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THEORETICAL PEARL

A bargain for intersection types: a simple strong normalization proof

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Abstract

This pearl gives a discount proof of the folklore theorem that every strongly β-normalizing λ-term is typable with an intersection type. (We consider typings that do not use the empty intersection ω which can type any term.) The proof uses the perpetual reduction strategy which finds a longest path. This is a simplification over existing proofs that consider any longest reduction path. The choice of reduction strategy avoids the need for weakening or strengthening of type derivations. The proof becomes a bargain because it works for more intersection type systems, while being simpler than existing proofs.

1 Introduction

Do we have a bargain for you! We prove that the set of λ-terms typable with an intersection type is exactly the strongly normalizing terms. The bargain is that we – to paraphrase Walmart – get more for less: our proof is simpler than existing proofs, but handles more systems of intersection types. The novel idea is that we use a specific reduction strategy rather than considering “any maximal reduction path”. A benefit of the approach is that our type system does not need weakening or strengthening of type derivations.

Intersection types have been around for about three decades. The original interest was theoretical: intersection types were introduced to characterize the set of solvable terms. Within the last decade they have gained traction for more practical purposes of program analysis.[1] The key idea is to introduce an intersection type operator $\wedge$ with the meaning that a term of type $\tau \wedge \sigma$ can be used at both types τ and σ. This provides a finite polymorphism where the various types of a term is listed explicitly. Moreover, strictly more terms can be typed than with the well-known universal polymorphism. In fact, it is a folklore theorem that the set of strongly β-normalizing λ-terms is exactly the set of terms that can be given an intersection type. This was

[1] Kfoury (2000) gives references on the history and Kfoury & Wells (2004) provide references on the practical usage.

first presented by Pottinger (1980). According to Kfoury (2000) Betty Venneri found subtle errors in the part that all strongly normalizing terms have an intersection types in that and a large number of subsequent proofs. Kfoury attributes the first correct proof in the published literature to Amadio and Curien some fifteen years later (Amadio & Curien, 1998).

In this pearl we focus on the tricky direction and prove that all strongly β-normalizing λ-terms have an intersection type. The general approach taken in many proofs, including the proof by Amadio and Curien, is the following:

1. establish that all β-normal forms have an intersection type, and
2. show that the type system has subject expansion under non-erasing reductions (I-redexes), i.e. that if the term M reduces to N (written $M \to_\beta N$) and N is typable, then M is typable *with the same typing*.

By definition, all strongly normalizing terms have a longest reduction path. We consider any such reduction path

$$M_0 \to_\beta M_1 \to_\beta \cdots \to_\beta M_n$$

where M_n is a β-normal form. We want to prove that M_0 is typable. The base case where $n = 0$ and $M_n = M_0$ is covered by Point 1. In the inductive step, we use Point 2 when $M_0 \to_\beta M_1$ is non-erasing. When $M_0 \to_\beta M_1$ erases a subterm of M_0, both the contractum and the erased subterm have shorter longest reduction paths than M_0. Therefore, they are typable by induction hypothesis and it is easy to construct a type for M_0.

It is straightforward to prove Point 1. However, Point 2 requires weakening and strengthening of type derivations, i.e. that we can add or remove unused variables to the type environment. This poses a problem as the most rigid intersection type systems do not have weakening or strengthening, e.g. System-$\mathbb{I}$ proposed by Kfoury & Wells (2004). Consequently, we cannot transfer Amadio and Curien's proof to these systems, but need indirect means. For instance, Kfoury and Wells prove that all System-$\mathbb{I}$ terms are strongly normalizing by a translation into a type system that Kfoury has previously proved strongly normalizing (Kfoury, 2000).

In this pearl, we circumvent weakening and strengthening by considering a specific longest reduction path, the perpetual strategy. The longest reduction path is chosen so we can transform a typing of the reduced term to a typing of the original term without strengthening and weakening. Combining this result with Point 1, we conclude that all strongly normalizable terms are typable. We get a simpler proof that works for more type systems. Now, that could be considered a bargain.

2 Outline

In the following section we recall a few notions of the λ-calculus and define intersections types. In Section 4, we introduce our reduction strategy and establish the properties outlined above. We conclude the pearl with a discussion of related work.

All omitted details of the proof can be found in my dissertation (Møller Neergaard, 2004), where the proof method is applied to System-$\mathbb{I}$.

Table 1. *Terminology*

Concept	Syntax	Meta variables
Term variable	$\mathscr{V}_\Lambda$	x, y, z, v, u, f, g
Term	$\Lambda \ni M ::= x \mid \lambda x.M \mid M\,M$	M, N, P, Q, R
Type variable	$\mathscr{V}_\mathscr{T}$	a, b
Strict type	$\overline{\mathscr{T}} \ni \overline{\tau} ::= a \mid \tau \to \overline{\tau}$	$\overline{\tau}$
Type	$\mathscr{T} \ni \tau ::= \overline{\tau} \mid \tau \wedge \tau$	τ
Type enviroment		Γ

$$\frac{}{x : \overline{\tau} \vdash x : \overline{\tau}}\,\text{Var}$$

$$\frac{\Gamma, x : \tau \vdash P : \overline{\tau}}{\Gamma \vdash \lambda^I x.P : \tau \to \overline{\tau}}\,\lambda\text{I} \qquad \frac{\Gamma \vdash P : \overline{\tau}'}{\Gamma \vdash \lambda^K x.P : \overline{\tau} \to \overline{\tau}'}\,\lambda\text{K}$$

$$\frac{\Gamma \vdash P : \tau \quad \Gamma' \vdash P : \tau'}{\Gamma \wedge \Gamma' \vdash P : \tau \wedge \tau'}\,\wedge \quad \frac{\Gamma \vdash P : \tau \to \overline{\tau} \quad \Gamma' \vdash Q : \tau}{\Gamma \wedge \Gamma' \vdash P\,Q : \overline{\tau}}\,@$$

Fig. 1. Intersection type typing rules,

3 Preliminaries on rigid intersection types

We consider λ-terms typed with intersection types. Let $\mathscr{V}_\Lambda$ and $\mathscr{V}_\mathscr{T}$ be a countably infinite sets of (term) variables and type variables, resp. Table 1 presents the term and type syntax and meta variable conventions for the remainder of the paper. We use $\mathsf{fv}(P)$ for the set of free variables in P. We use $\lambda x^I.P$ to denote an abstraction where $x \in \mathsf{fv}(P)$ and $\lambda x^K.P$ when $x \notin \mathsf{fv}(P)$. We adopt Barendregt's variable convention (Barendregt, 1984) and assume implicitly that an abstraction variable is not mentioned elsewhere in the current context (proof, discussion, etc.). We define β-reduction in the usual way:

$$(\lambda x.P)\,Q \quad \beta \quad P[Q/x]$$

where $P[Q/x]$ is the capture-free substitution of Q for x in P. The compatible closure of β is $\to_\beta$. The set of normal forms under $\to_\beta$ is NF_β.

As usual, a type environment is a finite mapping from $\mathscr{V}_\Lambda$ to $\mathscr{T}$. We write $x : \tau$ for $\Gamma(x) = \tau$. When $x \notin \text{dom}(\Gamma)$, we write $\Gamma, x : \tau$ for the extension of Γ with $x : \tau$. We extend $\wedge$ to a binary operation on type environments Γ_0 and Γ_1 by intersecting the type of common variables:

$$\Gamma_0 \wedge \Gamma_1 = \{x : \tau \mid x : \tau \in \Gamma_i, x \notin \text{dom}\,\Gamma_{1-i}\} \cup \{x : \tau_0 \wedge \tau_1 \mid x : \tau_i \in \Gamma_i\}.$$

We give the typing rules in Figure 1. When there is a derivation of $\Gamma \vdash M : \tau$, we call $\langle \Gamma; \tau \rangle$ a typing of M.

There are some subtleties to be aware of: all rules but $\wedge$ conclude with a strict type, the $\wedge$-rule can only be used on the operand of an application or at the very

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bottom of a type derivation. There is exactly one variable in the type environment of the Var rule – this prevents weakening.[2] Moreover, due to the definition of $\Gamma \wedge \Gamma'$ and the formulation of the @-rule, the type of a variable with n occurrences is the intersection of n types. This makes our type system less flexible than most presentations where the intersection operator is taken to be *associative*, *commutative*, and *idempotent*, i.e.

$$\tau \wedge \sigma = \sigma \wedge \tau \qquad \tau \wedge (\sigma \wedge \rho) = (\tau \wedge \sigma) \wedge \rho \qquad \tau \wedge \tau = \tau. \tag{1}$$

We do not need (1) and therefore refer to the intersection operator as *rigid*. The result is a system without some of the usual features of type systems. For instance, the system does not enjoy subject reduction:[3]

$$\left(\lambda x^{a\wedge(a\to b)}.(\lambda y^{a\to b}.y^{a\to b}\, x^a)\, x^{a\to b}\right)^{(a\wedge(a\to b))\to b} \to_\beta \left(\lambda x^{(a\to b)\wedge a}.x^{a\to b}\, x^a\right)^{((a\to b)\wedge a)\to b}$$
$$(\lambda x^{a\wedge b}.(\lambda z^b.x^a)\, x^b)^{(a\wedge b)\to a} \to_\beta (\lambda x^a.x^a)^{a\to a}.$$

In the first example, we lack commutativity in the second, weakening. Since rigid intersections are more restrictive than other intersection systems, all the proofs below carry through if we adopt any (or all) of the identities in (1). Therefore, it is a strength that our proof works for rigid intersections.

With the system at hand, we readily prove Point 1 mentioned in the introduction.

Lemma 1 (*All Normal Forms Are Typable With a Strict Type*)
Let $M \in \Lambda$ be a term. If $M \in \mathrm{NF}_\beta$, then M is typable with a strict type, i.e. there is a derivation of $\Gamma \vdash M : \overline{\tau}$ for some environment Γ and strict type $\overline{\tau}$.

Proof
As M is a normal form, we have either $M = \lambda x.N$ or $M = x\, N_1\, \dots\, N_n$ where $n \geqslant 0$ and $N, N_1, \dots, N_n$ are normal forms. We use induction on the structure of M. □

4 All strongly normalizing terms are typable

We can now turn to the proof that all strongly normalizing terms have an intersection type. We want a longest reduction path where we can fold the typing back over each step in the reduction. The following perpetual strategy[4] defined by Barendregt *et al.* (1976) serves our purpose:

Definition 2
The *perpetual* reduction strategy F_∞ is defined as $F_\infty(M) = M$ when $M \in \mathrm{NF}_\beta$ and otherwise

1. $F_\infty(x\, P_1\, \dots\, P_n) = x\, P_1\, \dots\, P_{m-1}\, F_\infty(P_m)\, P_{m+1}\, \dots\, P_n$ when $P_i \in \mathrm{NF}_\beta$ for $1 \leqslant i \leqslant m-1$ and $P_m \notin \mathrm{NF}_\beta$;
2. $F_\infty(\lambda x.P) = \lambda x.F_\infty(P)$;

[2] Formally, weakening is that $\Gamma \vdash M : \tau$ implies $\Gamma, x : \tau' \vdash M : \tau$. Strengthening is that $\Gamma, x : \tau' \vdash M : \tau$ implies $\Gamma \vdash M : \tau$ when $x \notin \mathsf{fv}(M)$.
[3] Subject reduction is the property that $\Gamma \vdash M : \tau$ and $M \to_\beta N$ implies $\Gamma \vdash N : \tau$.
[4] A reduction strategy is *perpetual* (Barendregt, 1984) if it preserves the existence of an infinite reduction path.

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3. $F_\infty((\lambda x.P_0)\, P_1\, \dots\, P_n) = P_0[P_1/x]\, P_2\, \dots\, P_n$ when $x \in \mathsf{fv}(P_0)$ or $P_1 \in \mathrm{NF}_\beta$;
4. $F_\infty((\lambda x.P_0)\, P_1\, \dots\, P_n) = (\lambda x.P_0)\, F_\infty(P_1)\, P_2\, \dots\, P_n$ when $x \notin \mathsf{fv}(P_0)$ and $P_1 \notin \mathrm{NF}_\beta$

where $n \geqslant 1$.

We note that F_∞ is a reduction strategy as $M \to_\beta F_\infty(M)$ if $M \notin \mathrm{NF}_\beta$ and $F_\infty(M) = M$ otherwise.

Remark 3
There are several proofs that F_∞ picks a reduction path that is the longest possible. It is implicit in de Vrijer's functionals for the length of β-reduction paths (de Vrijer, 1987). It is proved explicitly by Regnier & Danos (Regnier, 1994), Khasidashvili (1994), van Raamsdonk & Severi (1995), and Sørensen (1996).

The idea of the proof is to use induction on the length of the longest reduction path. In the inductive case, one considers a term M with a longest reduction path of length n and show that $F_\infty(M)$ has a reduction path of length $n-1$.

As mentioned, we have lost subject reduction due to the rigidity of the intersection operator. Likewise, we do not have subject expansion, i.e., unlike most intersection type systems, $M \to_\beta N$ and $\Gamma \vdash N : \tau$ do not imply $\Gamma \vdash M : \tau$. However, we can establish a weaker form of subject expansion where we only consider whether a typing exists: if the contractum of a term under F_∞ is typable, then the term is typable. It hinges on the following lemma:

Lemma 4 (Typability Is Preserved Under Substitution)
Let M and N be terms and let $x \in \mathsf{fv}(M)$. If $M[N/x]$ is typable with typing $\langle \Gamma; \tau \rangle$ then there are Γ' and Γ'' and a type τ' such that N is typable with $\langle \Gamma'; \tau' \rangle$ and M is typable with $\langle (\Gamma'', x : \tau'); \tau \rangle$ where $\Gamma''(y) = \Gamma(y)$ for $y \notin \mathsf{fv}(N)$.

Proof
Since $M[N/x]$ is typable, there is a derivation Δ of $\Gamma \vdash M[N/x] : \tau$. We use induction on the height of Δ. In the case of an application $M = P\,Q$ we split on whether $x \in \mathsf{fv}(P) \cap \mathsf{fv}(Q)$, $x \in \mathsf{fv}(P) \setminus \mathsf{fv}(Q)$, or $x \in \mathsf{fv}(Q) \setminus \mathsf{fv}(P)$. $\square$

Corollary 5 (Weak Subject Expansion Under Substitution)
Let M and N be terms and let $x \in \mathsf{fv}(M)$. If $M[N/x]$ is typable with typing $\langle \Gamma; \tau \rangle$ then $(\lambda x.M)\, N$ is typable with $\langle \Gamma'; \tau \rangle$ for some Γ'.

Using the corollary, we establish weak subject expansion under F_∞.

Proposition 6 (Weak Subject Expansion Under F_∞)
Let $M \in \Lambda$ be a term. If $F_\infty(M)$ is typable, then M is typable.

Proof
We note that if $F_\infty(M)$ is typable then there is a derivation Δ of $\Gamma \vdash F_\infty(M) : \tau$. When $M \in \mathrm{NF}_\beta$, then Δ is a typing of M as $F_\infty(M) = M$. When $M \notin \mathrm{NF}_\beta$, we use induction on the height of Δ to prove that there is a typing $\langle \Gamma', \tau' \rangle$ of M and that $\tau \in \overline{\mathscr{T}}$ if and only if $\tau' \in \overline{\mathscr{T}}$:

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1. If the derivation Δ ends in

$$\frac{\Gamma_1 \vdash F_\infty(M) : \tau_1 \quad \Gamma_2 \vdash F_\infty(M) : \tau_2}{\Gamma_1 \wedge \Gamma_2 \vdash F_\infty(M) : \tau_1 \wedge \tau_2}$$

   we get

$$\frac{\Gamma_1' \vdash M : \tau_1' \quad \Gamma_2' \vdash M : \tau_2'}{\Gamma_1' \wedge \Gamma_2' \vdash M : \tau_1' \wedge \tau_2'}$$

   from the induction hypothesis. We note that $\tau_1 \wedge \tau_2, \tau_1' \wedge \tau_2' \notin \overline{\mathscr{T}}$.

2. $M \equiv x\, P_1 \,\ldots\, P_n$ and $\tau = \overline{\tau}$. The type derivation for $F_\infty(M)$ ends in:

$$\frac{x : \tau_0 \vdash x : \tau_0 \quad \Delta_1 \quad \cdots \quad \Delta_{m-1} \quad \Gamma_m \vdash F_\infty(P_m) : \tau_m \quad \Delta_{m+1} \quad \cdots \quad \Delta_n}{(\cdots((x : \tau_0) \wedge \Gamma_1) \wedge \cdots \wedge \Gamma_n) \vdash x\, P_1 \,\ldots\, P_{m-1}\, F_\infty(P_m)\, P_{m+1} \,\ldots\, P_n : \overline{\tau}}$$

   where Δ_i derives $\Gamma_i \vdash P_i : \tau_i$ for $i = 1, \ldots, m-1, m+1, \ldots n$ and $\tau_0 = \tau_1 \to \cdots \to \tau_n \to \overline{\tau}$. By the induction hypothesis, we have

$$\frac{x : \tau_0' \vdash x : \tau_0' \quad \Delta_1 \quad \cdots \quad \Delta_{m-1} \quad \Gamma_m' \vdash P_m : \tau_m' \quad \Delta_{m+1} \quad \cdots \quad \Delta_n}{(\cdots((x : \tau_0') \wedge \Gamma_1) \wedge \cdots \wedge \Gamma_{m-1}) \wedge \Gamma_m') \wedge \Gamma_{m+1}) \wedge \cdots \wedge \Gamma_n) \vdash x\, P_1 \,\ldots\, P_n : \overline{\tau}}$$

   where $\tau_0' \equiv \tau_1 \to \cdots \to \tau_{m-1} \to \tau_m' \to \tau_{m+1} \to \ldots \to \tau_n \to \overline{\tau}$.

3. $M \equiv (\lambda x.P_0)\, P_1 \,\ldots\, P_n$ and $x \in \mathsf{fv}(P_0)$ and $\tau = \overline{\tau}$. The type derivation for $F_\infty(M)$ ends in

$$\frac{\Gamma_0 \vdash P_0[P_1/x] : \tau_0 \quad \Gamma_2 \vdash P_2 : \tau_2 \quad \cdots \quad \Gamma_n \vdash P_n : \tau_n}{(\cdots(\Gamma_0 \wedge \Gamma_2) \wedge \cdots \wedge \Gamma_n) \vdash P_0[P_1/x]\, P_2 \,\ldots\, P_n : \overline{\tau}}$$

   where $\tau_0 \equiv \tau_2 \to \cdots \to \tau_n \to \overline{\tau}$. Using Corollary 5, we have

$$\frac{\Gamma_0' \vdash (\lambda x.P_0)\, P_1 : \tau_0 \quad \Gamma_2 \vdash P_2 : \tau_2 \quad \cdots \quad \Gamma_n \vdash P_n : \tau_n}{(\cdots(\Gamma_0' \wedge \Gamma_2) \wedge \cdots \wedge \Gamma_n) \vdash (\lambda x.P_0)\, P_1 \cdots P_n : \overline{\tau}}$$

4. $M \equiv (\lambda x.P_0)\, P_1 \,\ldots\, P_n$ where $x \notin \mathsf{fv}(P_0)$, $P_1 \in \mathrm{NF}_\beta$, and $\tau = \overline{\tau}$. The type derivation for $F_\infty(M)$ ends in

$$\frac{\Gamma_0 \vdash P_0 : \tau_0 \quad \Gamma_2 \vdash P_2 : \tau_2 \quad \cdots \quad \Gamma_n \vdash P_n : \tau_n}{(\cdots(\Gamma_0 \wedge \Gamma_2) \wedge \cdots \wedge \Gamma_n) \vdash P_0\, P_2 \cdots P_n : \overline{\tau}}$$

   where $\tau_0 \equiv \tau_2 \to \cdots \to \tau_n \to \overline{\tau}$. As $P_1 \in \mathrm{NF}_\beta$, we have a typing $\Gamma_1 \vdash P_1 : \overline{\tau}_1$ by Lemma 1. We obtain

$$\frac{\dfrac{\Gamma_0 \vdash P_0 : \tau_0}{\Gamma_0 \vdash \lambda x.P_0 : \overline{\tau}_1 \to \tau_0} \quad \Gamma_1 \vdash P_1 : \overline{\tau}_1 \quad \Gamma_2 \vdash P_2 : \tau_2 \quad \cdots \quad \Gamma_n \vdash P_n : \tau_n}{(\cdots((\Gamma_0 \wedge \Gamma_1) \wedge \Gamma_2) \wedge \cdots \wedge \Gamma_n) \vdash (\lambda x.P_0)\, P_1 \cdots P_n : \overline{\tau}}$$

5. $M \equiv (\lambda x.P_0)\, P_1 \,\ldots\, P_n$ where $x \notin \mathsf{fv}(P_0)$, $P_1 \notin \mathrm{NF}_\beta$, and $\tau = \overline{\tau}$. The type derivation for $F_\infty(M)$ ends in

$$\frac{\dfrac{\Gamma_0 \vdash P_0 : \tau_0}{\Gamma_0 \vdash (\lambda x.P_0) : \overline{\tau}_1 \to \tau_0} \quad \Gamma_1 \vdash F_\infty(P_1) : \overline{\tau}_1 \quad \Gamma_2 \vdash P_2 : \tau_2 \quad \cdots \quad \Gamma_n \vdash P_n : \tau_n}{(\cdots(\Gamma_0 \wedge \Gamma_1) \wedge \cdots \wedge \Gamma_n) \vdash (\lambda x.P_0)\, F_\infty(P_1) \cdots P_n : \overline{\tau}}$$

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where $\tau_0 \equiv \tau_2 \to \cdots \to \tau_n \to \overline{\tau}$. Using the induction hypothesis, we have

$$\cfrac{\cfrac{\Gamma_0 \vdash P_0 : \tau_0}{\Gamma_0 \vdash (\lambda x.P_0) : \overline{\tau}'_1 \to \tau_0} \quad \Gamma'_1 \vdash P_1 : \overline{\tau}'_1 \quad \Gamma_2 \vdash P_2 : \tau_2 \quad \cdots \quad \Gamma_n \vdash P_n : \tau_n}{(\cdots((\Gamma_0 \wedge \Gamma'_1) \wedge \Gamma_2) \wedge \cdots \wedge \Gamma_n) \vdash (\lambda x.P_0)\, P_1 \,\cdots\, P_n : \overline{\tau}}$$

This exhausts the cases so we conclude weak subject expansion under F_∞. □

We now have the following standard theorem.

Corollary 7
Let M be a strongly normalizing term, then M is typable.

Proof
We consider any strongly normalizing term $M \in \Lambda$. There exists an n such that $F^n_\infty(M) \in \mathrm{NF}_\beta$. Now, $F^n_\infty(M)$ is typable by Lemma 1. By induction on n using the proposition it follows that M is typable. □

The opposite direction, strong normalization of every term typable with an intersection type, is usually done by the realizability method due to Tait (1975). This is a semantic method where each type is interpreted as a suitable strongly normalizing set of terms and the type derivation is shown to be sound with respect to the interpretation.

Theorem 8
Let M be typable, then M is strongly normalizing.

Proof (sketch)
Define the following sets by induction on types $\tau \in \mathscr{T}$: $[\![\alpha]\!] = \mathrm{SN}_\beta$, $[\![\rho \to \tau]\!] = \{F \in \Lambda \mid \forall a \in [\![\rho]\!].F\ a \in [\![\tau]\!]\}$, and $[\![\rho \wedge \tau]\!] = \{M \in \Lambda \mid M \in [\![\rho]\!], [\![\tau]\!]\}$. By induction on the structure of the types, we show that $[\![\tau]\!] \subseteq \mathrm{SN}_\beta$ and $x \in [\![\tau]\!]$ for all types τ.

Let a *valuation* be a map $v : V \to \Lambda$. Let $[\![M]\!]_v = M[v(x_1)/x_1, \ldots, v(x_n)/x_n]$ where $\mathsf{fv}(M) = \{x_1, \ldots, x_n\}$. We define the model relation $\models$ as follows: $v \models M : \tau$ if and only if $[\![M]\!]_v \in [\![\tau]\!]$. Moreover, $v \models \Gamma$ if and only if $v(x) \in [\![\rho]\!]$ for all $x : \tau \in \Gamma$. Finally, $\Gamma \models M : \rho$ if and only if $v \models \Gamma$ implies $v \models M : \rho$ for all valuations v.

We prove that $\Gamma \vdash M : \tau$ implies $\Gamma \models M : \tau$ by induction on the derivation. We consider the trivial valuation $v(x) = x$. We have $v \models \Gamma$. If follows that $M = [\![M]\!]_v \in [\![\tau]\!] \subseteq \mathrm{SN}_\beta$. □

An alternative approach is taken by Kfoury and Wells who give a proof-theoretic proof (Kfoury & Wells, 1995).

5 Concluding remarks and related work

We have with simple means reproved the well-known theorem that all strongly β-normalizing λ-terms have an intersection type. The simplicity stems from the fact that we consider a concrete reduction strategy and therefore can use weak subject expansion rather than subject expansion. In particular, we do not need such properties as weakening and strengthening of type derivations. I conjecture that the

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method works for all intersection type systems without nullary intersections, i.e. the type constant ω typing all terms in all contexts.

Of the two essential properties needed in the proof, typability of all normal forms and weak subject expansion, it is the weak subject expansion that sets intersection type systems apart from other type systems. There are many type systems where all β-normal forms are typable, e.g. System F (Reynolds, 1974; Girard, 1972). On the other hand, Urzyczyn proves that the following λ-term

$$\big(\lambda x.z\,(x\,(\lambda f\,\lambda u.f\;u))\,(x\,(\lambda v g.v\;g))\big)\,(\lambda y.y\;y\;y)$$

is not typable in the extremely powerful type system $\mathbf{F}_\omega$, while its only reduct

$$z\,((\lambda y.y\;y\;y)\,(\lambda f\,\lambda u.f\;u))\,((\lambda y.y\;y\;y)\,(\lambda v g.v\;g))$$

is typable (Urzyczyn, 1997).

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