OSCILLATORY BEHAVIOR OF NONLINEAR DIFFERENTIAL EQUATIONS WITH DEVIATING ARGUMENTS

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New oscillation criteria for nonlinear differential equations with deviating arguments of the form

\[ \frac{d}{dt} \left[ a_{n-1}(t) \frac{d}{dt} \left[ \cdots \left[ a_2(t) \frac{d}{dt} \left[ a_1(t) \frac{d}{dx(t)} \right] \right] \cdots \right] \right] + q(t)f(x[g(t)]) = 0, \]

n even, are established.

1. Introduction

Recently Kamenev [4] considered the linear equation

\[ (\alpha_1) \quad \ddot{x} + q(t)x = 0 \quad \left( \dot{x} = \frac{d}{dt} \right), \]

where \( q \) is a continuous real-valued function on the interval \([t_0, \infty)\) without any restriction on its sign, and proved that the condition

\[ (\ast) \quad \limsup_{t \to \infty} \frac{1}{t^m-1} \int_{t_0}^{t} (t-s)^{-m-1} q(s)ds = \infty, \]

for some integer \( m \geq 3 \), is sufficient so that all the solutions of \((\alpha_1)\) are oscillatory.

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are oscillatory. His criterion includes as a special case the well-known Wintner result of [8]. Kamenev’s criterion has been extended in various directions by Philos [6, 7] and Yeh [9, 10]. The present authors [1, 2] discussed this criterion for general functional equations of the form

$$(a_2) \quad x^{(n)} + q(t)f[x[t], x[g(t)]] = 0, \quad n \text{ even},$$

and

$$(a_3) \quad [a(t)x^{(n-1)}]^\bullet + p(t)|x^{(n-1)}|^\beta + q(t)f[x[g(t)]] = 0, \quad n \text{ even and } \beta \geq 0,$$

with $p$ and $q$ nonnegative continuous functions on the interval $[t_0, \infty)$.

The purpose of this paper is to extend some of the previously mentioned results and obtain new oscillation criteria for the equation

$$(1) \quad L_n x(t) + q(t)f[x[g(t)]] = 0, \quad n \text{ even},$$

where

$L_0 x(t) = x(t), \quad L_k x(t) = a_k(t)[L_{k-1} x(t)]^\bullet, \quad k = 1, 2, \ldots, n,$

with $a_0(t) = a_n(t) = 1$.

2. Main results

Consider the equation

$$(1) \quad L_n x(t) + q(t)f[x[g(t)]] = 0, \quad n \text{ even},$$

where

$L_0 x(t) = x(t), \quad L_k x(t) = a_k(t)[L_{k-1} x(t)]^\bullet, \quad k = 1, 2, \ldots, n,$

with $a_0(t) = a_n(t) = 1$, $a_\ell$, $q$, $g : [t_0, \infty) \to R$, $f : R \to R$ are continuous, $a_\ell(t) > 0$ ($i = 1, 2, \ldots, n-1$), $q(t)$ nonnegative and not identically zero on any ray $[t^*, \infty)$, $t^* \geq t_0$ and $g(t) \to \infty$ as $t \to \infty$.

We assume that
Oscillatory behavior

\[ (2) \quad \int_{t^*}^{\infty} \frac{1}{\mu^*_t(s)} \, ds = \infty, \text{ where } \mu^*_t(t) = \max_{t^* \leq s \leq t} \alpha^*_t(s) \text{ for } t \geq t^* \geq t_0 \]

and, for \( i = 1, 2, \ldots, n-1 \),

\[ (3) \quad x^f(x) > 0 \text{ and } f'(x) \geq k > 0 \text{ for } x \neq 0 \quad \left( ' = \frac{d}{dx} \right). \]

We further assume that there exists a real-valued function \( \sigma \in C^1\left( [0, \infty), (0, \infty) \right) \) such that

\[ \sigma(t) \leq \inf_{s \geq t} \{ s, g(s) \}, \]

\[ \sigma(t) > 0, \]

\[ \sigma(t) \to \infty \text{ as } t \to \infty. \]

The domain \( D(L_n) \) of \( L_n \) is defined to be the set of all functions \( x : [t_0, \infty) \to \mathbb{R} \) such that \( L_j x(t) \), \( 0 \leq j \leq n \), exist and are continuous on \([t_0, \infty)\). By a solution of (1) we mean a function \( x \in D(L_n) \) which satisfies (1) on \([t_0, \infty)\). A nontrivial solution of (1) is called oscillatory if the set of its zeros is unbounded and it is called non-oscillatory otherwise.

The following lemma generalizes a well-known Kiguradze's lemma and can be proved similarly.

**Lemma 1.** Let condition (2) hold and let \( x \in D(L_n) \) be a positive function. If \( L_n x(t) \) is of constant sign and not identically zero for all large \( t \), then there exist \( t_x \geq t_0 \) and an integer \( l \), \( 0 \leq l \leq n \), with \( n + l \) even for \( L_n x \) nonnegative or \( n + l \) odd for \( L_n x \) nonpositive and such that, for every \( t \geq t_x \),

\[ l > 0 \text{ implies } L_k^l x(t) > 0 \quad (k = 0, 1, \ldots, l-1) \]

and

\[ l \leq n - 1 \text{ implies } (-1)^{l+k} L_k^l x(t) > 0 \quad (k = l, l+1, \ldots, n-1). \]

The following lemma appears in [3] and is needed in the sequel.
LEMMA 2. Let \( x \in D(L_n) \) with \( x(t) > 0 \) for \( t \geq t_0 \). If
\[
L_{n-1}x(t)L_nx(t) \leq 0 \quad \text{for all} \quad t \geq t_1 \geq t_0,
\]
t_1 is sufficiently large, then there exist \( T \geq t_1 \) and a positive constant \( M \) such that, for each \( t \geq T \),
\[
x^*(t) \geq \frac{M}{\mu_1(t)} \omega(T, \mu, t)|L_{n-1}x(t)|,
\]
where
\[
\omega(T, \mu, t) = \int_T^{t_1} \frac{1}{s_2} \int_T^{s_2} \ldots \int_T^{s_{n-2}} \frac{1}{s_{n-1}} ds_{n-1} \ldots ds_2.
\]

THEOREM 1. Let conditions (2)-(4) hold. Suppose that there exists a continuously differentiable function
\[
\rho : [t_0, \infty) \to (0, \infty)
\]
such that for all sufficiently large \( T \) with \( \sigma(t_1) > 2T \) for some \( T_1 > T \) we have
\[
\limsup_{t \to \infty} \int_{t_1}^{t} \left[ \rho(s)q(s) - \frac{\mu_1[\sigma(s)]\rho^2(s)}{4M\sigma(s)\rho(s)\omega(T, \mu, \sigma(s))} \right] ds = \infty,
\]
where \( M \) is as in Lemma 2. Then every solution of (1) is oscillatory.

Proof. Let \( x(t) \) be a nonoscillatory solution of (1). Assume that \( x(t) > 0 \) for \( t \geq t_1 \geq t_0 \), and choose a \( t_2 \geq t_1 \) so that \( \sigma(t) \geq t_1 \) for \( t \geq t_2 \) and \( x[\sigma(t)] > 0 \) for \( t \geq t_2 \). By Lemma 1, there exists a \( t_3 \geq t_2 \) such that
\[
x^*(t) > 0 \quad \text{and} \quad L_{n-1}x(t) > 0 \quad \text{for all} \quad t \geq t_3.
\]
Notice next that the hypotheses of Lemma 2 are satisfied on \([t_3, \infty)\) which implies that there exists a \( t_4 \geq t_3 \) and a positive constant \( M \) so that
\[
x^*(t) \geq \frac{M}{\mu_1(t)} \omega(t_4, \mu, t)L_{n-1}x(t) \quad \text{for all} \quad t \geq t_4.
\]
Choose \( t_5 \geq t_4 \) so that \( \sigma(t) > t^4 \) for all \( t \geq t_5 \). Thus

\[
\dot{x}(\sigma(t)) \geq \frac{M}{\mu_1[\sigma(t)]} \omega(t_4, \mu, \sigma(t)) L_{n-1} x[\sigma(t)]
\]

\[
\geq \frac{M}{\mu_1[\sigma(t)]} \omega(t_4, \mu, \sigma(t)) L_{n-1} x(t) , \text{ for } t \geq t_5 .
\]

Let \( W(t) = \rho(t) L_{n-1} x(t)/f[x(\sigma(t))] \). Thus \( W(t) \) satisfies

\[
\dot{W}(t) = -\rho(t) q(t) \frac{f[x(\sigma(t))]}{f'_x[\sigma(t)]} + \frac{\dot{\rho}(t)}{\rho(t)} W(t) - \frac{\dot{\sigma}(t) x(\sigma(t)) f'_x[\sigma(t)]}{f[x(\sigma(t))]} W(t) .
\]

From (6) and (7) we have

\[
\dot{W}(t) \leq -\rho(t) q(t) + \frac{\dot{\rho}(t)}{\rho(t)} W(t) - \frac{Mk \omega(t_4, \mu, \sigma(t)) \dot{\sigma}(t)}{\mu_1[\sigma(t)] \rho(t)} W^2(t) - \frac{\dot{\sigma}(t)[x(\sigma(t)) f'_x[\sigma(t)]]}{f[x(\sigma(t))]} W(t) - \frac{\dot{\rho}(t)/\rho(t)}{2(Mk \omega(t_4, \mu, \sigma(t)) \dot{\sigma}(t)/\mu_1[\sigma(t)] \rho(t))^\frac{1}{2}} .
\]

Thus

\[
\dot{W}(t) \leq -\rho(t) q(t) + \frac{\mu_1[\sigma(t)] \rho^2(t)}{4Mk \dot{\sigma}(t) \omega(t_4, \mu, \sigma(t))} .
\]

Integrating (9) from \( t_5 \) to \( t \) we obtain

\[
\int_{t_5}^{t} \left[ \rho(s) q(s) - \frac{\mu_1[\sigma(s)] \rho^2(s)}{4Mk \dot{\sigma}(s) \omega(t_4, \mu, \sigma(s))} \right] ds \leq W(t_5) - W(t) \leq W(t_5) < \infty ,
\]

which contradicts (5) and the proof of the theorem is complete.

**THEOREM 2.** Let conditions (3) and (5) of Theorem 1 be replaced by
(10) \[ \frac{f(x)}{x} \geq \gamma > 0 \text{ for } x \neq 0 \]

and

(11) \[ \limsup_{t \to \infty} \int_{T_1}^{t} \left[ \gamma \rho(s)q(s) - \frac{\mu_1[\sigma(s)]\dot{\rho}^2(s)}{4M\rho(s)\omega(T,\mu,\sigma(s))} \right] ds = \infty, \]

respectively. Then the conclusion of Theorem 1 holds.

Proof. The proof of Theorem 2 is similar to that of Theorem 1 except that we let \( W(t) = \rho(t)\frac{x(t)}{x[0(t)]} \), and hence is omitted.

COROLLARY 1. In Theorem 1 (respectively Theorem 2), let conditions (5) (respectively (11)) be replaced by

(12) \[ \limsup_{t \to \infty} \int_{T_1}^{t} \rho(s)q(s) ds = \infty \]

and

(13) \[ \limsup_{t \to \infty} \int_{T_1}^{\infty} \frac{\mu_1[\sigma(s)]\dot{\rho}^2(s)}{\rho(s)\dot{\sigma}(s)\omega(T,\mu,\sigma(s))} ds < \infty. \]

Then the conclusion of Theorem 1 (respectively Theorem 2) holds.

THEOREM 3. Let conditions (2)-(4) hold, \( m \) be an integer with \( m \geq 3 \) and \( \rho \) be a positive continuously differentiable function on the interval \([t_0, \infty)\) such that

(14) \[ \limsup_{t \to \infty} \frac{1}{t^{m-1}} \int_{T_1}^{t} (t-s)^{m-3} \]

\[ \cdot \left[ (t-s)^2 \rho(s)q(s) - \frac{[(t-s)^2\rho(s)-(m-1)\rho(s)]^2\mu_1[\sigma(s)]}{4M\rho(s)\dot{\sigma}(s)\omega(T,\mu,\sigma(s))} \right] ds = \infty, \]

for all large \( T \) with \( \sigma(T_1) > T \) for some \( T_1 > T \), where \( M \) is as in Lemma 2. Then equation (1) is oscillatory.

Proof. Let \( x(t) \) be a non-oscillatory solution of (1), say \( x(t) > 0 \) for \( t \geq t_1 \geq t_0 \). As in the proof of Theorem 1, we obtain (8). Now we
multiply both sides of (8) by \((t-s)^{m-1}\) and integrate from \(t_5\) to \(t\) to obtain

\[
\int_{t_5}^{t} (t-s)^{m-1} p(s)q(s)\,ds
\]

\[
\leq (t-t_5)^{m-1} \mathcal{W}(t_5) + \int_{t_5}^{t} (t-s)^{m-2}\left[\frac{\dot{\rho}(s)}{\rho(s)} - (m-1)\right] \mathcal{W}(s)\,ds
\]

\[
- \int_{t_5}^{t} (t-s)^{m-1} \frac{M\omega(t,u,\sigma(s))\sigma(s)}{\mu_1(\sigma(s))\rho(s)} \mathcal{W}^2(s)\,ds
\]

\[
= (t-t_5)^{m-1} \mathcal{W}(t_5)
\]

\[
+ \int_{t_5}^{t} \frac{(t-s)^{m-3}\left[(t-s)^2\hat{\rho}(s)-(m-1)\rho(s)\right]^2}{4Mk\rho(s)\sigma(s)}\,ds
\]

\[
- \int_{t_5}^{t} \left[\frac{Mk(t-s)^{m-1}\omega(t,u,\sigma(s))\sigma(s)}{\mu_1(\sigma(s))\rho(s)}\right]^\frac{1}{2} \mathcal{W}(s)
\]

\[
- \frac{(t-s)^{m-2}\left[(t-s)^2\hat{\rho}(s)/(\rho(s))-(m-1)\right]}{2\left[Mk(t-s)^{m-1}\omega(t,u,\sigma(s))\sigma(s)/\mu_1(\sigma(s))\rho(s)\right]^\frac{1}{2}}\,ds.
\]

Thus

\[
\frac{1}{t^{m-1}} \int_{t_5}^{t} (t-s)^{m-3}\left[(t-s)^2\rho(s)q(s) - \left[(t-s)^2\hat{\rho}(s)-(m-1)\rho(s)\right] \frac{\mu_1(\sigma(s))}{4Mk\rho(s)\sigma(s)}\right]\,ds
\]

\[
\leq 1 - \left(t_5/t\right)^{m-1} \mathcal{W}(t_5) + \mathcal{W}(t_5) < \infty \quad \text{as} \quad t \to \infty,
\]

which contradict (14). A similar proof holds if \(x(t) < 0\) for \(t \geq t_1 \geq t_0\).

**Theorem 4.** In Theorem 3, let conditions (3) and (14) be replaced respectively by condition (10) and
(15) \[ \limsup_{t \to \infty} \frac{1}{t^{m-1}} \int_{T_1}^{t} (t-s)^{m-3} \cdot \left[ (t-s)^2 \rho(s)q(s) - \frac{[(t-s)\rho(s)-(m-1)\rho(s)]^2 \mu_1[\sigma(s)]}{\rho(s)\sigma'(s)\omega(t,\mu,\sigma(s))} \right] ds = \infty, \]
	hen the conclusion of Theorem 3 holds.

**Proof.** The proof of Theorem 4 is similar to that of Theorem 3 except that we let \( W(t) = \rho(t)\frac{L_{n-1}}{x(t)/x(\sigma(t))} \), and hence we omit it.

**COROLLARY 2.** In Theorem 3 (respectively Theorem 4), let condition (16) (respectively (11)) be replaced by

(16) \[ \limsup_{t \to \infty} \frac{1}{t^{m-1}} \int_{T_1}^{t} (t-s)^{m-1} \rho(s)q(s) ds = \infty \]

and

(17) \[ \limsup_{t \to \infty} \frac{1}{t^{m-1}} \int_{T_1}^{t} \frac{[(t-s)\rho(s)-(m-1)\rho(s)]^2 \mu_1[\sigma(s)]}{\rho(s)\sigma'(s)\omega(t,\mu,\sigma(s))} ds < \infty. \]

Then the conclusion of Theorem 1 (respectively Theorem 2) holds.

For illustration we consider the following examples.

**EXAMPLE 1.** The differential equation

(18) \[ \frac{1}{t} \left( \frac{1}{t} [(1/t)\dot{x}]^* \right)^* + \frac{23}{16t^2} x(t) = 0, \quad t \geq 1, \]

has the nonoscillatory solution \( x(t) = \sqrt{t} \). Only condition (5) of Theorem 1 is violated.

**EXAMPLE 2.** Consider the equation

(19) \[ \frac{1}{t} \left( \frac{1}{t} [(1/t)\dot{x}]^* \right)^* + \frac{1}{t^3} x[g(t)] \exp(\sin x[g(t)]) = 0, \quad t \geq 1, \]

g(t) = ct \quad \text{or} \quad t \pm \sin t. \quad \text{All conditions of Theorem 2 and Theorem 4 are satisfied for} \quad \rho(t) = t^2 \quad \text{and} \quad \sigma(t) = ct, \quad 0 < c \leq 1 \quad \text{for the first case and} \quad \sigma(t) = t - 1 \quad \text{for the second. Hence all solutions of (19) are oscillatory.}
EXAMPLE 3. Consider the equation

\[(t(x')')' + \frac{1}{t^2} x[g(t)] = 0 , \quad t \geq 1 ,\]

where \(g(t) = ct\) or \(g(t) = t^\alpha\) or \(g(t) = t \pm c \cos t , \quad c > 0\). All conditions of Corollaries 1 and 2 are satisfied for \(\rho(t) = t\), \(\sigma(t) \leq ct\) or \(t^\alpha\), \(0 < c \leq 1\) or \(t - c\) (respectively for the three cases) and \(m = 3\), and hence all solutions of (20) are oscillatory. We note that Theorem 1 in [5] can be applied to (20), however in [5] only bounded solutions of (20) are discussed. Thus our results are more general than those in [5].

REMARK 1. The main results of Kamenev [4], Philos [6, 7], Wintner [8] and Yeh [9, 10] are included in our Corollary 2, for \(n = 2\) and \(g(t) = t\). Those criteria are not applicable to equations of the form

\[\ddot{x} + \frac{k^2}{t^2} x = 0 , \quad k > \frac{1}{2} ,\]

however Theorems 1 and 2 can be applied. Hence our results are a substantial improvement on the above mentioned results.

REMARK 2. The results obtained here are new, and we do not stipulate that the function \(g\) in equation (1) is either retarded or advanced. Hence our theorems hold for ordinary, retarded, advanced, and mixed type equations.

References


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