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OSCILLAIORY BEHAVIOR OF NONLINEAR DIFFERENTIAL EQUATIONS WITH DEVIATING ARGUMENTS

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New oscillation criteria for nonlinear differential equations with deviating arguments of the form

$$\frac{d}{dt}\left[a_{n-1}(t)\frac{d}{dt}\left[\cdots\left[a_2(t)\frac{d}{dt}\left[a_1(t)\frac{dx(t)}{dt}\right]\right]\cdots\right]\right] + q(t)f\left(x[g(t)]\right) = 0 ,$$

n even, are established.

1. Introduction

Recently Kamenev [4] considered the linear equation

$$(\alpha_1) \qquad \ddot{x} + q(t)x = 0 \quad \left(\dot{} = \frac{d}{dt}\right) ,$$

where q is a continuous real-valued function on the interval $[t_0, \infty)$ without any restriction on its sign, and proved that the condition

$$(*) \qquad \limsup_{t\to\infty} \frac{1}{t^{m-1}}\int_{t_0}^{t} (t-s)^{m-1}q(s)ds = \infty ,$$

for some integer $m \geq 3$, is sufficient so that all the solutions of (α_1)

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are oscillatory. His criterion includes as a special case the well-known Wintner result of [8]. Kamenev's criterion has been extended in various directions by Philos [6, 7] and Yeh [9, 10]. The present authors [1, 2] discussed this criterion for general functional equations of the form

$$(\alpha_2) \qquad x^{(n)} + q(t)f\left(x[t], x[g(t)]\right) = 0 , \quad n \text{ even},$$

and

$$(\alpha_3) \qquad \left(a(t)x^{(n-1)}\right)^{\bullet} + p(t)\left|x^{(n-1)}\right|^{\beta} x^{(n-1)} + q(t)f\left(x[g(t)]\right) = 0 , \quad n \text{ even and } \beta \ge 0 ,$$

with p and q nonnegative continuous functions on the interval $[t_0, \infty)$.

The purpose of this paper is to extend some of the previously mentioned results and obtain new oscillation criteria for the equation

$$(1) \qquad L_n x(t) + q(t)f\left(x[g(t)]\right) = 0 , \quad n \text{ even},$$

where

$$L_0 x(t) = x(t) , \quad L_k x(t) = a_k(t)\left(L_{k-1}x(t)\right)^{\bullet} , \text{ for } k = 1, 2, \ldots, n ,$$

with $a_0(t) = a_n(t) = 1$.

2. Main results

Consider the equation

$$(1) \qquad L_n x(t) + q(t)f\left(x[g(t)]\right) = 0 , \quad n \text{ even},$$

where

$$L_0 x(t) = x(t) , \quad L_k x(t) = a_k(t)\left(L_{k-1}x(t)\right)^{\bullet} , \quad k = 1, 2, \ldots, n ,$$

with $a_0(t) = a_n(t) = 1$, $a_i, q, g : [t_0, \infty) \to R$, $f : R \to R$ are continuous, $a_i(t) > 0$ $(i = 1, 2, \ldots, n-1)$, $q(t)$ nonnegative and not identically zero on any ray $[t^*, \infty)$, $t^* \ge t_0$ and $g(t) \to \infty$ as $t \to \infty$.

We assume that

$$(2)\quad \int^{\infty} \frac{1}{\mu_i(s)}\, ds = \infty \text{ , where } \mu_i(t) = \max_{t^*\le s\le t} a_i(s) \quad \text{for} \quad t \ge t^* \ge t_0$$

and, for $i = 1, 2, \ldots, n-1$,

$$(3)\qquad xf(x) > 0 \quad \text{and} \quad f'(x) \ge k > 0 \quad \text{for} \quad x \ne 0 \quad \left(' = \frac{d}{dx}\right) .$$

We further assume that there exists a real-valued function $\sigma \in C^1[[\,_0, \infty), (0, \infty))$ such that

$$\sigma(t) \le \inf_{s\ge t} \{s, g(s)\} ,$$

$$(4)\qquad \sigma^{\bullet}(t) > 0 ,$$

$$\sigma(t) \to \infty \quad \text{as} \quad t \to \infty .$$

The domain $D(L_n)$ of L_n is defined to be the set of all functions $x : [t_0, \infty) \to R$ such that $L_j x(t)$, $0 \le j \le n$, exist and are continuous on $[t_0, \infty)$. By a solution of (1) we mean a function $x \in D(L_n)$ which satisfies (1) on $[t_0, \infty)$. A nontrivial solution of (1) is called oscillatory if the set of its zeros is unbounded and it is called non-oscillatory otherwise.

The following lemma generalizes a well-known Kiguradze's lemma and can be proved similarly.

LEMMA 1. *Let condition* (2) *hold and let* $x \in D(L_n)$ *be a positive function. If* $L_n x(t)$ *is of constant sign and not identically zero for all large* t *, then there exist* $t_x \ge t_0$ *and an integer* l *,* $0 \le l \le n$ *, with* $n + l$ *even for* $L_n x$ *nonnegative or* $n + l$ *odd for* $L_n x$ *nonpositive and such that, for every* $t \ge t_x$ *,*

$$l > 0 \quad \textit{implies} \quad L_k x(t) > 0 \quad (k = 0, 1, \ldots, l-1)$$

and

$$l \le n - 1 \quad \textit{implies} \quad (-1)^{l+k} L_k x(t) > 0 \quad (k = l, l+1, \ldots, n-1) .$$

The following lemma appears in [3] and is needed in the sequel.

LEMMA 2. *Let $x \in D(L_n)$ with $x(t) > 0$ for $t \geq t_0$. If*

$$L_{n-1}x(t)L_n x(t) \leq 0 \text{ for all } t \geq t_1 \geq t_0 ,$$

t_1 is sufficiently large, then there exist $T \geq t_1$ and a positive constant M such that, for each $t \geq T$,

$$\dot{x}(t) \geq \frac{M}{\mu_1(t)} \omega(T, \mu, t) |L_{n-1}x(t)| ,$$

where

$$\omega(T, \mu, t) = \int_T^t \frac{1}{\mu_1(s_2)} \int_T^{s_2} \cdots \int_T^{s_{n-2}} \frac{1}{\mu_{n-1}(s_{n-1})} ds_{n-1} \cdots ds_2 .$$

THEOREM 1. *Let conditions* (2)-(4) *hold. Suppose that there exists a continuously differentiable function*

$$\rho : [t_0, \infty) \to (0, \infty)$$

such that for all sufficiently large T with $\sigma(T_1) > T$ for some $T_1 > T$ we have

$$\text{(5)} \qquad \limsup_{t\to\infty} \int_{T_1}^t \left[\rho(s)q(s) - \frac{\mu_1[\sigma(s)]\dot{\rho}^2(s)}{4Mk\rho(s)\dot{\sigma}(s)\omega(T,\mu,\sigma[s])}\right] ds = \infty ,$$

where M is as in Lemma 2. Then every solution of (1) *is oscillatory.*

Proof. Let $x(t)$ be a nonoscillatory solution of (1). Assume that $x(t) > 0$ for $t \geq t_1 \geq t_0$, and choose a $t_2 \geq t_1$ so that $\sigma(t) \geq t_1$ for $t \geq t_2$ and $x[\sigma(t)] > 0$ for $t \geq t_2$. By Lemma 1, there exists a $t_3 \geq t_2$ such that

$$\text{(6)} \qquad \dot{x}(t) > 0 \text{ and } L_{n-1}x(t) > 0 \text{ for all } t \geq t_3 .$$

Notice next that the hypotheses of Lemma 2 are satisfied on $[t_3, \infty)$ which implies that there exists a $t_4 \geq t_3$ and a positive constant M so that

$$\dot{x}(t) \geq \frac{M}{\mu_1(t)} \omega(t_4, \mu, t) L_{n-1}x(t) \text{ for all } t \geq t_4 .$$

Choose $t_5 \ge t_4$ so that $\sigma(t) > t_4$ for all $t \ge t_5$. Thus

$$(7)\qquad \dot{x}[\sigma(t)] \ge \frac{M}{\mu_1[\sigma(t)]}\,\omega\big(t_4,\,\mu,\,\sigma(t)\big)L_{n-1}x[\sigma(t)]$$

$$\ge \frac{M}{\mu_1[\sigma(t)]}\,\omega\big(t_4,\,\mu,\,\sigma(t)\big)L_{n-1}x(t)\ ,\ \text{for}\ \ t \ge t_5\ .$$

Let $W(t) = \rho(t)L_{n-1}x(t)/f\big(x[\sigma(t)]\big)$. Thus $W(t)$ satisfies

$$\dot{W}(t) = -\rho(t)q(t)\,\frac{f\big(x[g(t)]\big)}{f\big(x[\sigma(t)]\big)} + \frac{\dot{\rho}(t)}{\rho(t)}\,W(t) - \frac{\dot{\sigma}(t)\dot{x}[\sigma(t)]f'\big(x[\sigma(t)]\big)}{f\big(x[\sigma(t)]\big)}\,W(t)\ .$$

From (6) and (7) we have

$$(8)\qquad \dot{W}(t) \le -\rho(t)q(t) + \frac{\dot{\rho}(t)}{\rho(t)}\,W(t) - \frac{Mk\omega\big(t_4,\mu,\sigma(t)\big)\dot{\sigma}(t)}{\mu_1[\sigma(t)]\rho(t)}\,W^2(t)$$

$$= -\rho(t)q(t) + \frac{\mu_1[\sigma(t)]\dot{\rho}^2(t)}{4Mk\rho(t)\dot{\sigma}(t)\omega\big(t_4,\mu,\sigma(t)\big)} - \left[\left(\frac{Mk\omega\big(t_4,\mu,\sigma(t)\big)\dot{\sigma}(t)}{\mu_1[\sigma(t)]\rho(t)}\right)^{\frac{1}{2}} W(t) - \frac{\dot{\rho}(t)/\rho(t)}{2\big(Mk\omega\big(t_4,\mu,\sigma(t)\big)\dot{\sigma}(t)/\mu_1[\sigma(t)]\rho(t)\big)^{\frac{1}{2}}}\right]^2 .$$

Thus

$$(9)\qquad \dot{W}(t) \le -\rho(t)q(t) + \frac{\mu_1[\sigma(t)]\dot{\rho}^2(t)}{4Mk\rho(t)\dot{\sigma}(t)\omega\big(t_4,\mu,\sigma(t)\big)}\ .$$

Integrating (9) from t_5 to t we obtain

$$\int_{t_5}^{t}\left[\rho(s)q(s) - \frac{\mu_1[\sigma(s)]\dot{\rho}^2(s)}{4Mk\rho(s)\dot{\sigma}(s)\omega\big(t_4,\mu,\sigma(s)\big)}\right]ds \le W\big(t_5\big) - W(t)$$

$$\le W\big(t_5\big) < \infty\ ,$$

which contradicts (5) and the proof of the theorem is complete.

THEOREM 2. *Let conditions* (3) *and* (5) *of Theorem* 1 *be replaced by*

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$$\frac{f(x)}{x} \geq \gamma > 0 \quad \text{for} \quad x \neq 0 \tag{10}$$

and

$$\limsup_{t\to\infty} \int_{T_1}^{t} \left[\gamma\rho(s)q(s) - \frac{\mu_1[\sigma(s)]\dot{\rho}^2(s)}{4M\dot{\sigma}(s)\rho(s)\omega(T,\mu,\sigma(s))} \right] ds = \infty , \tag{11}$$

respectively. Then the conclusion of Theorem 1 *holds.*

Proof. The proof of Theorem 2 is similar to that of Theorem 1 except that we let $W(t) = \rho(t)L_{n-1}x(t)/x[\sigma(t)]$, and hence is omitted.

COROLLARY 1. *In Theorem* 1 (*respectively Theorem* 2), *let conditions* (5) (*respectively* (11)) *be replaced by*

$$\limsup_{t\to\infty} \int_{T_1}^{t} \rho(s)q(s)ds = \infty \tag{12}$$

and

$$\limsup_{t\to\infty} \int_{T_1}^{\infty} \frac{\mu_1[\sigma(s)]\dot{\rho}^2(s)}{\rho(s)\dot{\sigma}(s)\omega(T,\mu,\sigma(s))} ds < \infty . \tag{13}$$

Then the conclusion of Theorem 1 (*respectively Theorem* 2) *holds.*

THEOREM 3. *Let conditions* (2)-(4) *hold,* m *be an integer with* $m \geq 3$ *and* ρ *be a positive continuously differentiable function on the interval* $[t_0, \infty)$ *such that*

$$\limsup_{t\to\infty} \frac{1}{t^{m-1}} \int_{T_1}^{t} (t-s)^{m-3} \cdot \left[(t-s)^2\rho(s)q(s) - \frac{[(t-s)\dot{\rho}(s)-(m-1)\rho(s)]^2\mu_1[\sigma(s)]}{4Mk\rho(s)\dot{\sigma}(s)\omega(T,\mu,\sigma(s))} \right] ds = \infty , \tag{14}$$

for all large T *with* $\sigma(T_1) > T$ *for some* $T_1 > T$, *where* M *is as in Lemma* 2. *Then equation* (1) *is oscillatory.*

Proof. Let $x(t)$ be a non-oscillatory solution of (1), say $x(t) > 0$ for $t \geq t_1 \geq t_0$. As in the proof of Theorem 1, we obtain (8). Now we multiply both sides of (8) by $(t-s)^{m-1}$ and integrate from t_5 to t to obtain

$$\int_{t_5}^{t} (t-s)^{m-1}\rho(s)q(s)ds$$

$$\le \left(t-t_5\right)^{m-1}W\left(t_5\right) + \int_{t_5}^{t} (t-s)^{m-2}\left[(t-s)\frac{\dot\rho(s)}{\rho(s)} - (m-1)\right]W(s)ds$$

$$- \int_{t_5}^{t} (t-s)^{m-1}\frac{Mk\omega\left(t_4,\mu,\sigma(s)\right)\dot\sigma(s)}{\mu_1\left(\sigma(s)\right)\rho(s)}W^2(s)ds$$

$$= \left(t-t_5\right)^{m-1}W\left(t_5\right)$$

$$+ \int_{t_5}^{t} \frac{(t-s)^{m-3}[(t-s)\dot\rho(s)-(m-1)\rho(s)]^2\mu_1[\sigma(s)]}{4Mk\rho(s)\dot\sigma(s)\omega\left(t_4,\mu,\sigma(s)\right)}ds$$

$$- \int_{t_5}^{t} \left(\left[\frac{Mk(t-s)^{m-1}\omega\left(t_4,\mu,\sigma(s)\right)\dot\sigma(s)}{\mu_1[\sigma(s)]\rho(s)}\right]^{\frac12}W(s) - \frac{(t-s)^{m-2}[(t-s)\left(\dot\rho(s)/\rho(s)\right)-(m-1)]}{2\left[\left(Mk(t-s)^{m-1}\omega\left(t_4,\mu,\sigma(s)\right)\dot\sigma(s)\right)/\mu_1[\sigma(s)]\rho(s)\right]^{\frac12}}\right)^2 ds .$$

Thus

$$\frac{1}{t^{m-1}}\int_{t_5}^{t} (t-s)^{m-3}\left[(t-s)^2\rho(s)q(s) - \frac{[(t-s)\dot\rho(s)-(m-1)\rho(s)]^2\mu_1[\sigma(s)]}{4Mk\rho(s)\dot\sigma(s)\omega\left(t,\mu,\sigma(s)\right)}\right]ds$$

$$\le 1 - \left(t_5/t\right)^{m-1}\dot W\left(t_5\right) \to W\left(t_5\right) < \infty \quad \text{as} \quad t \to \infty ,$$

which contradict (14). A similar proof holds if $x(t) < 0$ for $t \ge t_1 \ge t_0$.

THEOREM 4. *In Theorem* 3, *let conditions* (3) *and* (14) *be replaced respectively by condition* (10) *and*

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$$\text{(15)}\quad \limsup_{t\to\infty} \frac{1}{t^{m-1}} \int_{T_1}^{t} (t-s)^{m-3} \cdot \left[\gamma(t-s)^2\rho(s)q(s) - \frac{[(t-s)\dot{\rho}(s)-(m-1)\rho(s)]^2\mu_1[\sigma(s)]}{4M\rho(s)\dot{\sigma}(s)\omega(t,\mu,\sigma(s))}\right] ds = \infty ,$$

then the conclusion of Theorem 3 *holds.*

Proof. The proof of Theorem 4 is similar to that of Theorem 3 except that we let $W(t) = \rho(t)L_{n-1}x(t)/x[\sigma(t)]$, and hence we omit it.

COROLLARY 2. *In Theorem* 3 *(respectively Theorem* 4*), let condition* (4) *(respectively* (11)*) be replaced by*

$$\text{(16)}\quad \limsup_{t\to\infty} \frac{1}{t^{m-1}} \int_{T_1}^{t} (t-s)^{m-1}\rho(s)q(s)ds = \infty$$

and

$$\text{(17)}\quad \limsup_{t\to\infty} \frac{1}{t^{m-1}} \int_{T_1}^{t} \frac{[(t-s)\dot{\rho}(s)-(m-1)\rho(s)]^2\mu_1[\sigma(s)]}{\rho(s)\dot{\sigma}(s)\omega(T,\mu,\sigma(s))} ds < \infty .$$

Then the conclusion of Theorem 1 *(respectively Theorem* 2*) holds.*

For illustration we consider the following examples.

EXAMPLE 1. The differential equation

$$\text{(18)}\quad \left(\frac{1}{t}\left(\frac{1}{t}\left((1/t)\dot{x}\right)^{\bullet}\right)^{\bullet}\right)^{\bullet} + \frac{231}{16t^7}x(t) = 0 , \quad t \ge 1 ,$$

has the nonoscillatory solution $x(t) = \sqrt{t}$. Only condition (5) of Theorem 1 is violated.

EXAMPLE 2. Consider the equation

$$\text{(19)}\quad \left(\frac{1}{t}\left(\frac{1}{t}\left((1/t)\dot{x}\right)^{\bullet}\right)^{\bullet}\right)^{\bullet} + \frac{1}{t^3}x[g(t)]\exp\left(\sin x[g(t)]\right) = 0 , \quad t \ge 1 ,$$

$g(t) = ct$ or $t \pm \sin t$. All conditions of Theorem 2 and Theorem 4 are satisfied for $\rho(t) = t^2$ and $\sigma(t) = ct$, $0 < c \le 1$ for the first case and $\sigma(t) = t - 1$ for the second. Hence all solutions of (19) are oscillatory.

EXAMPLE 3. Consider the equation

$$(t(t\ddot{x})^{\cdot})^{\cdot} + \frac{1}{t^2}x[g(t)] = 0 , \quad t \geq 1 , \tag{20}$$

where $g(t) = ct$ or $g(t) = t^c$ or $g(t) = t \pm c \cos t$, $c > 0$. All conditions of Corollaries 1 and 2 are satisfied for $\rho(t) = t$, $\sigma(t) \leq ct$ or t^c , $0 < c \leq 1$ or $t - c$ (respectively for the three cases) and $m = 3$, and hence all solutions of (20) are oscillatory. We note that Theorem 1 in [5] can be applied to (20), however in [5] only bounded solutions of (20) are discussed. Thus our results are more general than those in [5].

REMARK 1. The main results of Kamenev [4], Philos [6, 7], Wintner [8] and Yeh [9, 10] are included in our Corollary 2, for $n = 2$ and $g(t) = t$. Those criteria are not applicable to equations of the form

$$\ddot{x} + \frac{k^2}{t^2}x = 0 , \quad k > \tfrac{1}{2} ,$$

however Theorems 1 and 2 can be applied. Hence our results are a substantial improvement on the above mentioned results.

REMARK 2. The results obtained here are new, and we do not stipulate that the function g in equation (1) is either retarded or advanced. Hence our theorems hold for ordinary, retarded, advanced, and mixed type equations.

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