ON NORMED ALGEBRAS WHOSE NORMS SATISFY POLYNOMIAL IDENTITIES

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1. Introduction. In this note we are concerned with normed algebras over a non-discrete field with absolute value. The norm, N, of a normed algebra A is said to *satisfy a polynomial identity* on a subset B if there is a polynomial $P(\zeta_1, \ldots, \zeta_r)$ such that $P(N(x_1), \ldots, N(x_r)) = N(P(x_1, \ldots, x_r))$ whenever $x_1, \ldots, x_r$ are in B, where the polynomial has rational integer coefficients, degree greater than 1, constant term zero, and non-negative coefficients for each term of highest degree. It is shown in Theorem 1, following a method of proof used by Kadison in **(6**, § 7**)**, that if the norm of a normed algebra satisfies a polynomial identity on the entire algebra, then the norm is power multiplicative. (That is, then $N(x)^2 = N(x^2)$ for all x.)

As incidental by-products of this result, some corollaries are obtained which are related to Ostrowski's results in **(8)** and to Mazur's Theorem **(7**, Theorem 1**)**. For example, Corollary 3 of Theorem 1 shows that if a normed division algebra A, over the real field normed with some power of its ordinary absolute value, has a norm which is stable (in the sense of **(3)**) and which satisfies a polynomial identity on A, then A is isomorphic, as a real algebra, with the real field, or the complex field, or the division ring of all real quaternions. (Compare with Mazur's Theorem 1 in **(7)**.)

Gelfand's algebraic characterization **(5)** of a semi-simple, commutative, complex Banach algebra with unit element, as an algebra of some of the continuous complex-valued functions on a suitable compact space, is well known. (*Compact* spaces throughout this note will be Hausdorff spaces for which every open covering has a finite subcovering; that is, we follow the definition employed in **(4)**.) A related result is obtained in Theorem 2 of this note, where it is shown that if a connected, commutative normed algebra A over a non-discrete field contains a non-zero element j such that $j^2x + x = 0$ for all x, and if the norm of A satisfies a polynomial identity on A, then A is algebraically and topologically isomorphic to a ring of some of the continuous complex-valued functions on a suitable compact space. Here the assumption that the norm satisfies a polynomial identity is stronger than assuming semi-simplicity of the algebra, while the assumption that the element j exists is weaker than assuming that the algebra is a complex algebra with unit e, since $i \cdot e$ could be taken as j in a complex algebra with unit.

One particular consequence of Gelfand's result quoted above was that **(5**,

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Satz 16, Folgerung 3) a commutative, complex Banach algebra A with unit element, such that the norm of A satisfies the condition $N(x)^2 = N(x^2)$ for all x in A, may be identified with the algebra of all continuous complex-valued functions (with the usual supremum norm) on a suitable compact space, provided that A "contains conjugate functions" of all its elements. In Theorem 3 of this note a similar result is obtained under much weaker assumptions; it is shown that a complete normed algebra which satisfies the hypotheses of Theorem 2 (given in the preceding paragraph), and which "contains conjugate functions" in an appropriate sense, must be algebraically and topologically isomorphic to the ring of all continuous complex-valued functions on a suitable compact space.

An interesting feature of the proofs is the use of Ostrowski's results (**8**) at several points where the standard practice in obtaining comparable results for Banach algebras is to rely on Mazur's Theorem (**7**). This change appears necessary since our hypotheses do not seem to permit the immediate use of Mazur's Theorem.

2. Preliminaries. We shall consider in the sequel normed algebras, usually with unit element e, over non-discrete fields with absolute value. (For basic terminology, see (**4**, chapter IX, § 3, no. **7**).) The normed algebras to be studied will be subjected to conditions on their norms similar to those imposed in (**3**); it is assumed that the material contained in that paper is known to the reader.

Every normed algebra is a metric ring; such words as *homomorphism* and *isomorphism* will be understood to refer only to the ring structure of the algebras under consideration, unless the contrary is indicated. A norm-preserving isomorphism of a metric ring R into a metric ring R' will be called an *isometry* of R into R'; if there is an isometry of a metric ring R onto a metric ring R' we shall say that R is *isometric* to R'.

A polynomial $P(\zeta_1, \ldots, \zeta_r)$ in the r indeterminates $\zeta_1, \ldots, \zeta_r$ will be called *regular* if it has rational integer coefficients, constant term zero, degree greater than 1, and if every term of highest degree has a non-negative coefficient. The number of indeterminates which actually appear in a regular polynomial will be called the *rank* of the polynomial, while the degree of the terms of lowest degree in a regular polynomial will be called the *order* of the polynomial. It may be noted that the rank and the order of a regular polynomial are both positive integers.

In every ring it is possible to define, in an obvious way, multiplication of ring elements by rational integers. It follows easily that if $P(\zeta_1, \ldots, \zeta_r)$ is a regular polynomial, and if $x_1, \ldots, x_r$ are elements of a ring R, then there is defined an element $P(x_1, \ldots, x_r)$ belonging to R. If N is a pseudonorm for a ring R and B is a subset of R such that there is a regular polynomial $P(\zeta_1, \ldots, \zeta_r)$ for which $P(N(x_1), \ldots, N(x_r)) = N(P(x_1, \ldots, x_r))$ whenever $x_1, \ldots, x_r$ are in B, then we shall say that N *satisfies a polynomial identity* on B. Any terminology applicable to regular polynomials will also be applied to situations in which a pseudonorm satisfies a polynomial identity, in order to describe the polynomial involved in the identity.

Polynomial identities have been imposed upon the norm of a metric ring in several instances in the past. For example, Ostrowski confined himself in **(8)** to fields having a norm N such that $N(x_1)N(x_2) = N(x_1x_2)$ for all x_1 and x_2; this is equivalent to assuming that N satisfies on the whole field the polynomial identity involving the regular polynomial $P(\zeta_1, \zeta_2) \equiv \zeta_1 \cdot \zeta_2$. In **(3)** pseudonorms N were considered such that $N(x)^2 = N(x^2)$ for all x, and this is the same as saying that N satisfies on the whole ring the polynomial identity corresponding to the polynomial $P(\zeta) \equiv \zeta^2$. This latter polynomial identity is quite typical, and we shall show that, under reasonably general conditions, the norm of a normed algebra must satisfy this particular polynomial identity on the algebra if it satisfies a polynomial identity on the algebra.

First, we note that if a pseudonorm N satisfies a polynomial identity on a set B, then N satisfies a polynomial identity of rank one on B. For, if $P(N(x_1), \ldots, N(x_r)) = N(P(x_1, \ldots, x_r))$ whenever $x_1, \ldots, x_r$ are in B, then $P(N(x), \ldots, N(x)) = N(P(x, \ldots, x))$ for all x in B; that is, N satisfies the polynomial identity $Q(N(x)) = N(Q(x))$ on B, where $Q(\zeta) \equiv P(\zeta, \ldots, \zeta)$ is clearly a regular polynomial having the same degree as P, and with order at least as large as that of P. Thus, we may always pass from a general polynomial identity to one of rank 1. The next step is to show that, under appropriate conditions, we may then pass to the specific polynomial identity $N(x)^2 = N(x^2)$, which has already been investigated in **(3)**.

THEOREM 1. *Let A be a normed algebra, with norm N, over a non-discrete field K with absolute value, such that N satisfies a polynomial identity on A (or a polynomial identity of order greater than* 1 *on a neighbourhood of zero in A). Then N is power multiplicative; that is, $N(x)^2 = N(x^2)$ for all x in A.*

Proof. Because of the remarks made above, we may assume that the polynomial identity satisfied by N has rank one, say $P(N(x)) = N(P(x))$ for all x. If P has degree m and order n, with $P(\zeta) \equiv \alpha_0 \cdot \zeta^m + \ldots + \alpha_s \cdot \zeta^n$, where $s = m - n$, then for x in A and non-zero k in K, kx is in A, and therefore $P(N(kx)) = N(P(kx))$. If $|| \; ||$ is the absolute value for K, then $N(kx) = ||k|| \cdot N(x)$, so that $P(||k|| \cdot N(x)) = P(N(kx)) = N(P(kx))$ for x in A and k a non-zero element of K. If $P(||k|| \cdot N(x)) = N(P(kx))$ is written explicitly, we obtain:

$$(1) \quad \alpha_0 \cdot ||k||^m \cdot N(x)^m + \ldots + \alpha_s \cdot ||k||^n \cdot N(x)^n = N(\alpha_0 \cdot k^m \cdot x^m + \ldots + \alpha_s \cdot k^n \cdot x^n).$$

If we divide by $||k||^m = ||k^m||$, we have:

$$(2) \quad \alpha_0 \cdot N(x)^m + \ldots + \alpha_s \cdot ||k||^{-s} \cdot N(x)^n = N(\alpha_0 \cdot x^m + \ldots + \alpha_s \cdot k^{-s} \cdot x^n),$$

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where division by $||k^m||$ was effected on the right side of the equation by moving k^m within the parentheses and there carrying out the division by k^m.

Let $||k||$ tend to infinity in (2), so that $\alpha_0 \cdot N(x)^m = N(\alpha_0 \cdot x^m)$ for every fixed x in A. Then $\alpha_0 \cdot N(x)^m = N(\alpha_0 \cdot x^m) \leqslant \alpha_0 \cdot N(x^m) \leqslant \alpha_0 \cdot N(x)^{m-2} \cdot N(x^2)$, and division by $\alpha_0 \cdot N(x)^{m-2}$ yields $N(x)^2 \leqslant N(x^2)$. But $N(x^2) \leqslant N(x)^2$ always, whence $N(x)^2 = N(x^2)$ for all x; thus, N is power multiplicative.

In case we make the alternative assumption that N satisfies a polynomial identity of order greater than 1 on a neighbourhood U of zero in A, then for a fixed x in A we have kx in U whenever $||k||$ is sufficiently small. This means that (1) applies in this situation, with $n > 1$. The procedure is now similar to that followed previously, except that we work at the other end of the polynomial. Thus, we divide (1) by $||k||^n = ||k^n||$ and then let $||k||$ tend to zero. This gives us $\alpha_s \cdot N(x)^n = N(\alpha_s \cdot x^n)$ for all x.

N is obviously power multiplicative if A consists solely of the zero element, and it will therefore suffice to complete the proof only for the case in which A contains an element different from zero. If there is a non-zero x in A, then $\alpha_s \cdot N(x)^n \neq 0$, and therefore $N(\alpha_s \cdot x^n) \neq 0$ for such an x. Thus, the right side of the equation $\alpha_s \cdot N(x)^n = N(\alpha_s \cdot x^n)$ is positive, so that the left side is positive, and α_s is then positive. Then for every x in A we have $\alpha_s \cdot N(x)^n = N(\alpha_s \cdot x^n) \leqslant \alpha_s \cdot N(x^n) \leqslant \alpha_s \cdot N(x)^{n-2} N(x^2)$. Division by $\alpha_s \cdot N(x)^{n-2}$ shows that $N(x)^2 \leqslant N(x^2)$, and consequently $N(x)^2 = N(x^2)$ for all x. That is, N is power multiplicative.

COROLLARY 1. *Let A be an archimedean normed division algebra, over a non-discrete field K with absolute value. If the norm of A is stable and satisfies a polynomial identity on A (or a polynomial identity of order greater than 1 on a neighbourhood of zero in A), then A is isomorphic to a division subring of the division ring $\mathfrak{Q}$ of all real quaternions. If, in addition, A is commutative, then A is isomorphic to a subfield of the field $\mathfrak{C}$ of all complex numbers.*

COROLLARY 2. *Let A be a connected normed division algebra, over a non-discrete field K with absolute value. If the norm for A is stable and satisfies a polynomial identity on A (or a polynomial identity of order greater than* 1 *on a neighbourhood of zero in A), then A is isomorphic to a division subring of $\mathfrak{Q}$. If, in addition, A is commutative, then A is isomorphic to a subfield of $\mathfrak{C}$.*

The theorem shows that the norm is power multiplicative in both corollaries, so that the conclusions in the second sentences of these corollaries follow from (**3**, Theorem 8, and Corollary 2 of Theorem 6). The final sentence in each corollary follows from the fact that every commutative subring of $\mathfrak{Q}$ is contained in a maximal commutative subring, for it is easily demonstrated that every maximal commutative subring of $\mathfrak{Q}$ is isomorphic to $\mathfrak{C}$.

An interesting special case of Corollary 2 occurs when the algebra is a normed algebra over the real field. First, if ρ is a real number such that $0 < \sigma \leqslant 1$, the symbol $\mathfrak{R}^{(\rho)}$ will denote the field of all real numbers, with the ρth power of the ordinary absolute value taken as a norm for $\mathfrak{R}^{(\rho)}$. It is easily established that each $\mathfrak{R}^{(\rho)}$ is a connected, complete field with absolute value. Then every normed algebra over $\mathfrak{R}^{(\rho)}$ is connected since $\mathfrak{R}^{(\rho)}$ is connected. This would yield a special case of Corollary 2, but it is possible to obtain a stronger conclusion when connectedness of the algebra is replaced as a hypothesis by the assumption that the field K of scalars is a field $\mathfrak{R}^{(\rho)}$.

COROLLARY 3. *Let ρ be a real number such that $0 < \rho \leqslant 1$, and let A be a normed division algebra over $\mathfrak{R}^{(\rho)}$, such that the norm for A is stable and satisfies a polynomial identity on A (or a polynomial identity of order greater than 1 on a neighbourhood of zero in A). Then A is isomorphic, as a real algebra, to the algebra $\mathfrak{R}$ of all real numbers, or to $\mathfrak{C}$, or to $\mathfrak{Q}$.*

Proof. If N is the norm for A then N is power multiplicative, by Theorem 1, so that Corollary 2 of Theorem 2 in **(3)** shows that there is an absolute value N' subordinate to N.

Let α be a non-zero real number, and let e be the unit element of A. Then $N'(\alpha\cdot e) \leqslant N(\alpha\cdot e) = |\alpha|^\rho\cdot N(e) = |\alpha|^\rho$. Similarly, $N'(\alpha^{-1}\cdot e) \leqslant |\alpha^{-1}|^\rho = |\alpha|^{-\rho}$. But $1 = N'(e) = N'((\alpha\cdot e)(\alpha^{-1}\cdot e)) = N'(\alpha\cdot e)N'(\alpha^{-1}\cdot e) \leqslant |\alpha|^\rho\cdot|\alpha|^{-\rho} = 1$, so that $N'(\alpha\cdot e) = |\alpha|^\rho$. Thus, $N'(\alpha\cdot x) = N'((\alpha\cdot e)x) = N'(\alpha\cdot e)N'(x) = |\alpha|^\rho\cdot N(x)$ for every x in A. It follows easily that $N'(\alpha\cdot x) = |\alpha|^\rho\cdot N'(x)$ for every x in A and for every real scalar α. If A' is the structure obtained from A by using N' instead of N as the norm, then we have just shown that A' is a normed algebra over $\mathfrak{R}^{(\rho)}$.

Let A'' be the completion of A', so that A'' is also a normed algebra over $\mathfrak{R}^{(\rho)}$. Since A' is a division ring with absolute value, A'' is a division ring with absolute value. In addition, A'' is connected since it is a normed algebra over $\mathfrak{R}^{(\rho)}$; furthermore, A'' is complete. Theorem 11 of **(2)** then shows that there exists an algebraic and topological isomorphism σ of A'' onto $\mathfrak{R}$, or $\mathfrak{C}$, or $\mathfrak{Q}$. The continuity of σ implies that σ is not only a ring-isomorphism, but also an isomorphism as algebras over $\mathfrak{R}$. Since A'' is therefore a finite-dimensional algebra over $\mathfrak{R}$, it follows that the subalgebra A' is also a finite-dimensional algebra over $\mathfrak{R}$. The theorem of Frobenius then implies that the division algebra A' is isomorphic, as an algebra over $\mathfrak{R}$, to one of the algebras $\mathfrak{R}$, $\mathfrak{C}$, and $\mathfrak{Q}$. But A has the same underlying algebra as A', so that A is also isomorphic, as an algebra over $\mathfrak{R}$, to one of the algebras $\mathfrak{R}$, $\mathfrak{C}$, and $\mathfrak{Q}$.

Corollary 3 bears some similarity to Mazur's Theorem **(7**, Theorem 1**)**, but the latter also assumes that $\rho = 1$, while the hypotheses of Corollary 3 are stronger than those of Mazur's Theorem in assuming that the norm is stable and satisfies a polynomial identity. The standard proofs of Mazur's Theorem do not seem to be adaptable for proving the corollary, however, despite its similarity to Mazur's Theorem. Gelfand's proof in **(5)**, using analytic function theory, and Tornheim's elementary proof in **(10)** both require that the field of scalars be the real field or the complex field, equipped with the ordinary absolute value. The method of proof employed by Arens in **(1)** also seems to require stronger conditions than are available in Corollary 3.

3. Representations as rings of continuous functions. Gelfand showed in **(5)** that a semi-simple, commutative, complex Banach algebra with unit may be identified algebraically with an algebra of some of the continuous complex-valued functions defined on a suitable compact space. We shall obtain a result which is somewhat similar, although our hypotheses will be weaker in most particulars, except for replacing the assumption of semi-simplicity by the stronger assumption that the norm of the algebra satisfies a polynomial identity on the algebra. (It is easily seen that if the norm of a normed algebra over a non-discrete field satisfies a polynomial identity on the algebra, then Theorem 1 shows that the norm is power multiplicative, so that the algebra contains no non-zero nilpotents and is therefore semi-simple.) Before proceeding to our representation theorem, we first introduce some pertinent definitions.

A central, non-zero element j of a ring R will be called a *gaussian element* of R if $j^2x + x = 0$ for all x in R. The existence of a gaussian element in a ring always implies that the ring has a unit element; for, if j is a gaussian element in a ring R, then $-j^2$ is a unit for R. In a complex algebra with unit e, the elements $i \cdot e$ and $(-i) \cdot e$ are examples of gaussian elements.

If ρ is a real number with $0 < \rho \leqslant 1$, the symbol $\mathfrak{C}^{(\rho)}$ will denote the field of all complex numbers, with the ρth power of the ordinary absolute value taken as the norm. Then $\mathfrak{C}^{(\rho)}$ is a connected, complete field with absolute value; in particular, $\mathfrak{C}^{(\rho)}$ is *archimedean*, in the sense of **(8)** or **(2)**. If K is any archimedean field with absolute value, then there is a real number ρ, with $0 < \rho \leqslant 1$, such that K is isometric to a subfield of $\mathfrak{C}^{(\rho)}$. (This result occurs essentially in **(8)**; an easy proof is obtained by completing K and then applying Theorem 11 of **(2)**.) The isometry of K into a $\mathfrak{C}^{(\rho)}$ is not necessarily unique, but its restriction to the prime field of K is unique, and ρ is consequently determined uniquely by K. In fact, if n is any integer greater than 1, then ρ is determined by the condition $\rho = \log_n(||n||)$ if the absolute value in K is denoted by $||\ ||$. We shall refer to the number ρ as the *exponent* for the archimedean field K. It is clear that every subfield of an archimedean field K has the same exponent as K. (Lemma 14 of **(2)** clearly implies that a subfield of an archimedean field with absolute value must also be archimedean; for, whether a field with absolute value is archimedean or not is determined by the behaviour of the absolute value on the prime field.)

If ρ is a real number with $0 < \rho \leqslant 1$, and if Φ is a compact space, the symbol $C(\Phi; \mathfrak{C}; \rho)$ will designate the set of all continuous complex-valued functions $x(\phi)$ defined on Φ, with algebraic operations defined in the obvious way, and with the norm N such that $N(x) = \sup\{|x(\phi)|^\rho \,|\phi \in \Phi\}$ for each x in $C(\Phi; \mathfrak{C}; \rho)$. Clearly, the norm in $C(\Phi; \mathfrak{C}; \rho)$ is simply the ρth power of

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the usual supremum norm and is, therefore, power multiplicative. It is easily verified that $\mathfrak{C}(\Phi; \mathfrak{C}; \rho)$ is a connected, commutative, complete normed algebra with unit, over the field $C^{(\rho)}$. The topology in $C(\Phi; \mathfrak{C}; \rho)$ is evidently the topology of uniform convergence on Φ, and each $C(\Phi; \mathfrak{C}; \rho)$ clearly contains gaussian elements.

It is now possible to prove that a connected, commutative normed algebra which contains gaussian elements, and which has a norm that satisfies a polynomial identity on the algebra, is isometric to a subring of a $C(\Phi; \mathfrak{C}; \rho)$. This result is quite similar to the result of Gelfand which was described at the beginning of this section, but our hypotheses are clearly weaker than those of Gelfand, with the one exception already noted.

LEMMA 1. *Let A be a connected, commutative normed algebra, with norm N, over a non-discrete field K with absolute value. Suppose that N satisfies a polynomial identity on A (or a polynomial identity of order greater than 1 on a neighbourhood of zero in A). Then K is archimedean, and, if ρ is the exponent for K, it is possible to find for each non-zero c in A a non-zero homomorphism ϕ of A into $\mathfrak{C}^{(\rho)}$, such that $||\phi(x)|| \leqslant N(x)$ for all x in A, and $||\phi(c)|| = N(c)$.*

Proof. Theorem 1 shows that N is power multiplicative.

If c is a given non-zero element of A, let $\mathcal{N}$ be the set of all power multiplicative pseudonorms N' subordinate to N and such that: (i) $N'(c) = N(c)$, (ii) $N'(cx) = N'(c)\cdot N'(x)$ for all x in A, and (iii) $N'(kx) = ||k||\cdot N'(x)$ whenever k is in K and x is in A. It is easily shown that N_c belongs to $\mathcal{N}$ and that $\mathcal{N}$ is a hereditary system. The method of proof of Lemma 4 in **(3)** may be used to show that $\mathcal{N}$ contains a minimal element N', and **(3**, Lemma 2**)** shows that N' is a pseudo absolute value. Then $\bar{A} = A/I(N')$ is an algebra over K; if $\bar{N}$ is the function on $\bar{A}$ such that the value $\bar{N}(X)$ assumed by $\bar{N}$ on the residue class X (modulo $I(N')$) is equal to the constant value assumed by N' on the elements of X, then $\bar{A}$ becomes a commutative normed algebra, with norm $\bar{N}$, over K. Also, $\bar{N}$ is an absolute value for $\bar{A}$, so that $\bar{A}$ has no proper zero-divisors. The natural mapping η of A onto $\bar{A}$ is a homomorphism of A onto $\bar{A}$, as algebras over K, and we have $\bar{N}(\eta(x)) = N'(x)$ for all x in A. Thus, $\bar{N}(\eta(c)) = N'(c) = N(c) \neq 0$, so that $\eta(c)$ is a non-zero element of $\bar{A}$.

Since $\bar{A}$ is a commutative non-zero algebra without proper zero-divisors, a standard procedure permits us to embed $\bar{A}$ in a field E of quotients. With the obvious norm $\bar{N}$ chosen for E, we obtain a normed algebra E over K, such that E is a field with absolute value. The natural mapping ξ of $\bar{A}$ into E is a norm-preserving isomorphism of $\bar{A}$ into E, as algebras over K. The inequality $\bar{N}(\eta(x) - \eta(y)) = \bar{N}(\eta(x-y)) = N'(x-y) \leqslant N(x-y)$ shows that η is continuous, so that $\bar{A}$ is connected since it is the continuous image of the connected set A. Also, $\bar{A}$ contains the distinct points 0 and $\eta(c)$, and the image of $\bar{A}$ under the continuous one-to-one mapping ξ is therefore a connected subset of E containing more than one point. Then E is a field with absolute value and is not totally disconnected; Lemma 15 of **(2)** implies that E must then be archimedean. But K may be identified with a subfield of E in an obvious way, and it follows that K is archimedean.

Let ρ be the exponent for K, so that ρ is also the exponent for E. Then there exists an isometry θ of E into $\mathfrak{C}^{(\rho)}$, and we let ϕ be the mapping obtained by applying η, then ξ, and finally θ. Clearly, ϕ is a homomorphism of A into $\mathfrak{C}^{(\rho)}$, and $||\phi(x)|| = ||\theta(\xi(\eta(x)))|| = \tilde{N}(\xi(\eta(x))) = \bar{N}(\eta(x)) = N'(x) \leqslant N(x)$ for all x in A. Also, $||\phi(c)|| = ||\theta(\xi(\eta(c)))|| = \tilde{N}(\xi(\eta(c))) = \bar{N}(\eta(c)) = N'(c) = N(c) \neq 0$, and this shows that ϕ is not zero since $\phi(c) \neq 0$. The proof is complete.

THEOREM 2. *Let A be a connected, commutative normed algebra, over a non-discrete field K with absolute value, and let j be a gaussian element in A. Suppose that the norm of A satisfies a polynomial identity on A (or a polynomial identity of order greater than* 1 *on a neighbourhood of zero in A). Then K is archimedean, and, if ρ is the exponent for K, there exists a compact space Φ and an isometry σ of A into $C(\Phi;\ \mathfrak{C};\ \rho)$, such that $\sigma(j)$ is the constant function i.*

Proof. Lemma 1 shows that K is archimedean; let ρ be the exponent for K.

Let Φ be the set of all homomorphisms ϕ of A into $\mathfrak{C}^{(\rho)}$, such that $\phi(j) = i$ and $||\phi(x)|| \leqslant N(x)$ for all x in A, where N is the norm of A. If ϕ_0 is in Φ, if x is in A, and if ϵ is a positive number, the symbol $\{\phi_0;\ x;\ \epsilon\}$ will denote the set of all ϕ in Φ such that $|\phi(x) - \phi_0(x)| < \epsilon$. Then Φ becomes a topological space if the family of all sets of type $\{\phi_0;\ x;\ \epsilon\}$ is used as a sub-base for the topology.

For each non-zero x in A, let $\mathscr{D}_x$ be the circular disc $\{\zeta|\ |\zeta|^\rho \leqslant N(x)\}$ in the complex plane. The cartesian product, $\mathscr{D}$, of all the $\mathscr{D}_x$, as x ranges over the non-zero elements of A, is a compact space since every factor is compact. But the space Φ may be identified with a subset of $\mathscr{D}$, if the $\mathscr{D}_x$-co-ordinate of any ϕ in Φ is taken as $\phi(x)$; the relative topology of Φ as a subset of $\mathscr{D}$ is clearly the topology which was given to Φ in the preceding paragraph. It is easily shown that Φ is a closed subset of $\mathscr{D}$, whence Φ is compact since $\mathscr{D}$ is compact.

If x is in A, define $\hat{x}(\phi) = \phi(x)$ for all ϕ in Φ. Then each $\hat{x}$ is a complex-valued function defined on Φ, and each $\hat{x}$ is continuous on Φ because of the kind of topology which was introduced in Φ. The mapping $x \to \hat{x}$ is clearly a homomorphism of A into $C(\Phi;\ \mathfrak{C};\ \rho)$, and $\hat{j}$ is the constant function i. We have $N(x) \geqslant ||\phi(x)|| = |\phi(x)|^\rho = |\hat{x}(\phi)|^\rho$ for all ϕ in Φ, and it follows that $N(x) \geqslant \sup\{|\hat{x}(\phi)|^\rho|\phi \in \Phi\}$ for each x in A. On the other hand, if c is a non-zero element of A, Lemma 1 shows that there is a non-zero homomorphism χ of A into $\mathfrak{C}^{(\rho)}$, with $||\chi(c)|| = N(c)$ and $||\chi(x)|| \leqslant N(x)$ for all x in A. It is easily seen that $\chi(j)$ is a gaussian element of the non-zero ring $\chi(A)$, so that $\chi(j) = i$ or $\chi(j) = -i$. Let $\phi(x)$ equal $\chi(x)$ for all x if $\chi(j) = i$, and let $\phi(x)$ equal the complex-conjugate of $\chi(x)$ for all x if $\chi(j) = -i$. Then $\phi(j) = i$, and ϕ belongs to Φ. Now, $N(c) = ||\chi(c)|| = |\chi(c)|^\rho = |\phi(c)|^\rho = |\hat{c}(\phi)|^\rho$, so that $N(c) = \sup\{|\hat{c}(\phi)|^{\rho}\,|\phi \in \Phi\}$. The mapping $x \to \hat{x}$ thus preserves norms, and is therefore an isometry since its kernel must be zero.

If the mapping $x \to \hat{x}$ is denoted by σ, then the theorem is established.

THEOREM 3. *Let A be a complete normed algebra which satisfies the hypotheses of Theorem 2, and suppose there is a mapping $x \to x^*$ of A into itself such that $\phi(x^*)$ is the complex-conjugate of $\phi(x)$ whenever x is in A and ϕ is in Φ. Then the mapping σ is an isometry of A onto $C(\Phi;\, \mathfrak{C};\, \rho)$, such that $\sigma(x^*)$ is the conjugate function of $\sigma(x)$ for each x in A.*

Proof. Let $\hat{A} = \sigma(A)$, so that $\hat{A}$ is a subring of $C(\Phi;\, \mathfrak{C};\, \rho)$.

We note that $\hat{j} = \sigma(j)$ is the constant function i and belongs to $\hat{A}$. Also, if e is the unit element of A, then the mapping $k \to ke$ is an isometry of K into A, and the mapping $k \to \sigma(ke)$ is consequently an isometry of the archimedean field K into $\hat{A}$. Then $\hat{A}$ contains a field isomorphic to the field of rational numbers, and so $\hat{A}$ contains every constant function on Φ which assumes a real, rational value. But $\hat{A}$ is isometric to A and is therefore complete; thus, $\hat{A}$ is uniformly closed in $C(\Phi;\, \mathfrak{C};\, \rho)$. It follows that $\hat{A}$ contains every constant real-valued function on Φ. Since $\hat{A}$ is a ring and also contains the constant function i, every constant complex-valued function on Φ belongs to $\hat{A}$. But $\hat{A}$ is closed under multiplication, and $\hat{A}$ may therefore be considered a complex algebra and a subalgebra of the complex algebra $C(\Phi;\, \mathfrak{C};\, \rho)$.

Next, we note that if ϕ_1 and ϕ_2 are distinct elements of Φ, then $\phi_1(x) \neq \phi_2(x)$ for some x in A, and so $\hat{x}(\phi_1) \neq \hat{x}(\phi_2)$ for such an x. That is, there is an $\hat{x}$ in $\hat{A}$ which distinguishes ϕ_1 and ϕ_2; in the terminology of **(9)**, $\hat{A}$ is a *separating family* for Φ.

Finally, $[\sigma(x^*)](\phi) = \phi(x^*)$ is the complex-conjugate of $\phi(x) = [\sigma(x)](\phi)$, whenever x is in A and ϕ is in Φ. In other words, $\sigma(x^*)$ is the conjugate function of $\sigma(x)$, for each x in A. Then $\hat{A}$ contains the conjugate function $\sigma(x^*)$ of each of its elements $\sigma(x)$, $\hat{A}$ is a uniformly closed complex subalgebra of $C(\Phi;\, \mathfrak{C};\, \rho)$, and $\hat{A}$ is a separating family for Φ. Since $\hat{j}$ is in $\hat{A}$ and vanishes nowhere on Φ, it follows from Corollary 2 of Theorem 10 in **(9)** that $\hat{A}$ coincides with $C(\Phi;\, \mathfrak{C};\, \rho)$. Thus, σ is an isometry of A onto $C(\Phi;\, \mathfrak{C};\, \rho)$, and this completes the proof.

Theorems 2 and 3 describe the algebra A in terms of its ring structure rather than as an algebra; that is, the isometry σ is a ring isomorphism in those theorems, and not an isomorphism as algebras. This difficulty arises because A is not given as a complex algebra, and also because of the arbitrary selection of the gaussian element j to be carried by σ into the constant function i. However, in the case of a complex algebra with unit e it becomes natural to expect σ to carry $i\cdot e$ into the constant function i; when σ is chosen in such a way σ becomes an isomorphism relative to the structure as complex algebras.

LEMMA 2. *Let A and A' be complex topological algebras, such that A has a*

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unit element e, and let ϕ be a continuous homomorphism of A into A' such that $\phi(i\cdot e) = i\cdot\phi(e)$. Then ϕ is a homomorphism of A into A' as algebras over the complex field.

The proof of this lemma is routine, and is left to the reader.

THEOREM 4. *Let ρ be a real number with $0 < \rho \leqslant 1$, and let A be a commutative normed algebra, with unit e, over $\mathfrak{C}^{(\rho)}$. Suppose that the norm of A satisfies a polynomial identity on A (or a polynomial identity of order greater than 1 on a neighbourhood of zero in A). Then there exists a compact space Φ and a norm-preserving isomorphism σ of A into $C(\Phi;\ \mathfrak{C};\ \rho)$, as algebras over $\mathfrak{C}^{(\rho)}$. If, in addition, A is complete and there is a mapping $x \rightarrow x^*$ of A into itself such that $\sigma(x^*)$ is the conjugate function of $\sigma(x)$ for every x in A, then σ is an isomorphism of A onto $C(\Phi;\ \mathfrak{C};\ \rho)$.*

Proof. Apply Theorem 2, with $j = i\cdot e$. Then $\sigma(i\cdot e)$ is the constant function i, and σ is continuous since it is an isometry. Lemma 2 may be applied to show that σ is an isomorphism of A into $C(\Phi;\ \mathfrak{C};\ \rho)$ as algebras over $\mathfrak{C}^{(\rho)}$. The remainder of the theorem follows from Theorem 3.

An interesting special case of this theorem merits an explicit statement as a corollary.

COROLLARY. *Let A be a commutative, complete normed algebra with unit e, over $\mathfrak{C}^{(1)}$, and let the norm N for A satisfy the polynomial identity $N(x)^2 = N(x^2)$ on A. Then there exist a compact space Φ and a norm-preserving isomorphism σ of A into $C(\Phi;\ \mathfrak{C};\ 1)$, as algebras over $\mathfrak{C}^{(1)}$. If, in addition, there is a mapping $x \rightarrow x^*$ of A into itself such that $\sigma(x^*)$ is the conjugate function of $\sigma(x)$ for each x in A, then σ is a norm-preserving isomorphism of A onto $C(\Phi;\ \mathfrak{C};\ 1)$, as algebras over $\mathfrak{C}^{(1)}$.*

This corollary was first given by Gelfand in (**5**, Satz 16, Folgerung 3), but Φ was defined in a completely different way there. However, the use of Lemma 2 would enable us to show that in this corollary Φ can also be described as the set of all homomorphisms ϕ of A into $\mathfrak{C}^{(1)}$, as algebras over $\mathfrak{C}^{(1)}$, such that $\phi(e) = 1$, and such that $|\phi(x)| \leqslant N(x)$ for all x in A. It would then follow that the kernel of each ϕ in Φ is a maximal ideal in A, and that each maximal ideal in A is the kernel of some ϕ in Φ. Indeed, the mapping which associates with each ϕ the kernel of ϕ is a homeomorphism of Φ onto Gelfand's space of maximal ideals, so that there is really no difference between our corollary and Gelfand's result.

Note 1. In Lemma 1, Theorem 2, and Theorem 3, the only purpose of assuming that the field K is non-discrete is to make it possible to use Theorem 1, in order to show that the norm of the algebra is power multiplicative. Thus, it would be possible to drop the hypothesis that K is non-discrete in these three results, provided that the hypothesis that the norm of the algebra satisfies a polynomial identity is replaced by the assumption that the norm of the algebra is power multiplicative.

Note 2. In Theorems 2 and 3, the assumption that the algebra is connected can be replaced by the assumption that the field K of scalars is archimedean; since the completion of the algebra would be a normed algebra over the completion of K and would therefore be connected because of the connectedness of the completion of K, the conclusions of these theorems would apply to the completion of the algebra and therefore to the original algebra.

References

1. R. Arens, *Linear topological division algebras*, Bull. Amer. Math. Soc., *53* (1947), 623–630.

2. S. Aurora, *Multiplicative norms for metric rings*, Pacific J. Math., *7* (1957), 1279–1304.

3. ——— *On power multiplicative norms*, Amer. J. Math., *80* (1958), 879–894.

4. N. Bourbaki, Éléments de mathématique, Livre III. *Topologie générale*, Actualités Scientifiques et Industrielles, nos. 858, 916, 1029, 1045, 1084 (Paris, 1940–1951).

5. I. Gelfand, *Normierte Ringe*, Rec. Math. (Mat. Sb.) N.S., *9* (1941), 3–24.

6. R. Kadison, *A representation theory for commutative topological algebra*, Mem. Amer. Math. Soc., *7* (New York, 1951).

7. S. Mazur, *Sur les anneaux linéaires*, C.R. Acad. Sci. Paris, *207* (1938), 1025–1027.

8. A. Ostrowski, *Ueber einige Lösungen der Funktionalgleichung* $\phi(x)\cdot\phi(y) = \phi(xy)$, Acta Math., *41* (1918), 271–284.

9. M. H. Stone, *The generalized Weierstrass approximation theorem*, Math. Mag., *21* (1948), 167–184, 237–254.

10. L. Tornheim, *Normed fields over the real and complex fields*, Mich. Math. J., *1* (1952), 61–68.

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