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WEIGHTED HARDY INEQUALITIES FOR INCREASING FUNCTIONS

Dedicated to Professor P. G. Rooney on his sixty-fifth birthday

H. P. HEINIG AND V. D. STEPANOV

ABSTRACT The purpose of this paper is to characterize the weight functions for which the Hardy operator $(Pf)(x) = x^{-1}\int_0^x f(t)\,dt$, with non-decreasing function f, is bounded between various weighted L^p-spaces for a wide range of indices Our characterizations complement for the most part those of E T Sawyer [11] and V D Stepanov [15] for the Hardy operator of non-increasing function

1. **Introduction.** The classical Hardy inequality states that the averaging operator P defined for locally integrable function f on $(0,\infty)$ by

$$(1.1)\qquad (Pf)(x) = \frac{1}{x}\int_0^x f(t)\,dt, \quad x > 0$$

is bounded on $L^p(0,\infty)$, $p > 1$. The problem of characterizing the weights (and measures) w, ν for which the inequality

$$(1.2)\qquad \left[\int_0^\infty |(Pf)(x)|^q w(x)\,dx\right]^{1/q} \leq C\left[\int_0^\infty |f(x)|^p \nu(x)\,dx\right]^{1/p}$$

holds with $0 < p, q < \infty$, $p > 1$, has been widely studied during the past twenty years and has now been completely solved. (*cf.* [10] [12] and the literature cited there.)

More recently this problem focused on the Hardy operator defined on positive decreasing functions. In this context characterizations of the weights, w, ν for which (1.2) holds were obtained by E. T. Sawyer [11] for $1 < p, q < \infty$ via a general approach using duality. A different proof of these results as well as the weight characterizations in the index ranges $0 < p, q < \infty$ were given by V. D. Stepanov [15]. It should be noted that for $1 \leq p = q < \infty$ and $w = \nu$, a weight characterization for which (1.2) holds was already obtained by D. W. Boyd [3] and subsequently by S. G. Krein, Yu. I. Petunin and E. M. Semenov [4, Chapter 2; Theorem 6.6] in connection with their study of operators in rearrangement invariant spaces. For a different characterization of this result we refer to [2]. Other results of this type for more general operators defined on monotone functions have been obtained by K. F. Andersen [1], C. J. Neugebauer [9] and S. Lai [5],

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[6]. However if f is positive and increasing then the inequalities given by the last two authors were reversed except for the index raange $0 < p \leq q < 1$. In this case weight characterizations were given for which (1.2) holds. (*cf.* [5, Theorems 2.3; 2.4]).

The object of this paper is to consider the Hardy operator on positive increasing functions and to characterize the weight functions w and ν for which (1.2) holds. Theorem 2.1 contains these results for $1 < p \leq q < \infty$ and Theorem 2.2 for the index range $1 < q < p < \infty$. If $0 < q < 1 < p < \infty$, sufficient conditions are given for which (1.2) holds. In addition we characterize the weights in Proposition 2.1 for which the identity operator is bounded between weighted Lebesgue spaces for all positive indices. The proofs given here are akin to the corresponding results for the Hardy operator on decreasing functions [15] although most of the details are quite different.

We now introduce notation and conventions used in the sequel. $F \approx G$ means that F/G is bounded above and below by positive constants and $F \ll G$ means that F is dominated by G, *i.e.* $F \leq CG$. C denote constants which may be different at different places although in some instances we write $C_1, C_2, \ldots,$ to indicate different constants. The conjugate index p' of p is defined by $1/p + 1/p' = 1$, even if $0 < p \leq 1$. Similarly for other letters. A function f is said to be *increasing* (*decreasing*) if it is non-decreasing (non-increasing) and we write $f\uparrow$ ($f\downarrow$). We also adhere to the convention that expressions of the form $0.\infty$ are zero and inequalities (such as (1.2)) are interpreted in the sense that if the right side is finite, so is the left and the inequality holds. Finally χ_E denotes the characteristic function of the set E.

We wish to thank Professor K. F. Andersen for the correspondence we had regarding this paper. In particular his observations led to an easier and shorter proof of Proposition 2.1.

2. **Results.**

We shall use the following notation throughout:

$$W(t) = \int_t^\infty w, \quad V(t) = \int_t^\infty \nu, \quad t > 0.$$

Our first result characterizes the weight functions w and ν for which the identity operator is bounded in weighted L^p-spaces.

PROPOSITION 2.1. *(i) If* $0 < p \leq q < \infty$, *then*

$$\Big[\int_0^\infty f(x)^q w(x)\,dx\Big]^{1/q} \leq C\Big[\int_0^\infty f(x)^p \nu(x)\,dx\Big]^{1/p} \tag{2.1}$$

holds for all $0 \leq f\uparrow$, *if and only if* $A_0 = \sup_{t>0} W(t)^{1/q}V(t)^{-1/p} < \infty$. *Moreover, if C is the least constant for which (2.1) holds, then* $C = A_0$.

(ii) If $0 < q < p < \infty$, *then (2.1) holds for all* $0 \leq f\uparrow$, *if and only if*

$$A_1 = \Big[\int_0^\infty W^{r/p}V^{-r/p}w\Big]^{1/r} < \infty,$$

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where $1/r = 1/q - 1/p$. *Moreover, if* C *is the least constant for which (2.1) holds, then* $C \approx A_1$.

PROOF. The change of variable $x \longrightarrow 1/x$ shows that (2.1) is equivalent to

$$\left[\int_0^\infty g(x)^q \tilde{w}(x)\,dx\right]^{1/q} \le C\left[\int_0^\infty g(x)^p \tilde{v}(x)\,dx\right]^{1/p}, \tag{2.2}$$

where $\tilde{w}(x) = x^{-2}w(1/x)$, $\tilde{v}(x) = x^{-2}v(1/x)$ and $g(x) = f(1/x)$. But $f\uparrow$ if and only if $g\downarrow$, so the weight characterizations for which (2.2) holds follow from the known results of Sawyer [11, p. 148, Remark(i)] in the cases $1 < p, q < \infty$ and in the remaining cases from [15, Proposition 1] of Stepanov. Specifically, (2.2) holds for $1 < p \le q < \infty$, if and only if

$$\sup_{t>0}\left[\int_0^t \tilde{w}(x)\,dx\right]^{1/q}\left[\int_0^t \tilde{v}(x)\,dx\right]^{-1/p} < \infty,$$

which is equivalent to $A_0 < \infty$ as the change of variable $x \longrightarrow 1/x$ shows. An examination of the proof of [15, Proposition 1] also shows that if C is the least constant for which (2.1) holds then $C = A_0$. This proves (i).

If $0 < q < p < \infty$, then (2.2) ([11], [15]), holds if and only if

$$\left\{\int_0^\infty\left[\left[\int_0^x \tilde{w}(t)\,dt\right]^{1/p}\left[\int_0^x \tilde{v}(t)\,dt\right]^{-1/p}\right]^r \tilde{w}(x)\,dx\right\}^{1/r} < \infty.$$

But the changes $t \longrightarrow 1/t$ and $x \longrightarrow 1/x$ show that this is equivalent to $A_1 < \infty$. This proves the result.

If $0 \le f\uparrow$, then $(Pf)(x) \le f(x)$, so the conditions of Proposition 2.1 are always sufficient but not necessary for (1.2). In the next theorem we characterize the weights for which (1.2) holds in the index range $1 < p \le q < \infty$.

We shall use the notation

$$B_0(t) = \left[\int_t^\infty (x-t)^q x^{-q} w(x)\,dx\right]^{1/q} V(t)^{-1/p}$$

and

$$B_1(t) = \left[\int_t^\infty x^{-q} w(x)\,dx\right]^{1/q}\left[\int_0^t (t-x)^{p'} V(x)^{-p'} v(x)\,dx\right]^{1/p'}, \quad t > 0.$$

THEOREM 2.1. *Let* $1 < p \le q < \infty$, *then*

$$\left[\int_0^\infty (Pf)(x)^q w(x)\,dx\right]^{1/q} \le C\left[\int_0^\infty f(x)^p v(x)\,dx\right]^{1/p} \tag{2.3}$$

holds for $0 \le f\uparrow$, *if and only if* $B = \max(B_0, B_1) < \infty$, *where* $B_i = \sup_{t>0} B_i(t)$, $i = 0, 1$.

PROOF: SUFFICIENCY. Assume first that $V(0) = \infty$, then

$$B_0(t) = (p'-1)^{1/p'}\left[\int_t^\infty (x-t)^q x^{-q} w(x)\,dx\right]^{1/q}\left[\int_0^t V(x)^{-p'} v(x)\,dx\right]^{1/p'}.$$

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If the right side of (2.3) is finite, then $f(0) = 0$ implies that $f(x) = \int_0^x h$, where without loss of generality we assume h has compact support in $(0, \infty)$. Writing $H(y) = h(y)V(y)$ and $G(y) = \int_y^\infty H$, we obtain

(2.4)

$$\begin{aligned}
(Pf)(x) &= 1/x \int_0^x \Big[\int_0^y h\Big]\, dy = 1/x \int_0^x (x-y)h(y)\, dy = 1/x \int_0^x (x-y)H(y)V(y)^{-1}\, dy \\
&= 1/x \int_0^x (x-y)V(y)^{-1}\, d\big(-G(y)\big) \\
&= \frac{-(x-y)}{x} V(y)^{-1} G(y)\Big|_0^x - 1/x \int_0^x V(y)^{-1}G(y)\, dy + 1/x \int_0^x \frac{(x-y)\nu(y)}{V(y)^2} G(y)\, dy \\
&\leq V(0)^{-1}G(0) + 1/x \int_0^x (x-y)\,\Phi(y)\, dy,
\end{aligned}$$

where $\Phi(y) = G(y)\nu(y)/V(y)^2$. But since $V(0) = \infty$, the first term in (2.4) is zero. Therefore, applying [13, Theorem 1] we obtain

$$\Big[\int_0^\infty (Pf)(x)^q w(x)\, dx\Big]^{1/q} \leq C\Big[\int_0^\infty \Phi(y)^p u(y)^p\, dy\Big]^{1/p},$$

provided

$$\sup_{x>0} \Big[\int_x^\infty (t-x)^q t^{-q} w(t)\, dt\Big]^{1/q} \Big[\int_0^x u(t)^{p(1-p')}\, dt\Big]^{1/p'} < \infty$$

and

$$\sup_{x>0} \Big[\int_x^\infty t^{-q} w(t)\, dt\Big]^{1/q} \Big[\int_0^x (x-t)^{p'} u(t)^{p(1-p')}\, dt\Big] < \infty.$$

Taking $u^p = V^p \nu^{-p/p'}$, these two suprema are B_0 and B_1, respectively, and are finite by hypotheses. But now

$$\begin{aligned}
\int_0^\infty \Phi(y)^p u(y)^p\, dy &= \int_0^\infty \Phi(y)^p V(y)^p \nu(y)^{-p/p'}\, dy \\
&= \int_0^\infty G^p V^{-p} \nu^{p-p/p'} = \int_0^\infty \nu(y)V(y)^{-p}\Big[\int_y^\infty H\Big]^p dy \\
&= \int_0^\infty \nu(y)V(y)^{-p}\Big[\int_y^\infty h(s) \int_s^\infty \nu(t)\, dt\, ds\Big]^p dy \\
&= \int_0^\infty \nu(y)V(y)^{-p}\Big[\int_y^\infty \nu(t)\Big[\int_y^t h(s)\, ds\Big]\, dt\Big]^p dy \\
&\leq \int_0^\infty \nu(y)V(y)^{-p}\Big[\int_y^\infty \nu(t) f(t)\, dt\Big]^p dy,
\end{aligned}$$

and applying the dual form of the classical weighted Hardy inequality [8] we obtain

$$\int_0^\infty \Phi(y)^p u(y)^p\, dy \leq C \int_0^\infty f(t)^p \nu(t)\, dt$$

provided

$$\sup_{t>0} \Big[\int_0^t \nu V^{-p}\Big]^{1/p} \Big[\int_t^\infty \nu^{(1-p)(1-p')}\Big]^{1/p'} < \infty.$$

But since $(1-p)(1-p') = 1$ an integration shows the supremum is not larger than $(p-1)^{-1/p}$ for $V(0) \leq \infty$. Hence sufficiency follows in this case.

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If $V(0) < \infty$, then we estimate the first term of (2.4) by interchanging the order of integration and Hölder's inequality to obtain

$$V(0)^{-1}G(0) \leq V(0)^{-1}\int_0^\infty h(y)\int_y^\infty \nu(t)\,dt\,dy = V(0)^{-1}\int_0^\infty f(t)\nu(t)\,dt$$
$$\leq V(0)^{-1}\Big[\int_0^\infty f(t)^p\nu(t)\,dt\Big]^{1/p} V(0)^{1/p'}.$$

The result follows now under the assumption $f(0) = 0$. If $f(x) = k \neq 0$, then $V(0) < \infty$. But $B_0 < \infty$ implies the existence of a subsequence $\{t_j\}$ with $t_j \longrightarrow 0$ as $j \longrightarrow \infty$ such that $B_0 \geq \lim_{j\to\infty} B_0(t_j) = V(0)^{-1/p}W(0)^{1/q}$. Therefore,

$$\Big[\int_0^\infty (Pf)(x)^q w(x)\,dx\Big]^{1/q} = k\Big[\int_0^\infty w(x)\,dx\Big]^{1/q} \leq B_0 k V(0)^{1/p}$$
$$= B_0\Big[\int_0^\infty k^p\nu(x)\,dx\Big]^{1/p} = B_0\Big[\int_0^\infty f^p\nu\Big]^{1/p}.$$

Finally, if $f(0) \neq 0$, then $f(x) = k + \int_0^x h \equiv k + g(x)$, where $g \uparrow$ and $g(0) = 0$. Since now $V(0) < \infty$, Minkowski's inequality and the previous argument shows that

$$\Big[\int_0^\infty Pf(x)^q w(x)\,dx\Big]^{1/q} = \Big[\int_0^\infty \big[k + (Pg)(x)\big]^q w(x)\,dx\Big]^{1/q}$$
$$\leq kW(0)^{1/q} + \Big[\int_0^\infty (Pg)(x)^q w(x)\,dx\Big]^{1/q}$$
$$\leq kB_0V(0)^{1/p} + C\Big[\int_0^\infty g(x)^p\nu(x)\,dx\Big]^{1/p}$$
$$= B_0\Big[\int_0^\infty k^p\nu(x)\,dx\Big]^{1/p} + C\Big[\int_0^\infty g(x)^p v(x)\,dx\Big]^{1/p}$$
$$\leq (B_0 + C)\Big[\int_0^\infty f(x)^p v(x)\,dx\Big]^{1/p},$$

which proves the sufficiency part of the result.

To prove necessity, substitute $f = f_t$, $t > 0$ fixed where

$$f_t(x) = \Big[\int_0^x (t-s)^{p'} V(s)^{-p'-1}\nu(s)\chi_{[0,t]}(s)\,ds\Big]^{1/p}$$

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in (2.3). Then under the assumption $V(0)=\infty$,

(2.5)

$$\begin{aligned}
C\Big[\int_0^\infty f_t^p\nu\Big]^{1/p} &= C\Big[\int_0^\infty \nu(x)\int_0^x (t-s)^{p'}V(s)^{-p'-1}\nu(s)\chi_{[0,t]}(s)\,ds\,dx\Big]^{1/p}\\
&= C\Big[\int_0^\infty V(s)^{-p'}\nu(s)(t-s)^{p'}\chi_{[0,t]}(s)\,ds\Big]^{1/p}\\
&= C\Big[\int_0^t (t-s)^{p'}V(s)^{-p'}\nu(s)\,ds\Big]^{1/p}\\
&\geq \Big\{\int_t^\infty x^{-q}w(x)\Big[\int_0^x\Big[\int_0^s (t-y)^{p'}V(y)^{-p'-1}\nu(y)\chi_{[0,t]}(y)\,dy\Big]^{1/p}ds\Big]^q dx\Big\}^{1/q}\\
&\geq \Big[\int_t^\infty x^{-q}w(x)\,dx\Big]^{1/q}\Big[\int_0^t\Big[\int_0^s (t-y)^{p'}V(y)^{-p'-1}\nu(y)\,dy\Big]^{1/p}ds\Big]\\
&\geq \Big[\int_t^\infty x^{-q}w(x)\,dx\Big]^{1/q}\int_0^t (t-s)^{p'/p}\Big[\int_0^s V^{-p'-1}\nu\Big]^{1/p}ds\\
&= (p')^{-1/p}\Big[\int_t^\infty x^{-q}w(x)\,dx\Big]^{1/q}\int_0^t (t-s)^{p'/p}V(s)^{-p'/p}\,ds.
\end{aligned}$$

But on integrating, the last integral on the right is equal to

$$\begin{aligned}
(p'-1)\int_0^t (t-s)^{p'/p}\int_0^s V(\tau)^{-p'}\nu(\tau)\,d\tau\,ds &= (p'-1)\int_0^t V(\tau)^{-p'}\nu(\tau)\int_\tau^t (t-s)^{p'/p}\,ds\,d\tau\\
&= 1/p\int_0^t (t-\tau)^{p'}V(\tau)^{-p'}\nu(\tau)\,d\tau
\end{aligned}$$

and substituting into (2.5) shows that $B_1(t)\ll C$ for $t>0$ and hence $B_1<\infty$.

If $V(0)<\infty$, replace ν by $\nu_\epsilon=\nu+\epsilon x^{-2}$, $\epsilon>0$. Then $V_\epsilon(0)=\infty$ and the result follows with V replaced by V_ϵ. Now applying Fatou's lemma as $\epsilon\longrightarrow 0+$ and we obtain $B_1<\infty$ also in this case.

Finally $B_0<\infty$ follows at once on taking $f(x)=f_0(x)=\chi_{[t,\infty]}(x)$, $t>0$ fixed, in (2.3). This completes the proof.

In our next result we use the notation

$$D_0 = W(0)^{1/q}V(0)^{-1/p}$$

$$D_1 = \Big\{\int_0^\infty\Big[\int_t^\infty (x-t)^q x^{-q}w(x)\,dx\Big]^{r/q}\Big[\int_0^t V^{-p'}\nu\Big]^{r/q'}V(t)^{-p'}\nu(t)\,dt\Big\}^{1/r}$$

$$D_2 = \Big\{\int_0^\infty\Big[\int_t^\infty x^{-q}w(x)\,dx\Big]^{r/p}\Big[\int_0^t (t-x)^{p'}V(x)^{-p'}\nu(x)\,dx\Big]^{r/p'}t^{-q}w(t)\,dt\Big\}^{1/r},$$

where $1/r=1/q-1/p$.

THEOREM 2.2. *The inequality (2.3) holds for all* $0\leq f\uparrow$ *in the index range*

(i) $1<q<p<\infty$, *if and only if* $\max(D_0,D_1,D_2)<\infty$.

(ii) $0<q<1<p<\infty$, *if* $\max(D_0,D_1)<\infty$,

PROOF. The sufficiency part of this result is very similar to that of Theorem 2.1. Now, however, we apply [14, Theorem 3 and Theorem 2] for $0<q<1<p<\infty$, respectively, $1<q<p<\infty$, instead of [13, Theorem 1]. We omit the details.

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NECESSITY. (i): Observe that if (2.3) holds with weights w and ν, then it holds also for smaller w and larger ν. For example let $w_\epsilon(x) = \min\big(1/\epsilon, w(x)\big)\chi_{[\epsilon,1/\epsilon]}(x)$ and $\nu_\epsilon(x) = \nu(x) + \epsilon/(1+x^2)$, $0 < \epsilon < 1$, then

$$W_q^\epsilon(y) = \int_y^\infty (x-y)^{q-1}x^{-q}w_\epsilon(x)\,dx$$

is bounded and $W_q^\epsilon(y) = 0$ if $1/\epsilon \le y$. Therefore $D_1 = D_{1,\epsilon} < \infty$. Now if we show that $D_1 = D_{1,\epsilon}$ is uniformly bounded, by $C(p,q)$ say, then the restriction on w can be removed by Fatou's lemma. In fact, Fatou's lemma shows that $W_q(y) \le \lim_{\epsilon\to 0+}\inf W_q^\epsilon(y)$ and applying Fatou's lemma again yields

$$\begin{aligned}\int_0^\infty W_q(y)^{r/q}\Big[\int_0^y V^{-p'}\nu\Big]^{r/q'} V(y)^{-p'}\nu(y)\,dy \\ &\le \int_0^\infty \lim_{\epsilon\to 0+}\inf[W_q^\epsilon(y)]^{r/q}\Big[\int_0^y V^{-p'}\nu\Big]^{r/q'} V(y)^{-p'}\nu(y)\,dy \\ &\le \lim_{\epsilon\to 0+}\inf D_{1,\epsilon}^r \\ &\le C(p,q)\end{aligned}$$

and the result follows.

We therefore assume first that w is bounded and compactly supported and ν is positive in $(0,\infty)$, for then $D_1 < \infty$. We then show that if $1 < q < p < \infty$ and (2.3) holds, then D_1 is bounded by a constant depending only on C of (2.3), p and q. Hence D_1 is uniformly bounded and the compactness condition imposed on w can then be removed by Fatou's lemma.

Now let $U^{-p'} = V^{-p'}\nu$, then under the assumption $V(0) = \infty$,

$$\begin{aligned}D_1^r &= \int_0^\infty \Big[\int_t^\infty (x-t)^q x^{-q}w(x)\,dx\Big]^{r/q}\Big[\int_0^t U^{-p'}\Big]^{r/q'} U(t)^{-p'}V(t)^{-1}V(t)\,dt \\ &= \int_0^\infty \nu(s)\Big[\int_0^s\Big[\int_t^\infty (x-t)^q x^{-q}w(x)\,dx\Big]^{r/q}\Big[\int_0^t U^{-p'}\Big]^{r/q'} U(t)^{-p'}V(t)^{-1}\,dt\Big]\,ds \\ &\equiv \int_0^\infty \nu(s)f_0(s)^p\,ds\end{aligned}$$

where f_0^p is the bracketed term in the integrand. Now by (2.3) with $f = f_0$

$$\begin{aligned}CD_1^{r/p} &\ge \Big[\int_0^\infty w(y)y^{-q}\Big[\int_0^y f_0\Big]^q dy\Big]^{1/q} = \Big[\int_0^\infty w(y)y^{-q}\Big[\int_0^y f_0\Big]\Big[\int_0^y f_0\Big]^{q-1} dy\Big]^{1/q} \\ &= \Big[\int_0^\infty f_0(s)\int_s^\infty\Big[\int_0^y f_0\Big]^{q-1} w(y)\,y^{-q}\,dy\,ds\Big]^{1/q} \equiv \Big[\int_0^\infty f_0(s)J_0(s)\,ds\Big]^{1/q},\end{aligned}$$

where

$$J_0(s) = \int_s^\infty\Bigg[\int_0^y\Big[\int_0^\tau\Big[\int_t^\infty (x-t)^q x^{-q}w(x)\,dx\Big]^{r/q} \Big[\int_0^t U^{-p'}\Big]^{r/q'} U(t)^{-p'}V(t)^{-1}\,dt\Big]^{1/p} d\tau\Bigg]^{q-1} y^{-q}w(y)\,dy.$$

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Performing the inner integration first shows that

$$\int_0^s \Big[\int_0^t U^{-p'}\Big]^{r/q'} U(t)^{-p'} V(t)^{-1}\, dt = C_1 V(s)^{-r/q},$$

and since $0 < s < y < \infty,\ 0 < t < \tau < y < \infty$ we obtain

$$\begin{aligned}
J_0(s) \geq \int_s^\infty \Bigg[\int_s^y \Big[\int_0^\tau \Big[\int_t^\infty (x-t)^q x^{-q} w(x)\, dx \Big]^{r/q} \\
\Big[\int_0^t U^{-p'}\Big]^{r/q'} U(t)^{-p'} V(t)^{-1}\, dt \Big]^{1/p} d\tau \Bigg]^{q-1} y^{-q} w(y)\, dy \\
\geq \int_s^\infty \Bigg[\int_s^y \Big[\int_0^s \Big[\int_t^\infty (x-t)^q x^{-q} w(x)\, dx \Big]^{r/q} \\
\Big[\int_0^t U^{-p'}\Big]^{r/q'} U(t)^{-p'} V(t)^{-1}\, dt \Big]^{1/p} d\tau \Bigg]^{q-1} y^{-q} w(y)\, dy \\
\geq \Big[\int_s^\infty (x-s)^q x^{-q} w(x)\, dx \Big]^{r/(pq')} \Big[\int_s^\infty \Big[\int_s^y d\tau \Big]^{q-1} y^{-q} w(y)\, dy \Big] \\
\Big[\int_0^s \Big[\int_0^t U^{-p'}\Big]^{r/q'} U(t)^{-p'} V(t)^{-1}\, dt \Big]^{(q-1)/p} \\
= C_1^{(q-1)/p} \Big[\int_s^\infty (x-s)^q x^{-q} w(x)\, dx \Big]^{r/(pq')} \\
\Big[\int_s^\infty (y-s)^{q-1} y^{-q} w(y)\, dy \Big] V(s)^{-r/(pq')}.
\end{aligned}$$

Also from the definition of f_0 we have

$$\begin{aligned}
f_0(s) &\geq \Big[\int_s^\infty (x-s)^q x^{-q} w(x)\, dx \Big]^{r/(pq)} \Big[\int_0^s \Big[\int_0^t U^{-p'} \Big]^{r/q'} U(t)^{-p'} V(t)^{-1}\, dt \Big]^{1/p} \\
&= C_1^{1/p} \Big[\int_s^\infty (x-s)^q x^{-q} w(x)\, dx \Big]^{r/(pq)} V(s)^{-r/(pq)}
\end{aligned}$$

and so

(2.6)

$$CD_1^{r/p} \geq C_1^{1/p} \Big[\int_0^\infty \Big[\int_s^\infty (x-s)^q x^{-q} w(x)\, dx \Big]^{r/p} \Big[\int_s^\infty (y-s)^{q-1} y^q w(y)\, dy \Big] V(s)^{-r/p}\, ds \Big]^{1/q}.$$

Now an integration by parts also shows that

$$\begin{aligned}
D_1^r &= p'/r \int_0^\infty \Big[\int_t^\infty (x-t)^q x^{-q} w(x)\, dx \Big]^{r/q} d\Big[\int_0^t U^{-p'} \Big]^{r/p'} \\
&= p'/r \Big\{ \Big[\int_0^t U^{-p'} \Big]^{r/p'} \Big[\int_t^\infty (x-t)^q x^{-q} w(x)\, dx \Big]^{r/q} \Big|_{t=0}^{t=\infty} \\
&\quad + r \int_0^\infty \Big[\int_t^\infty (x-t)^q x^{-q} w(x)\, dx \Big]^{r/p} \Big[\int_t^\infty (x-t)^{q-1} x^{-q} w(x)\, dx \Big] \Big[\int_0^t U^{-p'} \Big]^{r/p'} dt \Big\}.
\end{aligned}$$

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But since $D_1^r < \infty$ it follows that the integrated term vanishes and so

$$D_1^r = C_2 \int_0^\infty \Big[\int_t^\infty (x-t)^q x^{-q} w(x)\,dx\Big]^{r/p} \Big[\int_t^\infty (x-t)^{q-1} x^{-q} w(x)\,dx\Big] \Big[\int_0^t U^{p'}\Big]^{r/p'} dt$$
$$= C_3 \int_0^\infty \Big[\int_t^\infty (x-t)^q x^{-q} w(x)\,dx\Big]^{r/p} \Big[\int_t^\infty (x-t)^{q-1} x^{-q} w(x)\,dx\Big] V(t)^{-r/p}\,dt,$$

where we used the fact that $\int_0^t U^{-p'} = V(t)^{1-p'}/(p'-1)$. Hence by (2.6) $CD_1^{r/p} \geq C_4 D_1^{r/q}$ and so $C \geq C_4 D_1$. Therefore D_1 is uniformly bounded and the general case follows from limiting arguments. If $V(0) < \infty$ we replace ν by $\nu_\epsilon(x) = \nu(x) + \epsilon x^{-2}$, $\epsilon > 0$ and use Fatou's lemma.

Next, we show that $D_2 < \infty$. Again assume first that w is compactly supported in $(0,\infty)$ and that $V(0) = \infty$. Applying Hölder's inequality we find from the inequality (2.3) for arbitrary $h \geq 0$ that

$$\int_0^\infty \Big[\int_0^x f\Big] h(x)\,dx \leq C\Big[\int_0^\infty f^p \nu\Big]^{1/p} \Big[\int_0^\infty h(x)^{q'} x^{q'} w(x)^{-q'/q}\,dx\Big]^{1/q'}. \tag{2.7}$$

Setting

$$h_0(x) = \Big[\int_0^x (x-y)^{p'} V(y)^{p'} \nu(y)\,dy\Big]^{r/(q'p')} \Big[\int_x^\infty y^{-q} w(y)\,dy\Big]^{r/(q'p)} x^{-q} w(x),$$

we see that

$$D_2^{r/q'} = \Big[\int_0^\infty h_0(x)^{q'} x^{q'} w(x)^{-q'/q}\,dx\Big]^{1/q'}.$$

Now let $f(x) = \int_0^x g$, where $g \geq 0$ has compact support, and applying Muckenhoupt's criterion for the Hardy's inequality [8] we obtain

$$\Big[\int_0^\infty g(y)^p V(y)^p \nu(y)^{-p/p'}\,dy\Big]^{1/p} \gg \Big[\int_0^\infty \Big[\int_0^x g\Big]^p \nu\Big]^{1/p}. \tag{2.8}$$

From (2.7) and (2.8) we get

$$\begin{aligned} CD_2^{r/q'} \Big[\int_0^\infty g(y)^p V(y)^p \nu(y)^{-p/p'}\,dy\Big]^{1/p} &\gg CD_2^{r/q'} \Big[\int_0^\infty \Big[\int_0^x g\Big]^p \nu\Big]^{1/p} \\ &= CD_2^{r/q'} \Big[\int_0^\infty f^p \nu\Big]^{1/p} \\ &= C\Big[\int_0^\infty h_0(x)^{q'} x^{q'} w(x)^{-q'/q}\,dx\Big]^{1/q'} \Big[\int_0^\infty f^p \nu\Big]^{1/p} \\ &\geq \int_0^\infty \Big[\int_0^x f\Big] h_0(x)\,dx \\ &= \int_0^\infty f(y) \Big[\int_y^\infty h_0\Big]\,dy \\ &= \int_0^\infty g(x) \Big[\int_x^\infty (y-x) h_0(y)\,dy\Big]\,dx. \end{aligned}$$

Now taking the supremum over all g with $\|g\|_{p,Q} \leq 1$, where $Q = V^p \nu^{-p/p'}$, we obtain by duality (*i.e.* the inverse Hölder inequality)

$$CD_2^{r/q'} \gg \Big[\int_0^\infty \Big[\int_x^\infty (y-x) h_0(y)\,dy\Big]^{p'} V(x)^{-p'} \nu(x)\,dx\Big]^{1/p'}.$$

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Repeating the steps of obtaining the lower bound for $CD_1^{r/p}$ above, we see that the right side of this is

$$\Big[\int_0^\infty V(x)^{-p'}\nu(x)\int_x^\infty (y-x)h_0(y)\,dy\Big[\int_x^\infty (y-x)h_0(y)\,dy\Big]^{p'-1}dx\Big]^{1/p'}$$
$$\geq \Big[\int_0^\infty h_0(y)\int_0^y (y-x)V(x)^{-p'}\nu(x)\Big[\int_x^\infty (t-x)h_0(t)\,dt\Big]^{p'-1}dx\,dy\Big]^{1/p'}$$
$$\geq \Big[\int_0^\infty h_0(y)\int_0^y (y-x)V(x)^{-p'}\nu(x)\Big[\int_y^\infty (t-x)h_0(t)\,dt\Big]^{p'-1}dx\,dy\Big]^{1/p'}$$
$$\geq \Big[\int_0^\infty h_0(y)\int_0^y (y-x)^{p'}V(x)^{-p'}\nu(x)\Big[\int_y^\infty h_0(t)\,dt\Big]^{p'-1}dx\,dy\Big]^{1/p'}$$
$$\geq \Bigg[\int_0^\infty\Big[\int_0^y (y-x)^{p'}V(x)^{-p'}\nu(x)\,dx\Big]^{r/(q'p')+1}\Big[\int_y^\infty s^{-q}w(s)\,ds\Big]^{r/(q'p)}y^{-q}w(y)$$
$$\Big[\int_y^\infty\Big[\int_0^t (t-z)^{p'}V(z)^{-p'}\nu(z)\,dz\Big]^{r/(q'p')}\Big[\int_t^\infty z^{-q}w(z)\,dz\Big]^{r/(q'p)}t^{-q}w(t)\,dt\Big]^{p'-1}dy\Bigg]^{1/p'}$$
$$\geq \Bigg[\int_0^\infty\Big[\int_0^y (y-x)^{p'}V(x)^{-p'}\nu(x)\,dx\Big]^{r/q'+1}\Big[\int_y^\infty s^{-q}w(s)\,ds\Big]^{r/(q'p)}y^{-q}w(y)$$
$$\Big[\int_y^\infty\Big[\int_t^\infty z^{-q}w(z)\,dz\Big]^{r/(q'p)}t^{-q}w(t)\,dt\Big]^{p'-1}dy\Bigg]^{1/p'}$$
$$\approx \Big[\int_0^\infty\Big[\int_0^y (y-x)^{p'}V(x)^{-p'}\nu(x)\,dx\Big]^{r/p'}\Big[\int_y^\infty s^{-q}w(s)\,ds\Big]^{r/p}y^{-q}w(y)\,dy\Big]^{1/p'}$$
$$= D_2^{r/p'}.$$

Note that we applied the definition of h_0 after the third inequality above. Hence $D_2 \ll C$, so that D_2 is uniformly bounded. The compactness restriction imposed on w can now be removed via limiting arguments in the usual way. If $V(0) < \infty$ the argument is as before so we omit the details. That $D_0 < \infty$ follows at once from (2.3) with $f \equiv 1$. This proves the theorem.

REMARKS. (i) It is easily seen that $A_0 < \infty$ implies $D_0 < \infty$, however the converse fails in general. For if $\nu(x) = e^{-x}$, $w(x) = xe^{-x}$, $x > 0$, then $V(x) = e^{-x}$ and $W(x) = (x+1)e^{-x}$ so that $D_0 < \infty$. However, if $p \leq q$ then

$$A_0 = \sup_{x>0} W(x)^{1/q}V(x)^{-1/p} = \sup_{x>0}(x+1)^{1/q}e^{x(1/p-1/q)} = \infty.$$

(ii) We also note that if $V(0) < \infty$ then $D_1 < \infty$ does not in general imply $D_0 < \infty$. For instance, let $0 < q < p < \infty$, $p > 1$ and $w(x)$ and $\nu(x)$ be given by

$$w(x) = \begin{cases} 1/x & \text{if } 0 < x < 1 \\ 0 & \text{if } x \geq 1 \end{cases}; \quad \nu(x) = \begin{cases} 1 & \text{if } 0 \leq x \leq 1 \\ 0 & \text{if } x > 1 \end{cases}$$

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Then $D_0 = \infty$ and

$$D_1 = \Big\{\int_0^1 \Big[\int_t^1 (x-t)^q x^{-q-1}\,dx\Big]^{r/q}\Big[\int_0^t (1-x)^{-p'}\,dx\Big]^{r/q'}(1-t)^{-p'}\,dt\Big\}^{1/r}.$$

We have for $0 < t < 1$

$$\int_0^t (1-x)^{-p'}\,dx = (p'-1)^{-1}[(1-t)^{1-p'}-1] \text{ and } \int_t^1 (x-t)^q x^{-q-1}\,dx \le -\ln t,$$

so that

$$\begin{aligned} D_1^r &\ll \int_0^1 \big(-\ln(1-u)\big)^{r/q} u^{(1-p')r/q'-p'}(1-u^{p'-1})^{r/q'}\,du \\ &= \int_0^1 \Big(\frac{\sum_{k=1}^{\infty}\frac{u^k}{k}}{u}\Big)^{r/q}(1-u^{p'-1})^{r/q'}\,du. \end{aligned} \tag{2.9}$$

If $q \ge 1$ then $q' > 1$ so that $(1-u^{p'-1})^{r/q'} \le 1$. Hence by Minkowski's inequality

$$\begin{aligned} D_1^r &\ll \int_0^1 \Big[\sum_{k=0}^{\infty}\frac{u^k}{k+1}\Big]^{r/q} du \le \Big\{\sum_{k=0}^{\infty}\Big[\int_0^1 \frac{u^{kr/q}}{(k+1)^{r/q}}\,du\Big]^{q/r}\Big\}^{r/q} \\ &= \Big\{\sum_{k=0}^{\infty}\frac{1}{(k+1)(kr/q+1)^{q/r}}\Big\}^{r/q} < \infty. \end{aligned}$$

If $q < 1$ then by (2.9) and Hölder's inequality

$$D_1^r \le \Big[\int_0^1 \Big[\sum_{k=0}^{\infty}\frac{u^k}{k+1}\Big]^{sr/q} du\Big]^{1/s}\Big[\int_0^1 (1-u^{p'-1})^{s'r/q'}\Big]^{1/s'}$$

where $s > 1$. The first integral is again finite by the same argument as before and for the second, let $1-u^{p'-1} = y$, then the integral is equivalent to

$$\{\int_0^1 y^{s'r/q'}(1-y)^{-1+1/(p'-1)}\,dy\}^{1/s'}.$$

But since $r/q' > -1$, we can choose s' sufficiently close to 1 so that $s'r/q' > -1$. Hence the integral is finite and consequently $D_1 < \infty$.

(iii) It is clear that $D_0 < \infty$ is necessary for (2.3) in Theorem 2.2(ii). However $D_1 \ll C$ is in general invalid in the case $0 < q < 1 < p < \infty$. For if in (2.3) $w(x) = \delta_1(x)$, the Dirac delta function at $x = 1$, and $\nu(x) = 0$ for $x \ge 1$ the inequality takes the form

$$\int_0^1 f \le C\Big[\int_0^1 f^p \nu\Big]^{1/p}.$$

But by Proposition 2.1(ii) with $q = 1$ and $w = \chi_{(0,1)}$ it follows that

$$\begin{aligned} C &\approx \Big[\int_0^1 \Big[\int_x^1 dt\Big]^{1/(p-1)}\Big[\int_x^1 \nu(t)\,dt\Big]^{-1/(p-1)} dx\Big]^{1/p'} \\ &= \Big[\int_0^1 (1-x)^{1/(p-1)}\Big[\int_x^1 \nu(t)\,dt\Big]^{-1/(p-1)} dx\Big]^{1/p'}. \end{aligned}$$

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On the other hand, with $w = \delta_1$ and $V(0) = \infty$,

$$D_1 \approx \Big\{\int_0^1 (1-x)^r \Big[\int_x^1 \nu(t)\,dt\Big]^{-r/q} \nu(x)\,dx\Big\}^{1/r}$$

and integrating by parts we find that

$$D_1 \approx \Big[\int_0^1 (1-x)^r\, d\Big[\Big[\int_x^1 \nu(t)\,dt\Big]^{-r/p}\Big]\Big]^{1/r} \gg \Big[\int_0^1 (1-x)^{r-1}\Big[\int_x^1 \nu(t)\,dt\Big]^{-r/p} dx\Big]^{1/r}.$$

But this shows that $C \gg D_1$ can fail because it implies

$$\Big[\int_0^1 (1-x)^{r-1}\Big[\int_x^1 \nu(t)\,dt\Big]^{-r/p} dx\Big]^{1/r} \ll \Big[\int_0^1 (1-x)^{1/(p-1)}\Big[\int_x^1 \nu(t)\,dt\Big]^{-1/(p-1)} dx\Big]^{1/p'}$$

and if we take

$$\int_x^1 \nu(t)\,dt = \begin{cases} (1-x)^p|\log(1-x)|^{p/r}, & 1/2 < x < 1,\\ (1/4x)^p|\log(1/2)|^{p/r}, & 0 < x \le 1/2,\end{cases}$$

then the right side of the inequality is finite, while the left side is not.

(iv) Necessary conditions can be derived in the case $0 < q < 1 < p < \infty$ as follows: By (2.3) and Minkowski's inequality we obtain

$$C\Big[\int_0^\infty f^p \nu\Big]^{1/p} \ge \Big[\int_0^\infty w(x)\Big[\frac{1}{x}\int_0^x f\Big]^q dx\Big]^{1/q} \ge \int_0^\infty f(t)\Big[\int_t^\infty x^{-q} w(x)\,dx\Big]^{1/q} dt.$$

Now apply Proposition 2.1(ii) with $q = 1$ and w replaced by

$$\tilde{w}(t) = \Big[\int_t^\infty x^{-q} w(x)\,dx\Big]^{1/q}$$

to obtain

$$C \approx \Big[\int_0^\infty \tilde{W}^{r/p} V^{-r/p} \tilde{w}\Big]^{1/r}$$

where $\tilde{W}(x) = \int_x^\infty \tilde{w}$ and $r = p'$.

Finally we note that S. Lai [5, Theorem 2.4] proved that for $0 < p \le q \le 1$, (2.3) holds for all $0 \le f \uparrow$ and only if

$$\sup_{r>0}\Big[\int_r^\infty (x-r)^q x^{-q} w(x)\,dx\Big]^{1/q}\Big[\int_r^\infty \nu(x)\,dx\Big]^{-1/p} < \infty.$$

This is of course one of the conditions—namely $B_0 < \infty$ of Theorem 2.1.

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