i) = 2$, and

$$\langle x_i\rangle^{\sigma_i} = \langle y_i\rangle, \quad \langle y_i\rangle^{\sigma_i} = \langle z_i\rangle \quad \text{and} \quad \langle z_i\rangle^{\sigma_i} = \langle x_i\rangle,$$
$$\langle x_i\rangle^{\tau_i} = \langle y_i\rangle, \quad \langle y_i\rangle^{\tau_i} = \langle x_i\rangle \quad \text{and} \quad \langle z_i\rangle^{\tau_i} = \langle z_i\rangle.$$

The group $G = (V_1 \times V_2 \times \cdots \times V_t) : H$ can be embedded in

$$(V_1 : H_1) \times \cdots \times (V_t : H_t),$$

as a subgroup such that $H = \langle x, y\rangle$, where $x = x_1 \cdots x_t$ and $y = y_1 \cdots y_t$. Let

$$\sigma = \sigma_1 \cdots \sigma_t \quad \text{and} \quad \tau = \tau_1 \cdots \tau_t.$$

Then $\sigma, \tau$ are automorphisms of $G$ such that

$$\langle x\rangle^{\sigma} = \langle y\rangle, \quad \langle y\rangle^{\sigma} = \langle xy\rangle, \quad \langle x\rangle^{\tau} = \langle y\rangle \quad \text{and} \quad \langle y\rangle^{\tau} = \langle x\rangle.$$

Let $N_1 = V : \langle x^2\rangle = V : C_2$. Then all elements of $G\backslash N_1$ are of order 4. The subgroup $\langle\sigma\rangle \cong C_3$ is transitive on the three subgroups of $H = Q_8$ of order 4, and, by Sylow's theorem, all subgroups of $G$ of order 4 are fused. For any two elements $a, b \in G\backslash N_1$, the subgroups $\langle a\rangle$ and $\langle b\rangle$ are of order 4 and fused, and so $\langle a\rangle^{\rho} = \langle b\rangle$ for some $\rho \in \mathsf{Aut}(G)$. Therefore, $a^{\rho} = b$ or $b^{-1}$, and so $G$ is an REA group relative to $N_1$, as in Theorem 1.6(iii).

Let $N_2 = V : \langle xy\rangle = V : C_4$. Let $a, b \in G\backslash N_2$ be such that $b \neq a$ or $a^{-1}$. Then $a$ and $b$ are of order 4. By Sylow's theorem, we may assume that $a, b$ belong to the same Sylow 2-subgroup $H = \langle x, y\rangle$. Without loss of generality, assume that $\langle a\rangle = \langle x\rangle$. Then $\langle b\rangle = \langle y\rangle$. Thus, the automorphism $\tau$ of $G$ mentioned above is such that

$$\langle a\rangle^{\tau} = \langle x\rangle^{\tau} = \langle y\rangle = \langle b\rangle,$$

and so $a^{\tau} = b$ or $b^{-1}$. Therefore, $G$ is an REA group relative to $N_2$, as in part (iv). □

This has an immediate consequence regarding the parameters $m$ and $b$.

Corollary 4.6. *For any odd integer $m$, the complete 4-partite graph $\mathbf{K}_{4[2m^2]}$ is a normal edge-transitive Cayley graph.*

Proof. By Theorem 1.6, there exists an REA group $G = C_m^2 : Q_8$ relative to $N = C_m^2 : C_2$. Thus, $\mathbf{K}_{4[2m^2]}$ is a normal edge-transitive Cayley graph of $G$. □

https://doi.org/10.1017/S1446788715000464 Published online by Cambridge University Press

## References


[1] N. Biggs, *Algebraic Graph Theory* (Cambridge University Press, Cambridge, 1993).
[2] A. R. Camina, 'Some conditions which almost characterize Frobenius groups', *Israel J. Math.* **31** (1978), 153–160.
[3] R. Dark and C. M. Scoppola, 'On Camina groups of prime power order', *J. Algebra* **181** (1996), 787–802.
[4] A. Devillers, M. Giudici, C. H. Li, G. Pearce and C. E. Praeger, 'On imprimitive rank 3 permutation groups', *J. Lond. Math. Soc.* (2) **84** (2011), 649–669.
[5] A. Devillers, W. Jin, C. H. Li and C. E. Praeger, 'On normal 2-geodesic transitive Cayley graphs', *J. Algebraic Combin.* **39** (2014), 903–918.
[6] J. D. Dixon and B. Mortimer, *Permutation Groups* (Springer, Hong Kong, New York, 1996).
[7] W. W. Fan, D. Leemans, C. H. Li and J. M. Pan, 'Locally 2-arc-transitive complete bipartite graphs', *J. Combin. Theory Ser. A* **120** (2013), 683–699.
[8] W. W. Fan, C. H. Li and J. M. Pan, 'Finite locally-primitive complete bipartite graphs', *J. Group Theory* **17** (2014), 111–129.
[9] C. D. Godsil, 'On the full automorphism group of a graph', *Combinatorica* **1** (1981), 243–256.
[10] M. L. Lewis, 'Generalizing Camina groups and their character tables', *J. Group Theory* **12** (2009), 209–218.
[11] M. L. Lewis, 'On $p$-group Camina pairs', *J. Group Theory* **15** (2012), 469–483.
[12] C. H. Li and S. H. Qiao, 'Finite groups of fourth-power free order', *J. Group Theory* **16** (2013), 275–298.
[13] N. M. Mlaiki, 'Camina triples', *Canad. Math. Bull.* **57** (2014), 125–131.
[14] C. E. Praeger, 'Finite normal edge-transitive Cayley graphs', *Bull. Aust. Math. Soc.* **60** (1999), 207–220.
[15] M. Y. Xu, 'Automorphism groups and isomorphisms of Cayley digraphs', *Discrete Math.* **182** (1998), 309–319.



CAI HENG LI, School of Mathematics and Statistics, Yunnan University, Kunming, Yunnan 650091, PR China
and
School of Mathematics and Statistics, The University of Western Australia, Crawley, WA 6009, Australia
e-mail: cai.heng.li@uwa.edu.au

LEI WANG, School of Mathematics and Statistics, Yunnan University, Kunming, Yunnan 650091, PR China
e-mail: wanglei-2468@163.com