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INTERIOR REGULARITY OF THE DEGENERATE MONGE-AMPÈRE EQUATION

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We study interior $C^{1,1}$ regularity of generalised solutions of the Monge-Ampére equation $\det D^2u = \psi$, $\psi \geqslant 0$, on a bounded convex domain Ω in $\mathbb{R}^n$ with $u = \varphi$ on $\partial\Omega$. We prove in particular that $u \in C^{1,1}(\Omega)$ if either i) $\varphi = 0$ and $\psi^{1/(n-1)} \in C^{1,1}(\Omega)$ or ii) Ω is $C^{1,1}$ strongly convex, $\varphi \in C^{1,1}(\overline{\Omega})$, $\psi^{1/(n-1)} \in C^{1,1}(\overline{\Omega})$ and $\psi > 0$ on $U \cap \Omega$, where U is a neighbourhood of $\partial\Omega$. The main tool is an improvement of Pogorelov's well known $C^{1,1}$ estimate so that it can be applied to the degenerate case.

1. INTRODUCTION

For an arbitrary convex function u one can define a nonnegative Borel measure $M(u)$ such that

$$M(u) = \det D^2u\, d\lambda$$

for smooth, and even $W^{2,n}_{loc}$ functions (see [17] for details). The Dirichlet problem for M is solvable in a fairly general situation: let Ω be an arbitrary bounded convex domain in $\mathbb{R}^n$ and $\varphi \in C(\partial\Omega)$ be such that it is convex on any line segment in $\partial\Omega$ (we shall call such a φ *admissible*). Then for any nonnegative Borel measure μ with $\mu(\Omega) < \infty$ the Dirichlet problem

$$(1.1) \qquad \begin{cases} u \ \text{ continuous and convex on } \overline{\Omega} \\ M(u) = \mu \ \text{ in } \ \Omega \\ u = \varphi \ \text{ on } \ \partial\Omega \end{cases}$$

has a unique solution. (This was proven for example in [17] for strictly convex Ω where all continuous φ are admissible, and the general case easily follows from this - see Proposition 2.1 below.)

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We shall primarily consider measures μ with continuous, nonnegative densities ψ in Ω. Unless otherwise stated, u will always denote the solution of (1.1) (with $\mu = \psi d\lambda$), whereas v will be the solution of the corresponding homogeneous problem:

$$\left\{\begin{array}{l} v \ \text{ continuous and convex on } \ \overline{\Omega} \\ M(v) = 0 \ \text{ in } \ \Omega \\ v = \varphi \ \text{ on } \ \partial\Omega. \end{array}\right. \tag{1.2}$$

Below we list several known regularity results for solutions of (1.1):

(1) $\varphi = 0$, $\psi \in C^\infty(\Omega)$, $\psi > 0 \Rightarrow u \in C^\infty(\Omega)$ (Cheng, Yau [**8**], see also [**9**]);

(2) Ω is $C^{1,1}$ strongly convex, $\varphi \in C^{1,1}(\overline{\Omega}) \Rightarrow v \in C^{0,1}(\overline{\Omega}) \cap C^{1,1}(\Omega)$ (Trudinger, Urbas [**19**]);

(3) Ω is $C^{1,1}$ strongly convex, $\varphi \in C^{1,1}(\overline{\Omega})$, $\psi \in C^{1,1}(\overline{\Omega})$, $\psi > 0 \Rightarrow u \in C^{1,1}(\Omega)$ (Trudinger, Urbas [**19**]);

(4) Ω is C^∞ strongly convex, $\varphi \in C^\infty(\partial\Omega)$, $\psi \in C^\infty(\overline{\Omega})$, $\psi > 0 \Rightarrow u \in C^\infty(\overline{\Omega})$ (Krylov [**15**] and Caffarelli, Nirenberg and Spruck [**6**]);

(5) Ω is $C^{3,1}$ strongly convex, $\varphi \in C^{3,1}(\overline{\Omega})$, $\psi^{1/(n-1)} \in C^{1,1}(\overline{\Omega}) \Rightarrow u \in C^{1,1}(\overline{\Omega})$ (Guan, Trudinger and Wang [**13**]). One should mention that in [**13**] the authors several times use that $\psi^{1/2(n-1)} \in C^{0,1}(\overline{\Omega})$ which is not always satisfied in this situation. The general case can be obtained by slight modifications of the method used in [**13**], except for Lemma 2.1 but this one had been earlier proved in [**12**]. All this will be explained in the upcoming correction to [**13**].

Pogorelov had claimed to prove (1) earlier but his proof had gaps. However, his interior $C^{1,1}$ estimate remained a crucial step in the proof of (1). We are going to improve it to the degenerate case (see Theorem 3.1 below).

The following results on the local regularity of the Monge-Ampère operator M are also known:

(6) u is strictly convex, $\psi \in C^\infty$, $\psi > 0 \Rightarrow u \in C^\infty$ (this follows easily from (1));

(7) either $u \in W^{2,p}_{loc}$ for some $p > n(n-1)/2$ or $u \in C^{1,\alpha}$ for some $\alpha > 1 - 2/n$, $\psi \in C^\infty$, $\psi > 0 \Rightarrow u \in C^\infty$ (Urbas [**20**]).

In this paper we prove two more regularity results:

THEOREM 1.1. *If $\varphi = 0$ and $\psi^{1/(n-1)} \in C^{1,1}(\Omega)$ then $u \in C^{1,1}(\Omega)$.*

THEOREM 1.2. *Suppose that Ω is a $C^{1,1}$ strongly convex domain, $\varphi \in C^{1,1}(\overline{\Omega})$ and $\psi^{1/(n-1)} \in C^{1,1}(\overline{\Omega})$. Assume moreover that*

(1.3) *every connected component of the set $\{\psi = 0\} \cap \Omega$ is compact.*

Then $u \in C^{1,1}(\Omega)$.

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Note that in Theorem 1.1 Ω is an arbitrary bounded convex domain in $\mathbb{R}^n$. By (7), Theorem 1.1 is a generalisation of (1). We also immediately get the following local regularity which generalises (6):

THEOREM 1.3. *If u is strictly convex and $\psi^{1/(n-1)} \in C^{1,1}$ then $u \in C^{1,1}$.*

This paper was mostly motivated by recent articles [**12**] and [**13**], where global a priori estimates for second derivatives of u depending on $\|\psi^{1/(n-1)}\|_{C^{1,1}(\overline{\Omega})}$ were established. The main tool in proving Theorems 1.1 and 1.2 will be a corresponding interior a priori estimate (Theorem 3.1). The importance of the exponent $1/(n-1)$ is that, by [**21**, Example 3], it is optimal in all of the above results.

Note that (1.3) follows if, for example, $\psi > 0$ on $U \cap \Omega$, where U is a neighbourhood of $\partial\Omega$. We believe that the assumption (1.3) in Theorem 1.2 is in fact superfluous. However, at the end of section 4 we give an example which shows that the application of our methods only does not allow to drop this assumption. Namely, what we really prove is that under the assumptions of Theorem 1.2 except for (1.3), we have $u \in C^{1,1}(\{u < v\})$ (Theorem 4.2). Moreover, $\{u < v\} = \Omega$ if (1.3) is satisfied (Proposition 4.1), but without (1.3) it may happen that $\{u < v\} \neq \Omega$.

The Pogorelov estimate from [**16**] has been improved in the non-degenerate case by Ivochkina [**14**], who used a different method. There have been attempts to obtain a similar interior regularity for the complex Monge-Ampère equation. In [**18**] the Ivochkina integral method and in [**11**] the original Pogorelov approach were used. However, both authors made the same mistake. Namely, they used a false formula

$$\left|\frac{\partial^3 u}{\partial z_i \partial \overline{z}_j \partial z_i}\right| = \left|\frac{\partial^3 u}{\partial z_i \partial \overline{z}_i \partial z_j}\right|.$$

This made the first inequality in [**18**, p. 91] and inequality (3.6) on p. 697 in [**11**] false. In fact the function

$$u(z) = -1 + a|z_2|^2 + (1 + 2\,Re\, z_2)|z_1|^2,$$

where $0 < a < 1$ and $z = (z_1, z_2)$ is in some neighbourhood of the origin in $\mathbb{C}^2$, is a counterexample to both inequalities. Actually, the falseness of the above formula is the sole reason why the methods used here cannot be applied in the complex case. The gap in the proof of the regularity of the pluricomplex Green's function in [**11**] was filled in by [**4**] (see also the correction to [**11**] and [**5**]). We are not sure if anything can be saved from [**18**] though.

This means that in the complex case the only valid interior $C^{1,1}$ estimate so far (that is an estimate for the second derivative of a solution in the interior not depending on the smoothness of the boundary) is from [**10**, Proposition 7.1]. This one however depends in particular on an L^∞ norm of $u^{z_i\bar{z}_j}u_{z_i}u_{\bar{z}_j}$ instead of the L^∞ norm the gradient of u. Therefore, in order to apply it to obtain regularity results for the complex Monge-Ampère equation one would have to find an interior estimate for $u^{z_i\bar{z}_j}u_{z_i}u_{\bar{z}_j}$. In [**10**] it is achieved under additional assumption that the injectivity radius of the Kähler metric $(u_{z_i\bar{z}_j})$ is locally bounded from below by a positive constant (thanks to an estimate from [**7**]).

In the complex case already the interior gradient estimate presents a challenge, quite contrary to the real one, where it is more or less trivial (see (4.1) below). It was obtained for convex domains in $\mathbb{C}^n$ ([**3**, Theorem 2.1]) but the general case remains open. One should add that unfortunately the proofs of [**3**, Theorems A and 4.1] cannot be considered valid, since the estimate from [**18**] was used there.

TERMINOLOGY AND NOTATION. We say that a convex domain Ω (respectively, convex function u) is *strictly convex* if $\partial\Omega$ (respectively, graph u) contains no line segment. A convex function v in Ω is called *maximal* if for any $D \Subset \Omega$ and u continuous on $\overline{D}$, convex in D, we have that $u \leqslant v$ on ∂D implies $u \leqslant v$ in D. This condition is in fact equivalent to $M(v) = 0$ (see [**17**]). A function u is called *strongly convex* if locally there exists $\lambda > 0$ such that $u - \lambda|x|^2$ is convex. We say that a bounded domain Ω is *strongly convex* if there exists w, a strongly convex function in a neighbourhood of $\overline{\Omega}$ such that $\Omega = \{w < 0\}$ and $Dw \neq 0$ near $\partial\Omega$ (that is there exists $\varepsilon > 0$ such that for any supporting hyperplane $H = \operatorname{graph} L$ of w at a point near $\partial\Omega$ we have $|\nabla L| \geqslant \varepsilon$). A domain is called $C^{k,\alpha}$ strongly convex if one can find a $C^{k,\alpha}$ defining function w. If we write $f \in C^{k,1}(\overline{\Omega})$ then we mean that $f \in C^{k,1}(\Omega)$ and $|D^{k+1}f|$ is globally bounded in Ω. Then

$$\|f\|_{C^{k,1}(\overline{\Omega})} = \sum_{i=0}^{k+1} \sup_{\Omega} |D^i f|$$

and the values of f at $\partial\Omega$ are uniquely determined. Finally, $\overline{B}(x,r)$ will denote a closed ball centred at x with radius r and $[x,y]$ will stand for the line segment joining x and y.

2. Existence of generalised solutions

In this section we shall show how to generalise the solution of the Dirichlet problem in [**17**, (1.1)] from strictly convex to arbitrary convex domains.

PROPOSITION 2.1. *Assume that Ω is an arbitrary bounded convex domain in $\mathbb{R}^n$. Let $\varphi \in C(\partial\Omega)$ be admissible (that is φ is convex on any line segment in $\partial\Omega$) and μ be a nonnegative Borel measure in Ω with $\mu(\Omega) < \infty$. Then the problem* (1.1) *has a unique solution.*

PROOF: The uniqueness follows from the comparison principle (see for example, [**17**]). If Ω is strictly convex then the proof can be found for example in [**17**]. We shall use this as well as other results from [**17**] in the proof.

We first solve the homogeneous problem (1.2) using the Perron method. Define

$$v := \sup\{w \text{ convex and continuous on } \overline{\Omega}, w \leqslant \varphi \text{ on } \partial\Omega\}.$$

Then v is convex in Ω and we have to show that v continuously extends to $\partial\Omega$ and equals φ there. Fix $x_0 \in \partial\Omega$ and $\varepsilon > 0$. Since φ is convex on any line segment contained in $\partial\Omega$ and passing through x_0, we can find an affine function L such that $L(x_0) \geqslant \varphi(x_0) - \varepsilon$ and $L \leqslant \varphi$ on $\partial\Omega$. Let h be a solution of the classical Dirichlet problem

$$\begin{cases} h \text{ continuous on } \overline{\Omega} \\ h \text{ harmonic in } \Omega \\ h = \varphi \text{ on } \partial\Omega. \end{cases}$$

Then we have $L \leqslant v \leqslant h$ in Ω and

$$\varphi(x_0) - \varepsilon \leqslant \liminf_{x \to x_0} v(x) \leqslant \limsup_{x \to x_0} v(x) \leqslant \varphi(x_0).$$

This shows that v is continuous on $\overline{\Omega}$ and $v = \varphi$ on $\partial\Omega$. The definition of v implies that v is maximal and thus $M(v) = 0$ there. This solves (1.2).

Next we solve (1.1) with $\varphi = 0$. This has already been done in [**2**, Theorem 4.1] but we include it for the convenience of the reader. Let Ω_j be strictly convex domains such that $\Omega_j \uparrow \Omega$ as $j \uparrow \infty$. Let u_j be continuous and convex on $\overline{\Omega}_j$ and such that $M(u_j) = \mu$ in Ω_j and $u_j = 0$ on $\partial\Omega_j$. By the comparison principle the sequence u_j is decreasing. Moreover, [**17**, Lemma 3.5] gives

$$(2.1) \qquad \big(-u_j(x)\big)^n \leqslant c_n (\operatorname{diam}\Omega)^{n-1} \operatorname{dist}(x, \partial\Omega)\mu(\Omega), \quad x \in \Omega_j,$$

where c_n is a constant depending only on n. This implies that u_j converges locally uniformly in Ω to a convex $\widetilde{u}$. The inequality (2.1) also shows that $\lim_{x \to \partial\Omega} \widetilde{u}(x) = 0$. By the continuity theorem for M ([**17**, Proposition 3.1]) we have $M(\widetilde{u}) = \mu$ in Ω. This solves (1.1) for $\varphi = 0$.

Now let μ and φ be arbitrary. Again, we approximate Ω by strictly convex domains Ω_j from inside. There we can find u'_j, continuous and convex on $\overline{\Omega}_j$, such that $M(u'_j) = \mu$ in Ω_j and $u'_j = v$ on $\partial\Omega_j$. [**17**, Proposition 3.3] gives $M(\widetilde{u} + v) \geqslant \mu$ and thus

$$\widetilde{u} + v \leqslant u'_{j+1} \leqslant u'_j \leqslant v \text{ in } \Omega_j$$

by the comparison principle. It now easily follows that u'_j decreases to a function u which solves (1.1). $\square$

3. A priori estimate for second derivatives

In this section we shall generalise the Pogorelov estimate (see [**16**]) so that it can be applied to the degenerate case. We shall modify Pogorelov's method using some ideas from [**12**]. We shall prove:

Theorem 3.1. *Let $u \in C^4(\Omega) \cap C^{1,1}(\overline{\Omega})$ be a strongly convex solution of*

$$\det D^2 u = \psi \tag{3.1}$$

in a bounded domain Ω in $\mathbb{R}^n$. Assume that w is a C^2 convex function in Ω such that $u \leqslant w$ in Ω and $\lim_{x \to \partial\Omega} (w(x) - u(x)) = 0$. Then for α such that

$$\alpha \begin{cases} = n-1 & \text{for } n \geqslant 3, \\ > 1 & \text{for } n = 2, \end{cases}$$

we have in Ω

$$(w-u)^\alpha |D^2 u| \leqslant C,$$

where C depends only on n (on α if $n = 2$) and on upper bounds of $\operatorname{diam}\Omega$, $\|\psi^{1/(n-1)}\|_{C^{1,1}(\overline{\Omega})}$ *and* $\|w-u\|_{C^{0,1}(\overline{\Omega})}$.

Proof: We may assume that $\Omega \subset B(0,R)$, where $R \leqslant \operatorname{diam}\Omega$. We shall use the standard notation: $u_i = \partial u / \partial x_i$, $(u_{ij}) = D^2 u$ and $(u^{ij}) = (D^2 u)^{-1}$. First, we differentiate the logarithm of both sides of (3.1) twice with respect to x_p. We get

$$u^{ij} u_{pij} = (\log \psi)_p \tag{3.2}$$

$$u^{ij} u_{ppij} = (\log \psi)_{pp} + u^{ik} u^{jl} u_{pij} u_{pkl}. \tag{3.3}$$

Consider the auxiliary function

$$h = (w-u)^\alpha e^{\beta |x|^2/2} |D^2 u|,$$

where $\beta > 0$ will be specified later. By the assumptions on u and w, h attains a maximum at some $y \in \Omega$. Since $|D^2 u|$ is equal to the maximal eigenvalue of $D^2 u$, after an orthonormal change of variables we may assume that at y the matrix $D^2 u$ is diagonal and $|D^2 u| = u_{11}$. Then the function

$$\widetilde{h} = (w-u)^\alpha e^{\beta |x|^2/2} u_{11}$$

also attains a maximum at y and $\widetilde{h}(y) = h(y)$. By the assumptions of the theorem it is enough to show that

$$u_{11}(y) \leqslant C_1, \tag{3.4}$$

where by $C_1, C_2, \dots$ we shall denote constants depending only on the desired quantities.

From now on all formulas are assumed to hold at y. We may assume that $u < w$. We shall use the fact that $D(\log\widetilde{h}) = 0$ and $D^2(\log\widetilde{h}) \leqslant 0$. Therefore

$$0 = \left(\log\widetilde{h}\right)_i = \alpha\frac{w_i - u_i}{w - u} + \beta x_i + \frac{u_{11i}}{u_{11}} \tag{3.5}$$

for every $i = 1, \dots, n$ and

$$u^{ii}\left(\log\widetilde{h}\right)_{ii} \leqslant 0.$$

Now we analyse the term where the third and fourth derivatives of u appear. Using (3.2), (3.3), (3.5) and denoting $g := \psi^{1/(n-1)}$ we compute

$$\begin{aligned}
u^{ii}(\log u_{11})_{ii} &= \frac{1}{u_{11}}\sum_i \frac{u_{11ii}}{u_{ii}} - \frac{1}{(u_{11})^2}\sum_i \frac{(u_{11i})^2}{u_{ii}} \\
&= \frac{(\log\psi)_{11}}{u_{11}} + \frac{1}{u_{11}}\sum_{i,j}\frac{(u_{1ij})^2}{u_{ii}u_{jj}} - \frac{1}{(u_{11})^2}\sum_i \frac{(u_{11i})^2}{u_{ii}} \\
&\geqslant \frac{(\log\psi)_{11}}{u_{11}} + \frac{1}{u_{11}}\sum_{i\geqslant 2}\frac{(u_{1ii})^2}{(u_{ii})^2} + \frac{1}{(u_{11})^2}\sum_{i\geqslant 2}\frac{(u_{11i})^2}{u_{ii}} \\
&\geqslant \frac{(\log\psi)_{11}}{u_{11}} + \frac{1}{(n-1)u_{11}}\left((\log\psi)_1 - \frac{u_{111}}{u_{11}}\right)^2 + \frac{1}{(u_{11})^2}\sum_{i\geqslant 2}\frac{(u_{11i})^2}{u_{ii}} \\
&\geqslant \frac{(n-1)g_{11}}{g\,u_{11}} - \frac{2g_1u_{111}}{g(u_{11})^2} + \frac{1}{(u_{11})^2}\sum_{i\geqslant 2}\frac{(u_{11i})^2}{u_{ii}} \\
&= \frac{(n-1)g_{11}}{g\,u_{11}} + \frac{2g_1}{g\,u_{11}}\left(\alpha\frac{w_1 - u_1}{w - u} + \beta x_1\right) + \sum_{i\geqslant 2}\frac{1}{u_{ii}}\left(\alpha\frac{w_i - u_i}{w - u} + \beta x_i\right)^2.
\end{aligned}$$

We thus obtain

$$\begin{aligned}
0 &\geqslant u^{ii}\left(\log\widetilde{h}\right)_{ii} \\
&= \frac{\alpha}{w - u}\sum_i \frac{w_{ii}}{u_{ii}} - \frac{\alpha\, n}{w - u} - \frac{\alpha}{(w - u)^2}\sum_i \frac{(w_i - u_i)^2}{u_{ii}} + \beta\sum_i \frac{1}{u_{ii}} \\
&\qquad + u^{ii}(\log u_{11})_{ii} \\
&\geqslant -\frac{\alpha\, n}{w - u} - \frac{\alpha(w_1 - u_1)^2}{(w - u)^2 u_{11}} + \frac{(n-1)g_{11}}{g\,u_{11}} + \frac{2g_1}{g\,u_{11}}\left(\alpha\frac{w_1 - u_1}{w - u} + \beta x_1\right) \\
&\qquad + \beta\sum_i \frac{1}{u_{ii}} + \sum_{i\geqslant 2}\frac{1}{u_{ii}}\left(\frac{\alpha(\alpha - 1)(w_i - u_i)^2}{(w - u)^2} + \frac{2\alpha\beta(w_i - u_i)x_i}{w - u} + \beta^2 x_i^2\right)
\end{aligned}$$

$$\geqslant -\frac{\alpha n}{w-u} - \frac{\alpha(w_1-u_1)^2}{(w-u)^2 u_{11}} + \frac{(n-1)g_{11}}{g\,u_{11}} + \frac{2g_1}{g\,u_{11}}\left(\alpha\frac{w_1-u_1}{w-u} + \beta x_1\right)$$
$$+ \left(\beta - \frac{\beta^2 R^2}{\alpha-1}\right)\sum_{i\geqslant 2}\frac{1}{u_{ii}}.$$

An optimal choice for β is therefore $(\alpha-1)/(2R^2)$. Multiplying both sides of the obtained inequality by

$$\frac{4R^2(w-u)^n g\,u_{11}}{(\alpha-1)(n-1)}$$

and observing that the inequality between arithmetic and geometric means gives

$$\sum_{i\geqslant 2}\frac{1}{u_{ii}} \geqslant \frac{(n-1)(u_{11})^{1/(n-1)}}{g},$$

we get

$$\left((w-u)^{n-1}u_{11}\right)^{n/(n-1)} - C_2(w-u)^{n-1}u_{11} - C_3 \leqslant 0.$$

From this (3.4) easily follows. ∎

Note, that in fact in Theorem 3.1 we could replace $\|\psi^{1/(n-1)}\|_{C^{1,1}(\overline{\Omega})}$ with $\|\psi^{1/(n-1)}\|_{C^{0,1}(\overline{\Omega})}$ and the maximal eigenvalue of $-D^2\left(\psi^{1/(n-1)}\right)$ in Ω.

4. $C^{1,1}$ REGULARITY OF GENERALISED SOLUTIONS

First we prove Theorem 1.1.

PROOF OF THEOREM 1.1: We may assume that $\psi \in C^{1,1}(\overline{\Omega})$ - otherwise shrink Ω a little. Fix $\varepsilon > 0$. Let Ω_j be a sequence of C^∞ strongly convex domains such that $\Omega_j \uparrow \Omega$ as $j \uparrow \infty$. We can find $\psi_j \in C^\infty(\overline{\Omega}_j)$ such that $\psi_j > 0$, ψ_j tends uniformly to ψ in Ω and

$$\|\psi_j^{1/(n-1)}\|_{C^{1,1}(\overline{\Omega}_j)} \leqslant C_1.$$

(By $C_1, C_2, \ldots$ we shall denote constants independent of j.) By (4) one can find $u_j \in C^\infty(\overline{\Omega}_j)$, convex in Ω_j such that $\det D^2 u_j = \psi_j$ in Ω_j and $u_j = 0$ on $\partial\Omega_j$.

Let A be so big that $\eta(x) := |x|^2 - A \leqslant 0$ for $x \in \Omega$. From the comparison principle and the superadditivity of the operator M it follows that on $\overline{\Omega}_j$

$$u + \|\psi - \psi_j\|_{L^\infty(\overline{\Omega}_j)}^{1/n}\eta \leqslant u_j \leqslant u - \|\psi - \psi_j\|_{L^\infty(\overline{\Omega}_j)}^{1/n}\eta + \|u\|_{L^\infty(\partial\Omega_j)}.$$

Therefore in particular u_j tends locally uniformly to u in Ω and

$$\sup_{\Omega_j}|u_j| \leqslant C_2.$$

Since u_j is convex,

$$|Du_j(x)| \leqslant \frac{-u_j(x)}{\operatorname{dist}(x, \partial\Omega_j)}, \quad x \in \Omega_j. \tag{4.1}$$

Thus

$$|Du_j| \leqslant C_3 \quad \text{in } \{u_j < -\varepsilon\}.$$

Theorem 3.1 now gives for some α

$$(-\varepsilon - u_j)^\alpha |D^2 u_j| \leqslant C_4 \quad \text{in } \{u_j < -\varepsilon\}.$$

Hence

$$|D^2 u_j| \leqslant C_5 \quad \text{in } \{u_j < -2\varepsilon\}$$

and $u \in C^{1,1}(\{u < -2\varepsilon\})$. Since ε can be chosen arbitrarily small, it follows that $u \in C^{1,1}(\Omega)$. ∎

Theorem 1.2 will be a direct consequence of (2) and the next two results.

PROPOSITION 4.1. *Assume that ψ is continuous and let φ be admissible and such that $v \in C^{1,1}(\Omega)$. Then $u < v$ on the domain consisting of $\{\psi > 0\} \cap \Omega$ and those connected components of $\{\psi = 0\} \cap \Omega$ which are compact.*

PROOF: By K denote a compact component of $\{\psi = 0\} \cap \Omega$. First we want to find an open U such that $K \subset U \Subset \Omega$ and $\psi > 0$ on ∂U. Let Ω' be open and such that $K \subset \Omega' \Subset \Omega$. Set $F := \{\psi = 0\} \cap \overline{\Omega'}$ and by $\mathcal{E}$ denote the family of open, closed (in F) subsets of F containing K. It is a known fact from the general topology that, since K is a connected component of a compact F, $K = \bigcap \mathcal{E}$. Since the family $\{F \setminus E\}_{E \in \mathcal{E}}$ is an open (in F) cover of a compact set $F \cap \partial\Omega'$, we can find $E_1, \ldots, E_k \in \mathcal{E}$ such that $E := E_1 \cap \cdots \cap E_k \in \mathcal{E}$ does not intersect $\partial\Omega'$. Then E and $F \setminus E$ are compact and we can find open U, V in Ω such that $U \cap V = \emptyset$, $F \subset U \cup V$, $K \subset U \subset \Omega'$ and $F \cap \partial\Omega' \subset V$. It follows that U has the required properties.

From the comparison principle (applied in U) and since v is a maximal convex function, it is now enough to show that $u < v$ on $\{\psi > 0\}$. As the problem is now purely local, we may assume that $\psi \geqslant a > 0$ and $|D^2 v| \leqslant M < \infty$ on $\overline{B}(x_0, r)$. We shall proceed in the same way as in [**19**, p. 329]. For $\varepsilon > 0$ define

$$w(x) := \varepsilon\big(|x - x_0|^2 - r^2\big).$$

Then

$$\det D^2(v + w) \leqslant c_n \sum_{i=1}^{n} \varepsilon^i M^{n-i} \leqslant a$$

if ε is small enough. From the comparison principle we get $u \leqslant v + w$ on $\overline{B}(x_0, r)$. ∎

REMARK. If $n = 2$ then the assumption that $v \in C^{1,1}(\Omega)$ in Proposition 4.1 is superfluous. For by an old result of Aleksandrov [**1**] u is then strictly convex in $\{\psi > 0\} \cap \Omega$. If $x_0 \in \{\psi > 0\} \cap \Omega$ and $H = \operatorname{graph} L$ is a supporting hyperplane of $\operatorname{graph} v$ at x_0 then by [**19**, Lemma 2.1] x_0 belongs to the convex hull of the set $\{v = L\} \cap \partial\Omega$. We can thus find $y_1, y_2 \in \partial\Omega$ such that v is affine on $[y_1, y_2]$ and $x_0 \in [y_1, y_2]$. From the strict convexity of u near x_0 it now follows that $u(x_0) < v(x_0)$ which proves the claim.

However, the following example due to Pogorelov shows that this assumption cannot be dropped if $n \geqslant 3$. Set

$$u(x) = \left(1 + x_1^2\right)\left(x_2^2 + \cdots + x_n^2\right)^{1-1/n}.$$

Then in a neighbourhood of the origin u is convex (if $n \geqslant 3$), $M(u) \in C^\infty$, $M(u) > 0$. But if Ω is a small ball centred at the origin, we shall always have $u = v = 0$ on $\{x_2 = \cdots = x_n = 0\} \cap \Omega$.

We shall now show that to get $u \in C^{1,1}(\{u < v\})$ one needs only very mild assumptions on Ω and φ:

THEOREM 4.2. *Assume that Ω is strongly convex and let $\varphi \in C(\partial\Omega)$ be such that $v \in C^{0,1}(\overline{\Omega})$. Then, if $\psi^{1/(n-1)} \in C^{1,1}(\overline{\Omega})$, we have $u \in C^{1,1}(\{u < v\})$.*

PROOF: Regularising the defining function for Ω we get C^∞ strongly convex domains $\Omega_j = \{w_j < 0\}$ such that

$$D^2 w_j \geqslant \frac{1}{C_1} I \quad \text{on } \overline{\Omega}_j, \tag{4.2}$$

and, since convex functions are locally Lipschitz,

$$|Dw_j| \leqslant C_2, \quad \text{on } \partial\Omega_j, \tag{4.3}$$

where $C_1, C_2, \ldots$ are positive constants independent of j and I is the unitary matrix. Regularising v we get $v_j \in C^\infty(\overline{\Omega}_j)$, convex, converging locally uniformly to v, such that

$$\|u - v_j\|_{L^\infty(\partial\Omega_j)} \to \infty \tag{4.4}$$

as $j \uparrow \infty$ and

$$|Dv_j| \leqslant C_3 \quad \text{on } \overline{\Omega}_j. \tag{4.5}$$

We can also find $\psi_j \in C^\infty(\overline{\Omega}_j)$, $\psi_j > 0$, ψ_j tending uniformly to ψ in Ω and such that

$$\|\psi_j^{1/(n-1)}\|_{C^{1,1}(\overline{\Omega}_j)} \leqslant C_4. \tag{4.6}$$

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Let $u_j \in C^\infty(\overline{\Omega}_j)$ be convex in Ω_j and such that $\det D^2 u_j = \psi_j$ in Ω_j, $u_j = v_j$ on $\partial\Omega_j$. We could choose ψ_j so that $u_j < v_j$ in Ω_j. Similarly as in in the proof of Theorem 1.1 we can get that on $\overline{\Omega}_j$

$$u + \|\psi - \psi_j\|^{1/n}_{L^\infty(\overline{\Omega}_j)}\eta - \|u - u_j\|_{L^\infty(\partial\Omega_j)} \leqslant u_j \leqslant u - \|\psi - \psi_j\|^{1/n}_{L^\infty(\overline{\Omega}_j)}\eta + \|u - u_j\|_{L^\infty(\partial\Omega_j)}$$

Therefore by (4.4) u_j tends locally uniformly to u. By (4.2) and the comparison principle

$$v_j + C_5 w_j \leqslant u_j \leqslant v_j \quad \text{on } \overline{\Omega}_j.$$

By (4.3), (4.5) and since $w_j = 0$ on $\partial\Omega_j$ we thus get

$$|Du_j| \leqslant C_6 \quad \text{on } \partial\Omega_j$$

and, since u_j is convex,

$$|Du_j| \leqslant C_6 \quad \text{on } \overline{\Omega}_j.$$

We may now use Theorem 3.1 to get

$$(v_j - u_j)^\alpha |D^2 u_j| \leqslant C_7 \quad \text{on } \overline{\Omega}_j.$$

The required result follows if we let $j \uparrow \infty$. ∎

Finally, we want to show that if we drop the assumption (1.3) in Theorem 1.2 then it may happen that $\{u = v\} \cap \Omega \neq \emptyset$. Let Ω be the unit ball B and set $\varphi(x) := x_1^2$ for $x \in \partial B$, so that $v(x) = x_1^2$ for $x \in \overline{B}$. Let ψ be such that $\psi^{1/(n-1)}$ is smooth and $\operatorname{supp}\psi \subset \{x_1 > 0\} \cap B$. For $\varepsilon > 0$ let u_ε denote the solution of (1.1) with $\mu = \varepsilon\psi\, d\lambda$. Then $u_\varepsilon \uparrow v$ uniformly as $\varepsilon \downarrow 0$. For ε sufficiently small we thus have $u_\varepsilon \geqslant 0$ on $\operatorname{supp}\psi$ and thus $u_\varepsilon \geqslant 0$ in B. Therefore $u_\varepsilon = 0$ on $\{x_1 = 0\} \cap B$ and $u_\varepsilon = v$ on $\{x_1 \leqslant 0\} \cap B$ by the comparison principle.

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