SEMIGROUPS OF CONTINUOUS SELFMAPS FOR WHICH GREEN'S $\mathcal{D}$ AND $\mathcal{J}$ RELATIONS COINCIDE

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For algebraic terms which are not defined, one may consult [**2**]. The symbol $S(X)$ denotes the semigroup, under composition, of all continuous selfmaps of the topological space X. When X is discrete, $S(X)$ is simply $\mathcal{T}_X$ the full transformation semigroup on the set X. It has long been known that Green's relations $\mathcal{D}$ and $\mathcal{J}$ coincide for $\mathcal{T}_X$ [**2**, p. 52] and F. A. Cezus has shown in his doctoral dissertation [**1**, p. 34] that $\mathcal{D}$ and $\mathcal{J}$ also coincide for $S(X)$ when X is the one-point compactification of the countably infinite discrete space. Our main purpose here is to point out the fact that among the 0-dimensional metric spaces, Cezus discovered the only nondiscrete space X with the property that $\mathcal{D}$ and $\mathcal{J}$ coincide on the semigroup $S(X)$. Because of a result in a previous paper [**6**] by S. Subbiah and the author, this property (for 0-dimensional metric spaces) is in turn equivalent to the semigroup being regular. We gather all this together in the following

THEOREM. *Let X be a 0-dimensional metric space. Then the following statements are equivalent:*

(1) *$S(X)$ is a regular semigroup;*

(2) *$\mathcal{D}=\mathcal{J}$ in $S(X)$;*

(3) *X is either discrete or is the one-point compactification of he countably infinite discrete space.*

Proof. The equivalence of (1) and (3) has been established in [**6**, Theorem 3.11]. It follows from Theorem 2.9 of [**2**, p. 52] and Proposition 2.19 of [**1**, p. 34] that (3) implies (2). Now suppose that (2) holds. We want to show that (3) must then hold. As a preliminary step, we show, by contradiction, that X has at most one limit point. Suppose to the contrary that X has more. Choose any two distinct limit points and denote them by a and b respectively. Now there exist sequences $\{x_n\}_{n=1}^{\infty}$ converging to a and $\{y_n\}_{n=1}^{\infty}$ converging to b and we assume without loss of generality that all of the points involved are distinct. Let

$$A=\{a,b\}\cup\{x_n\}_{n=1}^{\infty}\cup\{y_n\}_{n=1}^{\infty}$$

and choose mutually disjoint clopen sets $\{G_n\}_{n=1}^{\infty}$ and $\{H_n\}_{n=1}^{\infty}$ so that $G_n\cap A=\{x_n\}$, $H_n\cap A=\{y_n\}$, lim diam $G_n=0$ and lim diam $H_n=0$ where diam means diameter. Finally, let

$$B=X\setminus\cup[G_n\cup H_n]_{n=1}^{\infty},$$

and define four functions f, g, v, w from X into X as follows:

$$f(x)=\begin{cases} y_{2n} & \text{for} \quad x\in G_n, \\ y_{2n-1} & \text{for} \quad x\in H_n, \\ b & \text{for} \quad x\in B. \end{cases}$$

$$g(x)=\begin{cases} y_n & \text{for} \quad x\in G_n\cup H_n, \\ b & \text{for} \quad x\in B. \end{cases}$$

$$v(x)=\begin{cases} y_n & \text{for} \quad x\in G_n, \\ y_{n/2} & \text{for} \quad x\in H_n,\ n \text{ even}, \\ y_{(n+1)/2} & \text{for} \quad x\in H_n,\ n \text{ odd}, \\ b & \text{for} \quad x\in B. \end{cases}$$

$$w(x)=\begin{cases} y_{2n} & \text{for} \quad x\in G_n, \\ y_{2n-1} & \text{for} \quad x\in H_n, \\ b & \text{for} \quad x\in B. \end{cases}$$

Now f, g, v, and w are all continuous and, since the verifications are all somewhat similar, we give the details only in the case of the function f. Since G_n and H_n are clopen, it is immediate that f is continuous at all points of these sets. Let V be any open set containing the point b. Then there is a positive integer N such that $y_n\in V$ for $n\geq N$. One readily verifies that

$$X\setminus[\cup[G_i\cup H_i]_{i=1}^N]$$

is a neighborhood of both a and b which f maps into V. This means that f is continuous at both a and b. It remains for us to consider a point c different from a and b and not in any G_n or H_n. Choose any clopen set W which contains c but does not contain either a or b. Since $\lim x_n=a$, $x_n\in G_n$ and $\lim \operatorname{diam} G_n=0$, it follows that W intersects only finitely many G_n and, for similar reasons, W intersects only finitely many H_n as well. Thus there is a positive integer N such that $W\cap[G_n\cup H_n]=\varnothing$ for $n>N$ and it follows that $W\setminus\cup[G_i\cup H_i]_{i=1}^N$ is a clopen subset of X containing c which f maps into the point b. This establishes continuity at the point c and we now conclude that f and also the functions g, v and w are continuous. That is, they all belong to $S(X)$. Routine calculations will serve to verify that $f=g\circ w$ and $g=v\circ f$ and this means that *f and g are $\mathcal{J}$-equivalent.*

Now, according to theorem (3.1) of [**5**, p. 1490] a function in $S(X)$ is regular if and only if its range is a retract of X and it maps some subspace homeomorphically onto its range. The function g is not only regular but, in fact, is idempotent. We show that f is not regular by showing that it doesn't map any subspace homeomorphically onto its range. Let

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E be any subspace which f maps bijectively onto its range. Then $E \cap G_n$ consists of one point which we denote by a_n and similarly $E \cap H_n$ consists of exactly one point which we denote by b_n. Finally $E \cap f^{-1}(b)$ consists of one point and we denote it by t. Then

$$E = \{t\} \cup \{a_n\}_{n=1}^{\infty} \cup \{b_n\}_{n=1}^{\infty}.$$

Since $\lim \operatorname{diam} G_n = 0$, $\lim a_n = a$ and, for analogous reasons, $\lim b_n = b$. Since the only point of E which could possibly be a limit of E is the point t, it readily follows that E is not compact. Thus f does not map any subspace of X homeomorphically onto its range and we conclude that f is not a regular element of $S(X)$. Since g is regular, this means that f and g are not $\mathcal{D}$-equivalent even though they are $\mathcal{J}$-equivalent. We have been able to derive this contradiction because we assumed that X has more than one limit point. Thus X has at most one limit point.

We show next that X is either discrete or compact. Suppose it is neither. Then X has exactly one limit b with a sequence $\{y_n\}_{n=1}^{\infty}$ converging to it and another sequence $\{x_n\}_{n=1}^{\infty}$ with no limit points at all. We may assume that all of these points are distinct and, as we did previously, we will construct functions which are $\mathcal{J}$-equivalent but not $\mathcal{D}$-equivalent. In fact, the only difference in what we do now from what we did previously lies in the way we define the set A. This time, let

$$A = \{b\} \cup \{x_n\}_{n=1}^{\infty} \cup \{y_n\}_{n=1}^{\infty}.$$

Then define the sets G_n, H_n and B and the functions f, g, v and w just as before. They are all continuous and f and g are $\mathcal{J}$-equivalent since $f = g \circ w$ and $g = v \circ f$. However, they are not $\mathcal{D}$-equivalent since g is idempotent while f is not even regular since, as before, f maps no subspace homeomorphically onto its range. This contradiction was reached because we assumed that X is neither discrete nor compact so it must be one of the two. It remains for us to show that if it is not discrete, then it is the one-point compactification of the countably infinite discrete space. This is easily done, for if X is not discrete then it is a compact space with exactly one limit point. That is, it is the one-point compactification of a discrete space. But that space must be countably infinite for X is metrizable and it is well-known that the one-point compactification of an uncountable discrete space is not metrizable [**3**, p. 247]. This concludes the verification that (2) implies (3), and thus the theorem is proved.

A few closing remarks are in order. In view of the theorem, there are very few 0-dimensional metric spaces X such that $\mathcal{D} = \mathcal{J}$ on $S(X)$ or, equivalently, such that $S(X)$ is a regular semigroup. Instances outside the class of 0-dimensional metric spaces are also rare. To be sure, we have deGroot's spaces [**4**, p. 87] whose semigroups are all left zero semigroups with identities and, of course, such a semigroup is regular and $\mathcal{D}$ and $\mathcal{J}$ will coincide on it. However, if X is completely regular and Hausdorff and contains an arc, then $S(X)$ will not be regular [**5**, p. 1490] and $\mathcal{D}$ and $\mathcal{J}$ will be distinct on $S(X)$ [**5**, p. 1491].

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