

On the asymptotic periods of integral functions

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§ 1. A *period* of a function $f(z)$ is defined to be a number $\omega (\neq 0)$ such that

$$(1.1) \quad \Delta_\omega(z) = f(z + \omega) - f(z)$$

is identically zero; and it is not a difficult matter to show that an integral function may either have no periods or else a single sequence $k\lambda$, $k = \pm 1, \pm 2, \dots$

The question may be asked "Are these the only values of ω for which $\Delta_\omega(z)$ can be 'smaller' than $f(z)$, or can there be other values of ω for which $\Delta_\omega(z)$ is 'smaller' than $f(z)$, but is not identically zero?" The term "smaller" needs precise definition. If $f(z)$ is an integral or meromorphic function it is most natural to interpret it as meaning "of lower order," and it is in this sense that the word will be understood. With this interpretation it is easy to see that in fact there may be more than a single sequence $k\lambda$ of values of ω for which $\Delta_\omega(z)$ is smaller than $f(z)$. For example the integral function $R(z)$ defined by

$$(1.2) \quad R(z) = \sum_{n=1}^{\infty} \frac{e^{2\pi i n! z}}{n!!}$$

is of infinite order, but if p is any rational number

$$R(z + p) - R(z)$$

is of order 1.

Numbers with the property under discussion will be called *asymptotic periods*. Their definition is then as follows:

DEF. A number $\beta (\neq 0)$ is an *asymptotic period* of an integral or meromorphic function $f(z)$ if $\Delta_\beta(z)$ is of lower order than $f(z)$. The set of all asymptotic periods of $f(z)$ is denoted by B .

This definition includes the case of a function $f(z)$ of infinite order. $\Delta_\beta(z)$ must then be of finite order. It is evident that, if β, γ are asymptotic periods, then $-\beta, \beta + \gamma$ have the same property. The following result is proved.

If $f(z)$ is an integral function, B is a linear set of measure zero.¹

An important part in the proof is played by an extension of Guichard's theorem, that every integral function has a finite sum. As this result is of independent interest it is discussed at the beginning, instead of in its proper place in the proof of the main theorem.

Another part of the proof consists in showing that a function of order < 1 cannot have any asymptotic periods. More generally, this is true if

$$(1.3) \quad \frac{\log M(r)}{r} \rightarrow 0.$$

§ 2. *The finite sum of an integral function.* It was proved by Guichard² that if $f(z)$ is any given integral function and $\omega \neq 0$, an integral function $g(z)$ exists with the property

$$(2.1) \quad g(z + \omega) - g(z) = f(z).$$

Guichard's method is not well adapted to elucidate the relation between the orders of $f(z)$, $g(z)$, an essential matter for the application which we have in view; but by an extension of the alternative method by which Appell and Hurwitz established (2.1) it is possible to prove the following result.

THEOREM I. *If $f(z)$ is an integral function of finite order ρ , there is an integral function $g(z)$, of order $\leq \max(1, \rho)$, for which (2.1) is satisfied.*

There is no loss of generality in taking $\omega = 1$. Let

$$(2.2) \quad f(z) = \sum_{n=0}^{\infty} a_n z^n.$$

The function

$$(2.3) \quad \psi_{n,s}(z) = \frac{n!}{2\pi i} \int_{C_s} \frac{e^{zt}}{(e^t - 1)^{n+1}} dt,$$

when C_s is the circle $|t| = (2s + 1)\pi$ and s is an integer, has the property³

¹ *Added in proof.* B may be a non-denumerable set. See *Journal London Math. Soc.*, 8 (1933), 119-125.

² Guichard, 1. See Nörlund, 2, for an account of Guichard's work and also of the work of Appell and Hurwitz mentioned below.

³ Nörlund, 2, 8. $\Delta f(z) = f(z + 1) - f(z)$.

$$(2.4) \quad \Delta \psi_{n,s}(z) = nz^{n-1}, \quad (n = 1, 2, \dots),$$

so that a solution of (2.1) (with $\omega = 1$) is

$$(2.5) \quad g(z) = \sum_{n=0}^{\infty} \frac{a_n}{n+1} \psi_{n+1,s_n}(z),$$

provided that s_1, s_2, \dots can be chosen in such a way that the series converges uniformly in any finite region.

If $\rho < 1$ it is sufficient to take $s_1 = s_2 = \dots = 0$, and the order of $g(z)$ is then ≤ 1 . For, let $\rho^{-1} > \sigma > 1$, so that¹

$$(2.6) \quad |a_n| < n^{-n\sigma}, \quad (n \geq n_0).$$

Now, writing $t = (2s + 1)\pi e^{i\theta}$,

$$(2.7) \quad \begin{aligned} |\psi_{n,s}(z)| &\leq \frac{n!}{2\pi} \int_0^{2\pi} \frac{e^{(2s+1)\pi|z|} (2s+1)\pi d\theta}{|e^t - 1| \{(2s+1)\pi\}^n} \\ &< An! e^{(2s+1)\pi(|z|+1)} \{(2s+1)\pi\}^{1-n}, \end{aligned}$$

where A is independent of n, s, z . Hence

$$\sum_{n=0}^{\infty} \left| \frac{a_n \psi_{n+1,0}(z)}{n+1} \right| < K \sum_{n=1}^{\infty} n^{n(1-\sigma)} e^{\pi|z|} < Ke^{\pi|z|}.$$

K always denotes a constant, but not always the same constant. Denoting by $M_0(r)$ the maximum of $|g(z)|$ for $|z| = r$, the last inequality shows that

$$(2.8) \quad \overline{\lim}_{r \rightarrow \infty} \frac{\log M_0(r)}{r} \leq \pi,$$

so that $g(z)$ is of order ≤ 1 .

If $\rho \geq 1$, consider the function $\sigma(n)$ defined by

$$(2.9) \quad \sigma(n) = \min_{m \leq n} \left(\frac{\log |a_m|^{-1}}{m \log m} \right).$$

$\sigma(n)$ is an increasing function, and

$$(2.10) \quad \sigma(n) \rightarrow \lim_{n \rightarrow \infty} \frac{\log |a_n|^{-1}}{n \log n} = \frac{1}{\rho}.$$

Again, writing $\alpha(n) = 1 - \sigma(n)$,

$$(2.11) \quad n^{\alpha(n)} \{(2s_n + 1)\pi\}^{-n} < e^{-n}, \quad (n \geq 1),$$

provided that

$$(2s_n + 1)\pi > en^{\alpha(n)},$$

and this is so if

$$(2.12) \quad s_n = [n^{\alpha(n)}].$$

¹ Valiron, 4, 40.

It will be shown that with this choice of s_1, s_2, \dots , (2.5) defines an integral function of order ρ .

Corresponding to a given value of $r = |z|$ define the integer

$$(2.13) \quad N(r) = [(8r)^{1/\sigma(2)}].$$

Then (2.7), (2.9), (2.11), (2.12) show that

$$(2.14) \quad \begin{aligned} v_n(z) &= \left| \frac{a_n \psi_{n+1, s_n}(z)}{n+1} \right| \\ &< n^{-n\sigma(n)} n^n \{(2s_n + 1)\pi\}^{-n} \exp\{(2s_n + 1)\pi r\} \\ &< \exp\{-n + 7n^{\alpha(n)} r\} \\ &< e^{-n/8}, \quad (n > N), \end{aligned}$$

since

$$\begin{aligned} n - 7n^{\alpha(n)} r - \frac{n}{8} &> \frac{7}{8} n - \frac{7}{8} n^{\alpha(n) + \sigma(2)} \\ &= \frac{7}{8} n \{1 - n^{\sigma(2) - \sigma(n)}\} \geq 0. \end{aligned}$$

Now take a fixed integer p and write $a(p) = a$. Then

$$(2.15) \quad \begin{aligned} \sum_{n=p}^N v_n(z) &< \sum_{n=p}^N \exp\{-n + 7n^{\alpha(n)} r\} \\ &\leq \sum_{n=p}^N \exp\{-n + 7n^{\alpha} r\} \\ &< N(r) \mu(r), \end{aligned}$$

where $\mu(r)$ is the greatest term of the last series. $\mu(r)$ is determined by finding the maximum value of the function

$$-x + 7x^{\alpha} r,$$

namely

$$\left(\frac{1}{\alpha} - 1\right) (7\alpha r)^{1/(1-\alpha)},$$

so that

$$(2.16) \quad \mu(r) \leq \exp\left\{\left(\frac{1}{\alpha} - 1\right) (7\alpha r)^{1/(1-\alpha)}\right\}.$$

Now, making use of (2.14), (2.15),

$$M_0(r) < \sum_{n=0}^p v_n(z) + N(r) \mu(r) + \sum_{n=N+1}^{\infty} e^{-n/8},$$

whence, by (2.13), (2.16),

$$\overline{\lim}_{r \rightarrow \infty} \frac{\log \log M_0(r)}{\log r} \leq (1 - \alpha)^{-1} = \{\sigma(p)\}^{-1}.$$

Finally, make p tend to infinity, and use (2.10).

It follows that $g(z)$ is of order $\leq \rho$. On the other hand it is evident that $g(z)$ is of order $\geq \rho$, so that $g(z)$ must be of order ρ .

§ 3. THEOREM 2. *An integral function has no asymptotic periods if*

$$(3.1) \quad \overline{\lim}_{r \rightarrow \infty} \frac{\log M(r)}{r} = 0,$$

or if

$$(3.2) \quad \lim_{r \rightarrow \infty} \frac{(\log r)^2 \log M(r)}{r} = 0.$$

The importance of a hypothesis such as (3.2), involving a lower limit instead of an upper limit, will be seen at a later stage of the proof of the main theorem.

We need three lemmas of Phragmén-Lindelöf type.

LEMMA 1. *An integral function is a constant if it satisfies the conditions*

$$(3.3) \quad \overline{\lim}_{r \rightarrow \infty} \frac{\log M(r)}{\sqrt{r}} = 0,$$

$$(3.4) \quad |f(-x)| \leq C, \quad (x \geq 0).$$

This result is due to Pólya and Szegő.¹

LEMMA 2. *In Lemma 1 (3.3) may be replaced by*

$$(3.5) \quad \lim_{r \rightarrow \infty} \frac{(\log r)^2 \log M(r)}{\sqrt{r}} = 0.$$

The function

$$h(z) = f(z) \exp\left(-\frac{\sigma z^{\frac{1}{2}}}{\log z}\right), \quad (\sigma > 0),$$

is regular in the domain D defined by

$$100 < r < R, \quad -\pi < \theta < \pi, \quad (z = re^{i\theta}).$$

¹ Pólya and Szegő, 3, 149.

$z^{\frac{1}{2}}/\log z$ is taken to be the branch which is real and positive on the real positive axis. Now

$$\left| \exp\left(-\frac{\sigma z^{\frac{1}{2}}}{\log z}\right) \right| = \exp\left\{-\frac{\sigma\sqrt{r}(\log r \cos \frac{1}{2}\theta + \theta \sin \frac{1}{2}\theta)}{(\log r)^2 + \theta^2}\right\},$$

and

$$\left. \begin{aligned} \cos \frac{1}{2}\theta \log r + \theta \sin \frac{1}{2}\theta &\geq \frac{1}{\sqrt{2}} \log r, & \left(0 \leq \theta \leq \frac{\pi}{2}\right), \\ &\geq \frac{\pi}{2\sqrt{2}}, & \left(\frac{\pi}{2} \leq \theta \leq \pi\right), \end{aligned} \right\}$$

so that in D

$$|h(z)| \leq |f(z)| \exp\left\{-\frac{\sigma\sqrt{r}}{2(\log r)^2}\right\}.$$

By (3.5), R may be chosen so that

$$\log M(R) < \frac{\sigma\sqrt{R}}{2(\log R)^2}.$$

Hence, by the maximum modulus principle,

$$|h(z)| \leq \max\{C, 1, M(100)\}$$

for all points z in D . Finally, keeping z fixed, let $\sigma \rightarrow 0$. It follows that, for all values of z ,

$$|f(z)| \leq \max\{C, 1, M(100)\},$$

and hence $f(z)$ is a constant.

LEMMA 3. *If an integral function $f(z)$ satisfies either (3.3) or (3.5) and*

$$(3.6) \quad |f(-x)| \leq Ce^{2\alpha x}, \quad (0 < \alpha < \frac{1}{2}), \quad (x \geq 0),$$

there is a constant A such that for all values of z

$$(3.7) \quad |f(z)| \leq Ae^{r^\alpha}.$$

It is only necessary to apply the arguments of the preceding lemmas to the function

$$f(z) \exp\{-(-z)^\alpha\}.$$

Theorem 2 is a simple deduction. Suppose, if possible, that a function $f(z)$ of order ρ satisfies either (3.1) or (3.2) and has an asymptotic period β , so that $\Delta_\beta(z)$ is of order $\sigma < \rho$. In the former case $\rho \leq 1$, but in the latter ρ may be any finite number or infinity. There is no loss of generality in taking β real and positive. Moreover we may suppose that the zeros a_1, a_2, \dots , of $f(z)$ have ρ as

exponent of convergence. This can always be secured by adding a suitable constant to $f(z)$. Then, if $0 \leq t \leq \beta$,

$$f(t + n\beta) = f(t) + \sum_{s=0}^{n-1} \Delta_\beta(t + s\beta),$$

$$f(t - n\beta) = f(t) - \sum_{s=1}^n \Delta_\beta(t - s\beta),$$

so that

$$|f(t \pm n\beta)| \leq M(\beta) + nM_\beta\{(1 + n)\beta\},$$

when $M_\beta(r)$ denotes the maximum modulus of $\Delta_\beta(z)$. Hence if $\sigma < 2\tau < \rho$ and $x \geq 0$,

$$|f(\pm x)| \leq M(\beta) + \frac{2x}{\beta} M_\beta(2x) < Ke^{2x\tau},$$

and so the function $h(z)$ defined by

$$h(z^2) = f(z)f(-z)$$

is such that

$$|h(x)| = |f(\sqrt{x})f(-\sqrt{x})| < K^2 e^{2x\tau}, \quad (x \geq 0).$$

Now $h(z)$ satisfies (3.3) or (3.5) according as $f(z)$ satisfies (3.1) or (3.2), and it follows from Lemma 3 that the order of $h(z)$ cannot exceed τ .

On the other hand the zeros of $h(z)$ are a_1^2, a_2^2, \dots and these have exponent of convergence $\rho/2$. Hence the order of $h(z)$ cannot be less than $\rho/2$. The contradiction implies that $f(z)$ has no asymptotic periods.

§ 4. LEMMA 4. *The ratio of two asymptotic periods of an integral function is real.*

If possible, let β, γ be asymptotic periods whose ratio is not real. It follows from the identity

$$f(z + k\beta + l\gamma) = f(z) + \sum_{s=0}^{l-1} \Delta_\gamma(z + s\gamma) + \sum_{t=0}^{k-1} \Delta_\beta(z + l\gamma + t\beta)$$

that if $|z| = r$,

$$f(z) = f(z_0) + \sum_s \Delta_\gamma(z_s) + \sum_t \Delta_\beta(z'_t),$$

where z_0 is in a "period parallelogram" with the origin as one vertex, z_1, z_2, \dots are points inside the circle $|z| = r$, each in a different

“period parallelogram,” and the same is true of z_1', z_2', \dots . In each case the number of points does not exceed $K(\beta, \gamma)r$. Hence

$$M(r) \leq M(|\beta| + |\gamma|) + K(\beta, \gamma)r\{M_\beta(r) + M_\gamma(r)\},$$

and this is evidently incompatible with the hypothesis that $\Delta_\beta(z), \Delta_\gamma(z)$ are of lower order than $f(z)$.

§ 5. LEMMA 5. Let $f_1(z), f_2(z)$ be periodic integral functions of order ρ , whose periods $\omega, \omega/\mu$ are real, and let

$$g(z) = f_1(z) - f_2(z), \quad z = x + iy.$$

If, for some $\tau < \rho$,

$$(5.1) \quad |g(x + iy)| < Ke^{|y|^\tau}, \quad (|y| \geq a),$$

where K is independent of x , then μ is a rational number.

ρ may be ∞ . There is no loss of generality in taking $\omega = 2\pi$. If μ is not rational, consider the function $f_1(-i \log w)$, any branch of the logarithm being chosen. This function is regular and uniform in every annulus $0 < R \leq |w| \leq R'$, and so can be expanded in a Laurent series

$$\sum_{n=0}^{\infty} a_n w^n + \sum_{n=1}^{\infty} b_n w^{-n},$$

so that, writing $z = -i \log w$,

$$(5.2) \quad f_1(z) = \sum_{n=0}^{\infty} a_n e^{niz} + \sum_{n=1}^{\infty} b_n e^{-niz}.$$

In the same way

$$(5.3) \quad f_2(z) = \sum_{n=0}^{\infty} c_n e^{n\mu iz} + \sum_{n=1}^{\infty} d_n e^{-n\mu iz}.$$

As $f_1(z)$ is of order ρ at least one of the series in (5.2), say the first, is an integral function of order ρ .

Now, as μ is irrational, the exponentials in (5.2), (5.3) are all distinct, and so

$$a_n e^{niz} = \lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T e^{ni(x-t)} g(t + iy) dt,$$

and thus, if $y \geq 0$,

$$(5.4) \quad \begin{aligned} |a_n| e^{ny} &\leq \overline{\lim}_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T |g(t + iy)| dt \\ &\leq G(y), \end{aligned}$$

where

$$G(y) = \max_{-\infty < x < \infty} |g(x + iy)|.$$

By (5.1),

$$(5.5) \quad G(y) < Ke^{y^\tau}, \quad (y \geq a).$$

Now the function

$$F(w) = \sum_{n=0}^{\infty} a_n w^n, \quad w = r_1 e^{i\phi},$$

is an integral function. Let $A(r_1)$ denote its maximum modulus,

$$A(r_1) = \max_{|w|=r_1} |F(w)|,$$

γ its order, and

$$\alpha(r_1) = \max_{n \geq 0} |a_n| r_1^n$$

its maximum term. (5.4) shows that

$$(5.6) \quad \alpha(e^y) \leq G(y), \quad (y \geq 0).$$

Moreover if $M_0(r)$ denotes the maximum modulus of $F(e^{iz})$,

$$(5.7) \quad M_0(y) \leq A(e^y), \quad (y \geq 0).$$

For $M_0(y)$ is the maximum of $|F(w)|$ on the curve

$$(\log r_1)^2 + \phi^2 = y^2, \quad (-\pi \leq \phi \leq \pi),$$

in the w -plane, and this lies inside the curve

$$\log r_1 = y.$$

Now if γ is finite,¹

$$(5.8) \quad \frac{\log \alpha(r)}{\log A(r)} \rightarrow 1, \quad \text{as } r \rightarrow \infty,$$

and so by (5.6), (5.7),

$$\lim_{y \rightarrow \infty} \frac{\log G(y)}{\log M_0(y)} \geq 1.$$

Moreover

$$\overline{\lim}_{y \rightarrow \infty} \frac{\log \log M_0(y)}{\log y} = \rho.$$

Hence

$$\overline{\lim}_{y \rightarrow \infty} \frac{\log \log G(y)}{\log y} \geq \rho > \tau,$$

and this contradicts (5.5).

Again, if γ is infinite,²

$$(5.9) \quad \overline{\lim}_{r \rightarrow \infty} \frac{\log \log \alpha(r)}{\log r} = \infty,$$

whence by (5.6)

$$\overline{\lim}_{y \rightarrow \infty} \frac{\log \log G(y)}{y} = \infty,$$

which likewise contradicts (5.5).

¹ Valiron, 4, 32.

² Cf. Valiron 4. (5.9) is not explicitly stated but follows readily from equation (2.9) on p. 31.

§ 6. *Functions of order > 1.* It follows from Theorem 2 that an integral function of order < 1 cannot have any asymptotic periods. It is convenient to take next the case of a function of order $\rho > 1$ (including the case $\rho = \infty$).

By Lemma 4 the asymptotic periods all lie on a line through the origin, say the real axis. Let B_σ ($\sigma < \rho$), denote the set of real numbers β such that $\Delta_\beta(z)$ is of order $\leq \sigma$. The sets B_σ form an increasing sequence with limit B .

Take a fixed value of $\sigma > 1$. If B_σ is not null, let β be a member of it. In accordance with Theorem 1 $\Delta_\beta(z)$ can be expressed as the β -difference of a function $g_1(z)$ of order $\leq \sigma$, so that

$$f(z + \beta) - f(z) = g_1(z + \beta) - g_1(z).$$

Thus the function

$$(6.1) \quad f_1(z) = f(z) - g_1(z)$$

is periodic, of period β , and of order ρ . Write

$$F_1(y) = \max_{0 \leq x, x' \leq \beta} |f_1(x + iy) - f_1(x' + iy)|,$$

and choose τ so that $\sigma < \tau < \rho$. There are two cases to consider. Either

$$(6.2) \quad F_1(y) \leq e^{|y|^\tau}, \quad (|y| \geq a),$$

or

$$(6.3) \quad F_1(y) > e^{|y|^\tau}, \text{ for arbitrarily large values of } |y|.$$

Take the first case. Associated with γ , any other member of B_σ , there are functions $g_2(z), f_2(z), F_2(y)$ corresponding to $g_1(z), f_1(z), F_1(y)$. $f_2(z)$ has period γ , order ρ , and

$$F_2(y) = \max_{0 \leq x, x' \leq \gamma} |f_2(x + iy) - f_2(x' + iy)|.$$

It will be shown that (6.2) implies that

$$(6.4) \quad F_2(y) \leq 2e^{|y|^\tau}, \quad (|y| \geq b).$$

For consider the function $g(z)$, of order $\leq \sigma$, defined by

$$(6.5) \quad g(z) = f_1(z) - f_2(z) = g_2(z) - g_1(z).$$

If (6.4) is false, then corresponding to a given value of K , numbers x, x', y can be found such that

$$0 \leq x, x' \leq \max(\beta, \gamma), \quad |y| > K,$$

and

$$\begin{aligned} |g(x + iy) - g(x' + iy)| &\geq |f_2(x + iy) - f_2(x' + iy)| \\ &\quad - |f_1(x + iy) - f_1(x' + iy)| \\ &> e^{|y|^\tau}. \end{aligned}$$

Hence, writing

$$M_0(r) = \max_{|z|=r} |g(z)|,$$

$$2M_0(2|y|) \geq |g(x + iy)| + |g(x' + iy)| > e^{|\nu|\tau},$$

and this is impossible if K is sufficiently large, as the order of $g(z)$ does not exceed σ .

(6.4) being thus established, it follows that if x, x' are any real numbers then

$$\begin{aligned} |g(x + iy) - g(x' + iy)| &\leq |f_2(x + iy) - f_2(x' + iy)| + |f_1(x + iy) - f_1(x' + iy)| \\ &\leq F_2(y) + F_1(y) \\ &\leq 3e^{|\nu|\tau}, \quad (|y| \geq c), \end{aligned}$$

and so if $-\infty < x < \infty$,

$$\begin{aligned} |g(x + iy)| &\leq |g(iy)| + 3e^{|\nu|\tau} \\ &< 4e^{|\nu|\tau}, \quad (|y| \geq d), \end{aligned}$$

since $g(z)$ is of order $\leq \sigma < \tau$.

It now follows from Lemma 5 that γ is a rational multiple of β .

The alternative case, expressed by (6.3), is dealt with by a different argument.

Let N be a "large" positive integer and let $\sigma < \lambda < \tau$. At least one of $\pm \infty$, say ∞ , is a limit point of the numbers y for which (6.3) is satisfied. Hence there is a sequence y_1, y_2, \dots , tending to ∞ , such that

$$(6.6) \quad F_1(y_p) > \exp(y_p^\tau) \quad (p = 1, 2, \dots),$$

and also

$$(6.7) \quad y_p^\tau > (N + p + 1) y_p^\lambda.$$

Corresponding to y_p , a member of this sequence, let x_1, x_2 be points of $(0, \beta)$ for which

$$|f_1(x_1 + iy_p) - f_1(x_2 + iy_p)| = F_1(y_p).$$

Mark in the f_1 -plane the values of $f_1(z)$ corresponding to values of $z = x + iy_p$, $0 \leq x \leq \beta$. The resulting curve is continuous and closed, since $f_1(z)$ has period β , and is contained in the circle of centre $f_1(x_1 + iy_p)$, radius $F_1(y_p)$. Now divide this circle into annuli by drawing concentric circles of radii $sF_1(y_p)/S$, ($s = 1, 2, \dots, S - 1$),

where $S = 2^{N+p}$, and denote by E_s the set of points x for which the f_1 -curve lies in the s^{th} annulus. To make matters precise, let E_s consist of the points x in $(0, \beta)$ for which

$$(s-1) S^{-1} F_1(y_p) \leq |f_1(x_1 + iy_p) - f_1(x + iy_p)| < s S^{-1} F_1(y_p), (s = 1, 2, \dots, S-1) \\ \leq F_1(y_p), (s = S).$$

Evidently

$$m E_1 + m E_2 + \dots + m E_S = \beta,$$

so that there is at least one value of k for which

$$m E_k \leq \beta S^{-1}.$$

As the f_1 -curve is continuous, and passes from the centre of the circle of radius $F_1(y_p)$ to its circumference, it must cut the concentric circle of radius $(k - \frac{1}{2}) S^{-1} F_1(y_p)$. That is to say, there is a point ξ_p in $(0, \beta)$ such that

$$|f_1(\xi_p + iy_p) - f_1(x_1 + iy_p)| = (k - \frac{1}{2}) S^{-1} F_1(y_p).$$

On drawing a diagram it is evident that if x is a point of E_s , ($s \neq k$), then

$$|f_1(\xi_p + iy_p) - f_1(x + iy_p)| \geq \frac{1}{2} S^{-1} F_1(y_p),$$

and hence the values of x in $(0, \beta)$ for which this inequality is satisfied form a set $E^{(p)}$ of measure $\geq (1 - S^{-1})\beta$.

As $f_1(z)$ has period β this means that the values of t in $(0, \beta)$ for which

$$|f_1(\xi_p + iy_p) - f_1(\xi_p + t + iy_p)| \geq \frac{1}{2} S^{-1} F_1(y_p)$$

form a set T_p of measure $\geq (1 - S^{-1})\beta$. Let

$$T = T_1 T_2 T_3 \dots$$

so that

$$(6.8) \quad m T \geq (1 - \sum_{p=1}^{\infty} 2^{-p-N}) \beta = (1 - 2^{-N}) \beta.$$

It will now be shown that the sets T, B_σ have no members in common.

If $T B_\sigma$ is not null, let γ be a member of it. $g_2(z), f_2(z), g(z)$ are defined as before, and

$$g(\xi_p + iy_p + \gamma) - g(\xi_p + iy_p) = f_1(\xi_p + iy_p + \gamma) - f_1(\xi_p + iy_p),$$

since $f_2(z)$ has period γ . Hence, for all values of p ,

$$2M_0(2y_p) \geq |g(\xi_p + iy_p + \gamma) - g(\xi_p + iy_p)| \\ = |f_1(\xi_p + iy_p + \gamma) - f_1(\xi_p + iy_p)| \\ \geq \frac{1}{2} S^{-1} F_1(y_p) > \frac{1}{2} S^{-1} \exp y_p^\tau > \exp y_p^\lambda,$$

by (6.7), and this is false, since it implies that

$$\overline{\lim}_{r \rightarrow \infty} \frac{\log \log M_0(r)}{\log r} \geq \lambda > \sigma,$$

whereas $g(z)$ is of order $\leq \sigma$. Hence the set TB_σ is null, and so if B_σ^* denotes the part of B_σ in the interval $(0, \beta)$, by (6.8),

$$m B_\sigma^* \leq 2^{-N} \beta.$$

As N is independent of β, σ this implies that

$$m B_\sigma^* = 0,$$

and so

$$m B_\sigma = 0$$

for each σ . Hence, finally,

$$m B = 0.$$

§7. *Functions of order 1.* The remaining case, $\rho = 1$, can be reduced to the one just discussed. It is enough to consider a function of regular growth *i.e.* such that

$$(7.1) \quad \frac{\log \log M(r)}{\log r} \rightarrow 1, \text{ as } r \rightarrow \infty.$$

For if

$$\lim_{r \rightarrow \infty} \frac{\log \log M(r)}{\log r} < 1,$$

then

$$\lim_{r \rightarrow \infty} \frac{(\log r)^2 \log M(r)}{r} = 0,$$

and so, by Theorem 2, $f(z)$ has no asymptotic periods.

If $mB > 0$, then $mB_\sigma > 0$ for some $\sigma < 1$. It will be shown that this hypothesis leads to a contradiction.

LEMMA 6. *If $a > 1$ and*

$$(7.2) \quad G(r) = \sum_{n=1}^{\infty} e^{-n^a} r^n,$$

then

$$(7.3) \quad \frac{\log \log G(r)}{\log \log r} \rightarrow \frac{a}{a-1}, \text{ as } r \rightarrow \infty.$$

This is well known. It is easy to prove it by calculating the maximum term of (7.2) and using (5.8).

Let τ, δ be positive numbers satisfying the conditions

$$\sigma < \tau < 1, \quad \delta < \min\left(\frac{1}{9}, \frac{1-\tau}{2}\right).$$

(7.1) implies that

$$(7.4) \quad r^{1-\delta} < \log M(r) < r^{1+\delta}, \quad (r \geq r_\delta).$$

Now define an auxiliary function

$$(7.5) \quad F(z) = \sum_{p=1}^{\infty} e^{-c(p)} f(2^{2^p} z),$$

where

$$(7.6) \quad c(p) = 2^{(1-\delta)2^p+1} - 1.$$

If $r = |z| = 2^{2^q}$, (7.4) gives

$$\begin{aligned} \max_{0 \leq \theta \leq 2\pi} |e^{-c(q)} f(2^{2^q} z)| &= e^{-c(q)} M(2^{2^{2^q+1}}) \\ &> \exp\{-c(q) + 2^{(1-\delta)2^q+1}\} = e^{c(q)}, \quad (q \geq q_\delta), \end{aligned}$$

while

$$\begin{aligned} \max_{0 \leq \theta \leq 2\pi} |e^{-c(p)} f(2^{2^p} z)| &= e^{-c(p)} M(2^{2^p+2^q}) \\ &< \exp\{-c(p) + 2^{(1+\delta)(2^p+2^q)}\}. \end{aligned}$$

Now if $p < q$,

$$\begin{aligned} c(q) + c(p) - 2^{(1+\delta)(2^p+2^q)} &> c(q) - 2^{(1+\delta)(2^p+2^q)} \\ &> X^{(2-3\delta)} - X^{(1+\delta)2^p-2^q+1+\delta} \\ &> X^{2-4\delta}, \quad (q \geq q_\delta), \end{aligned}$$

where $X = 2^{2^q}$, since

$$2 - 3\delta > \frac{1 + \delta}{2} + 1 + \delta.$$

Again, if $p > q$,

$$c(q) + c(p) - 2^{(1+\delta)(2^p+2^q)} > Y_p^{2-4\delta}, \quad (q \geq q_\delta),$$

when $Y_p = 2^{2^p}$.

Denoting by $M_F(r)$ the maximum modulus of $F(z)$, and writing $\epsilon = 2 - 4\delta$, these inequalities show that

$$\begin{aligned} M_F(r) &> e^{c(q)} \left\{ 1 - qe^{-X^\epsilon} - \sum_{p=q+1}^{\infty} e^{-Y_p^\epsilon} \right\}, \quad (q \geq q_\delta) \\ &> \frac{1}{2} e^{c(q)}, \quad (q \geq q'_\delta). \end{aligned}$$

Hence

$$\begin{aligned} \log M_F(r) &> 2^{2(1-\delta)2^q-1} - \log 2 \\ &= \frac{1}{2} r^{2(1-\delta)} - \log 2, \end{aligned}$$

so that $F(z)$ is of order $\geq 2 - 2\delta > 1$.

Let β be a member of B_σ , so that $\Delta_\beta(z)$ is of order $\leq \sigma$, and

$$|\Delta_\beta(z)| < Ke^{r^\tau}, \quad (r \geq 0).$$

Now, writing $c_p = e^{-c(p)}$,

$$F(z + \beta) - F(z) = \sum_{p=1}^{\infty} c_p \sum_{s=0}^{Y_p-1} \Delta_\beta(Y_p z + s\beta),$$

so that

$$\begin{aligned} M_0(r) &= \max_{|z|=r} |F(z + \beta) - F(z)| \\ &< K \sum_{p=1}^{\infty} c_p Y_p \exp\{Y_p^\tau(r + |\beta|)^\tau\} \\ &< K \sum_{p=1}^{\infty} c_p Y_p \exp(2Y_p r^\tau) \\ &= K \sum_{p=1}^{\infty} Y_p \exp\{-\frac{1}{2} Y_p^{2-2\delta} + 2Y_p r^\tau\} \\ &< K \sum_{n=1}^{\infty} \exp\{-n^{2-2\delta} + 4nr^\tau\}. \end{aligned}$$

Hence, by (7.3),

$$\overline{\lim}_{r \rightarrow \infty} \frac{\log \log M_0(r)}{\log r^\tau} \leq \frac{2 - 2\delta}{1 - 2\delta},$$

or

$$\overline{\lim}_{r \rightarrow \infty} \frac{\log \log M_0(r)}{\log r} \leq \left(\frac{2 - 2\delta}{1 - 2\delta}\right) \tau < 2 - 2\delta.$$

Thus β is an asymptotic period of $F(z)$. $F(z)$ is therefore a function of order > 1 whose B -set contains a subset of B_σ of positive measure, and this is impossible.

§ 8. THEOREM 3. *If B denotes the set of asymptotic periods of an integral function $f(z)$, either*

- (i) B is null,
- or (ii) B consists of a set of points $k\lambda$, $k = \pm 1, \pm 2, \dots$,
- or (iii) B lies on a line through the origin, is everywhere dense, and has measure zero.

It has been seen that B lies on a line through the origin and that $mB = 0$. To show that the only possibilities are the cases enumerated above a further lemma is needed.

LEMMA 7. *Let E be a non-null linear set of points with the property that if x, y are any members of E , $-x, x + y$ are also members. Then either (a) E consists of a set of points $k\lambda$, or (b) E is everywhere dense.*

If E is not everywhere dense, let (α, β) be an interval not containing a member of E . Let α' be the upper bound of members of $E \leq \alpha$, and β' the lower bound of members of $E \geq \beta$. Then there are no members of E inside the interval $(\beta', 2\beta' - \alpha')$. For suppose that such a member, x , exists. As β' is a lower bound of members of E there is a member y satisfying the inequality

$$\beta' \leq y < \frac{x + \beta'}{2}.$$

By the addition property $2y - x$ is a member of E , and

$$\alpha' < 2\beta' - x \leq 2y - x < \beta',$$

giving a contradiction. Similarly there are no members of E inside any interval $\{\beta' + k(\beta' - \alpha'), \beta' + (k + 1)(\beta' - \alpha')\}$, and so E consists of the points $\beta' + k(\beta' - \alpha')$, $k = 0, \pm 1, \dots$. As the origin is a member of E it is evident that this set is of the form $k\lambda$, $k = 0, \pm 1, \pm 2, \dots$

Theorem 3 follows on observing that B has the requisite property (except that the origin is excluded).

§9. *Examples.* Theorem 2 shows that if $\rho < 1$, case (i) is the only possibility, and functions of all orders with no asymptotic periods can be constructed by making use of the following result.

If $f(z)$ is any integral function and n an integer > 2 , $f(z^n)$ has no asymptotic periods.

For if $s = 1, 2, \dots, n - 1$,

$$f\{(z + e^{2\pi is/n} \beta)^n\} - f(z^n) = \Delta_\beta(e^{-2\pi is/n} z),$$

so that if β is an asymptotic period, the numbers $e^{2\pi is/n} \beta$ must have the same property. This is impossible, since they are not collinear.

e^z is an obvious example of a function of order 1 which falls under case (ii). A method of constructing functions of higher order with but a single sequence of asymptotic periods is suggested by noticing that the gaps in the series (1.2) played an essential part.

THEOREM 4. *Let*

$$(9.1) \quad f(z) = \sum_{n=1}^{\infty} a(n) e^{niz}$$

be an integral function of order ρ , ($1 < \rho < \infty$), so that

$$(9.2) \quad \lim_{n \rightarrow \infty} \frac{\log \log |a(n)|^{-1}}{\log n} = \frac{\rho}{\rho - 1}.$$

If there is a sequence n_1, n_2, \dots , such that

$$(9.3) \quad n_{p+1} - n_p < K, \quad (p = 1, 2, \dots),$$

$$(9.4) \quad \frac{\log \log |a(n_p)|^{-1}}{\log n_p} \rightarrow \frac{\rho}{\rho - 1},$$

then $f(z)$ has no asymptotic periods except a set $k\lambda$, $k = \pm 1, \pm 2, \dots$

λ must be of the form $\frac{2\pi}{l}$, where λ is an integer. If $f(z)$ has other asymptotic periods they must be everywhere dense on the real axis. Let N be an integer $> K$ and β a period $< N^{-1}$. If q is any integer there are at least N of the points $k\beta$ between $2\pi(q + \frac{1}{4})$ and $2\pi(q + \frac{3}{4})$, and so at least one of the points $n_p\beta$. Hence

$$|1 - e^{in_p\beta}| \geq \sqrt{2}, \text{ for a sequence } p_1, p_2, \dots$$

Thus

$$\begin{aligned} & \max_{0 \leq x \leq 2\pi} |f(x - iy + \beta) - f(x - iy)| \\ & \geq |a(n)| \cdot e^{ny} \cdot |e^{in\beta} - 1|, \quad (n \geq 1), \\ & \geq \sqrt{2} |a(n)| e^{ny}, \quad n = m_1, m_2, \dots \\ & = \sqrt{2} \exp(-n^{a_n} + ny), \end{aligned}$$

where $a_n = \frac{\rho_n}{\rho_n - 1}$, $\rho_n \rightarrow \rho$, and m_1, m_2, \dots is a subsequence of n_1, n_2, \dots

Thus if $n = m_i$, $y = 2n^{a_n - 1}$,

$$\max_{0 \leq x \leq 2\pi} |f(x - iy + \beta) - f(x - iy)| \geq \sqrt{2} \exp\left\{\left(\frac{y}{2}\right)^{\rho_n}\right\},$$

so that β cannot be an asymptotic period.

Functions of infinite order presenting case (ii) can be constructed in the same way, e.g. (9.4) may be replaced by

$$(9.5) \quad \frac{\log |a(n_p)|^{-1}}{n_p \log n_p} \rightarrow \alpha > 0.$$

If

$$(9.6) \quad \lim_{n \rightarrow \infty} \frac{\log \log |c_n|^{-1}}{n \log n} = \frac{\rho}{\rho - 1}, \quad (1 < \rho < \infty),$$

the series

$$(9.7) \quad \sum_{n=1}^{\infty} c_n e^{2\pi n! iz}$$

defines a function of order ρ having every rational number as an asymptotic period. The construction breaks down if $\rho = 1$ and I

have been unable to find an example of a function of order 1 which falls under case (iii). It may well be that no such function exists.

To sum up:

Case (i). *Functions of all orders exist.*

Case (ii). *Functions of all orders ρ in $(1 \leq \rho \leq \infty)$ exist.*

Case (iii). *Functions of all orders ρ in $(1 < \rho \leq \infty)$ exist.*

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