Theorems in connection with lines drawn through a pair of points parallel and antiparallel to the sides of a triangle.

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Figure 4.

1. *Parallel to sides.*

(*a*) Let O and O′ be two reciprocal points,

x, y, z, the $\perp^{rs}$ from O on the sides,

x', y', z', ,, ,, ,, O′ ,, ,, ,, ;

DE(α), FG(β), HI(γ) the intercepts made on the sides by lines drawn parallel to the sides through O;

D′E′(α'), F′G′ (β'), H′I′ (γ') similar intercepts for O′.

Then the six triangles

OD′E′, O′DE, OF′G′, O′FG, OH′I′, O′HI are equal.

$\alpha^2 : a^2 :: \triangle\text{ODE} : \triangle\text{ABC}$ or 2S, and $\alpha . x = 2\triangle\text{ODE}$.

$$\therefore \quad \alpha = \frac{a^2x}{4S}, \quad \beta = \frac{b^2y}{4S}; \quad \alpha' = \frac{a^2x'}{4S}, \text{ etc.}$$

Let AO cut BC in M, AO′ in M′,

then BM = M′C, BM′ = MC (by definition of reciprocal points).

$$\begin{aligned} \text{BM} : \text{MC} &= \triangle\text{AOB} : \triangle\text{AOC} \\ &= cz : by. \\ \text{BM}' : \text{M}'\text{C} &= cz' : by'. \end{aligned}$$

$$\therefore \quad cz : by :: by' : cz'.$$

$$\therefore \quad c^2zz' = b^2yy' = a^2xx'.$$

But $\alpha x' = \dfrac{a^2xx'}{4S}$, etc.

$$\therefore \quad \alpha x' = \alpha' x = \beta y' = \beta' y = \gamma z' = \gamma' z.$$

But $\alpha x' = 2\triangle\text{O}'\text{DE}$, etc.

$\therefore$ the six triangles are equal.

(*b*) Since the perpendiculars from the incentre are equal, the intercepts cut off by parallels through the reciprocal point to the incentre must be equal.

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FIGURE 5.

2. *Antiparallel to sides.*

(*a*) Let O and O′ be two points, and let the intercepts cut off be by lines drawn antiparallel to the sides.

$\triangle$ODE is isosceles.

$$\therefore \quad \frac{a}{2x} = \cot ODE = \cot A.$$

$$\therefore \quad \alpha = 2x\cot A, \quad \beta = 2y\cot B, \quad \text{etc.}$$

$$\alpha' = 2x'\cot A, \quad \text{etc.}$$

$$\therefore \quad \alpha x' = 2xx'\cot A = \alpha' x \quad \text{or} \quad \triangle O'DE = \triangle OD'E',$$

$$\beta y' = 2yy'\cot B = \beta' y \quad \text{or} \quad \triangle O'FG = \triangle OF'G',$$

$$\gamma z' = 2zz'\cot C = \gamma' z \quad \text{or} \quad \triangle O'HI = \triangle OH'I';$$

and the six triangles will be equal if the conditions

$$xx'\cot A = yy'\cot B = zz'\cot C \text{ are fulfilled,}$$

$$\text{or if} \quad xx' : yy' : zz' \quad :: \quad \tan A : \tan B : \tan C.$$

In analogy to parallels and antiparallels such a pair of points might be called antireciprocal points.

Now if O be the orthocentre $x : y : z \quad :: \quad \frac{1}{\cos A} : \frac{1}{\cos B} : \frac{1}{\cos C}$,

and if O′ be the symmedian point $x' : y' : z' \quad :: \quad \sin A : \sin B : \sin C$. Hence for these two points the six triangles are equal.

(*b*) The intercepts made on the sides by lines drawn antiparallel to the sides will be equal, if the point through which they are drawn has its perpendiculars on the sides in the ratio

$$\tan A : \tan B : \tan C.$$

If $\alpha = \beta = \gamma$, then $x\cot A = y\cot B = z\cot C$,

$$\therefore \quad x : y : z = \frac{1}{\cot A} : \frac{1}{\cot B} : \frac{1}{\cot C}$$

$$= \tan A : \tan B : \tan C.$$

Hence such a point might be called antireciprocal point to the incentre.

On an instrument for trisecting any angle.*

By JAS. N. MILLER.

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