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SUMMING A COMMON TYPE OF SLOWLY CONVERGENT SERIES OF POSITIVE TERMS

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Abstract

If the terms of a series behave like n^{-k} where k is an exactly known constant, a formula using two terms transforms the series into a series of terms like n^{-k-2} provided $k \neq 1$. The multiple use of this transformation is demonstrated in summing three series.

1. Introduction

When the nth term of a series is a quotient of powers or polynomials or γ-functions in n and the ratio of successive terms is close to 1 the series is difficult to sum. The three main reasons for this are that a lot of terms of the series are needed, the individual terms are much smaller than the sum, and rounding off errors in a numerical calculation can be troublesome.

Two methods for transforming such series are Lubkin's [1] transformation and Wynn's [2] ρ-algorithm which may be used to repeatedly transform the series or its corresponding sequence into more rapidly convergent forms.

The following method bears a close resemblance to an improvement on Wynn's ρ-algorithm and is easier to use.

2. Derivation of the formula

Let u_n be the nth term of a series.

Let Z_n be the sum to infinity of the series starting with u_n, thus $Z_n = u_n + u_{n+1} + u_{n+2} + \cdots$.

We first take a special series for which

$$Z_n = \frac{A}{(n-k)(n-k+1)\cdots(n-2)}.$$

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Hence

$$u_n = \frac{A(k-1)}{(n-k)(n-k+1)\cdots(n-1)}$$

and

$$Z_n = \frac{k}{k-1}\,\frac{u_{n-1}u_n}{u_{n-1}-u_n} = u_n + \frac{k}{k-1}\,\frac{u_n u_{n+1}}{u_n - u_{n+1}}. \tag{1}$$

The particular series given above is summed exactly from a knowledge of k and the first two terms of the series. However if we have some other series for which An^{-k} is the dominant part of the nth term we may still transform the series as follows:

Let $Z_1 = u_1 + u_2 + u_3 + \cdots$ and take $u_0 = 0$ and define

$$T_n = \frac{k}{k-1}\,\frac{u_{n-1}u_n}{u_{n-1}-u_n} \quad \text{so} \quad T_1 = 0 \tag{2}$$

then by simple rearrangements we get

$$Z_1 = u_1^* + u_2^* + u_3^* + \cdots$$

where

$$u_n^* = u_n - T_n + T_{n+1} \tag{3}$$

provided that $k \neq 1$ and $\lim_{n\to\infty} T_n = 0$.

Also if $u_n \sim An^{-k}$ we substitute in equations (2) and (3) to get

$$u_n^* \sim -[A(k+1)/12](n-1)^{-k-2}$$

for $n > 1$ and $k \neq 1$.

The transformed series may be transformed a second time, but now the series behaves like Σn^{-k-2} so we change k to $(k+2)$ in equation (2) and continue to increase k by 2 for each successive transformation.

Several comments on the transformation formula are:

(i) Equations (2) and (3) can be written into a calculator programme which stores k and calculates u_n^* when u_{n+1} is inserted.

(ii) Rounding off errors are magnified because of the factor $(u_{n-1} - u_n)$ in the denominator of T_n so where possible it is advisable to do at least the first transformation of the series exactly and the early transformations as accurately as possible.

(iii) k may be estimated using three terms of the series. Thus when the equations (1) are approximately true we eliminate Z_n from them to obtain an approximator

TABLE 1

Summing a series for $10^9\pi$ knowing that the terms behave like $n^{-3/2}$

u_n	$3A_n$	u_n^*	$7A_n^*/5$	u_n^{**}	$11A_n^{**}/9$	u_n^{***}	$15A_n^{***}/13$
2000,000,000*							
333,333,333	1200,000,000*						
150,000,000	818,181,818	−48,484,848*					
89,285,714	661,764,706	−6,417,112	−10,354,391*				
60,763,889	570,652,174	−1,826,818	−3,575,381	361,898*			
44,744,318	509,159,483	−728,802	−1,697,558	51,006	72,568*		
34,705,528	464,062,500	−352,665	−956,649	12,106	19,401	−2,161*	
27,929,686	429,163,491	−193,480	−600,104	3,881	6,981	−314	−424*

$A_n = u_{n-1}u_n/(u_{n-1} - u_n)$ $\quad T_n = k A_n/(k-1)$ $\quad u_n^* = u_n - T_n + T_{n+1}$

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$$k \simeq \frac{(u_{n-1} - u_n)(u_n - u_{n+1})}{u_{n-1}u_{n+1} - u_n^2}.$$

(iv) If k is unknown it may be eliminated from equations (1) to give Lubkin's [1] formula (17) namely

$$Z_n = \frac{u_n(1 - R_{n+1})}{1 - 2R_{n+1} + R_n R_{n+1}} \quad \text{where} \quad R_n = \frac{u_n}{u_{n-1}}.$$

3. Demonstration of the formula

The use of equations (2) and (3) is demonstrated in table 1 on the first eight terms of a slowly convergent series for π namely

$$\pi = \sum_{n=0}^{\infty} \frac{(2n)!}{2^{2n-1}(n!)^2(2n+1)}$$

multiplied by 10^9 to avoid tabulating decimals. This series would require about 10^{16} terms to give an error of 10 in the tabulated values or 10^{-8} in π.

If it is known that the first, third, fifth and seventh columns are terms of series which behave like $n^{-3/2}$, $n^{-7/2}$, $n^{-11/2}$, $n^{-15/2}$ then we take $k = 3/2$, $7/2$, $11/2$, $15/2$ in successive uses of equation (2) or use the equivalent formulae at the bottom of table 1 in order to construct the table. The numbers marked with an asterisk at the top of each column are summed to give 3,141,592,641 which differs from $10^9\pi$ by 12.

4. Comparison with other formulae

The two transformations most closely related to this are those of Lubkin and Wynn.

If we do not know k we may use Lubkin's transformation, the difference being that we use three terms of the series to reduce the size of the terms by a factor $(k+1)/6(k-1)n^2$ instead of using two terms and a knowledge of k to reduce the size of the terms by a factor $(k+1)/12n^2$. The results of summing six terms of three series for which k is 1.5, 2, and 4 are given in table 2. A comparison of columns A and C demonstrates the improvement resulting from a knowledge of k.

Wynn's [2] ρ-algorithm for transforming a sequence at the start of his section 2 is

$$\rho_s(S_n) = \rho_{s-2}(S_{n+1}) + \frac{s}{\rho_{s-1}(S_{n+1}) - \rho_{s-1}(S_n)} \tag{4}$$

TABLE 2

Comparison of formulae

I Six terms of $\Sigma 2(2n)!/2^{2n}(n!)^2(2n+1)$

	A	B	C
Sum	2.6781273	2.6781273	2.6781273
1st transformation	3.1425424	3.1425424	3.1365079
2nd transformation	3.1415737	3.1415715	3.1413978
3rd transformation	3.1415952	3.1415967	—
Infinite sum		$\pi = 3.1415927$	

II Six terms of Σn^{-2}

	A	B	C
Sum	1.4913889	1.4913889	1.4913889
1st transformation	1.6454293	1.6454293	1.6436111
2nd transformation	1.6449244	1.6449226	1.6448949
3rd transformation	1.6449350	1.6449357	—
Infinite sum		$\pi^2/6 = 1.6449341$	

III Six terms of Σn^{-4}

	A	B	C
Sum	1.08112353	1.08112353	1.08112353
1st transformation	1.08233901	1.08233901	1.08230690
2nd transformation	1.08232242	1.08232208	1.08232213
3rd transformation	1.08232340	1.08232368	—
Infinite sum		$\pi^4/90 = 1.08232323$	

A Present formula, B Modified Wynn's ρ-algorithm, C Lubkin's formula.

where S_n is the sum of a series from u_0 to u_n inclusive and ρ_s for even s is an estimate of the sum of the infinite series. His initial conditions are ideal if $k = 2$ and his three examples are of this type. However if k is known and is not 2 we can take as initial conditions $\rho_{k-3}(S_n) = 0$, $\rho_{k-2}(S_n) = S_n$ hence $\rho_{k-1}(S_n) = (k-1)/u_{n+1}$ and

$$\rho_k(S_n) = S_{n+1} + \frac{k}{k-1}\,\frac{u_{n+1}u_{n+2}}{u_{n+1} - u_{n+2}}.$$

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The first two steps in this modification of Wynn's formula are now identical with the transformation derived in section 2 but thereafter the two transformations give slightly different results. In the examples in table 2 a comparison of columns A and B shows that the present formula gives results slightly closer to the infinite sum than does the ρ-algorithm. Also the amount of calculation done using equation (4) twice or equations (2) and (3) is comparable but in using a programmable desk calculator where most of the time taken is in inserting numbers the ρ-algorithm is longer.

References

[1] S. Lubkin, 'A method of summing infinite series', *J. Res. Nat. Bur. Stand.* 48 (1952), 228–254.
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