CONVERGENCE IN SEQUENCE SPACES

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In a perfect sequence space α, on which a norm is defined, we can consider three types of convergence, namely projective convergence, strong projective convergence and distance convergence. In the space σ_∞, when distance is defined in the usual way, the last two types of convergence coincide and are distinct from projective convergence ((2), p. 316). In the space σ_1 all three types of convergence coincide ((2), p. 316). It will be shown in this paper that, if distance convergence and projective convergence coincide, then all three types of convergence coincide. It will not be assumed that the limit under one convergence is also the limit under the other convergence.

If a norm is defined on a perfect space α in such a way that distance convergence and limit coincides with strong projective convergence and limit in α, then α is called a Köthe-Banach space ((1), p. 114). It has been proved by G. Köthe that, under these conditions, a set of points in α is projective-bounded if, and only if, it is bounded under the metric ((4), p. 205). The proof of the lemma which follows is a slight modification of the proof of this result.

We shall denote the scalar product $\sum_{i=1}^{\infty} u_i x_i$ of two vectors $x=\{x_i\}$ belonging to α, and $u=\{u_i\}$ belonging to the Köthe-Toeplitz dual space α^*, by (x, u).

Lemma. *If α is a perfect sequence space, and if a norm is defined on the vector space α in such a way that distance convergence and projective convergence coincide, then a set of points in α is projective-bounded if, and only if, it is bounded under the metric.*

Let $\| x \|$ denote the norm of a vector x belonging to α. Suppose that $\mathscr{R}$ is a projective-bounded set in α which is not bounded under the metric. Then there is a sequence of points $x^{(n)}$ in $\mathscr{R}$ such that $\| x^{(n)} \| > n^2$ $(n=1, 2, \ldots)$. It follows that the sequence $n^{-1}x^{(n)}$ $(n=1, 2, \ldots)$ is not bounded under the metric and is not distance convergent. On the other hand, since the sequence $x^{(n)}$ is projective-bounded, it follows that the sequence $x^{(n)}/n$ is projective convergent. The contradiction proves that every projective-bounded set is bounded under the metric.

Conversely, suppose that a set $\mathscr{S}$ is bounded under the metric but is not projective-bounded. Then there exists a vector u in α^*, and a sequence of points $y^{(n)}$ in $\mathscr{S}$, such that $|(y^{(n)}, u)| > n^2$ $(n=1\ 2, \ldots)$. The sequence of points $n^{-1}y^{(n)}$ $(n=1, 2, \ldots)$ is distance convergent but not projective convergent, which is a contradiction. This proves the lemma.

Theorem. *If α is a perfect sequence space, and if a norm is defined on the vector space α in such a way that distance convergence and projective convergence coincide, then α is a Köthe-Banach space under this norm.*

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Let the given norm of a vector $x \in \alpha$ be denoted by $\|x\|$. The set of all $x \in \alpha$ such that $\| x \| \leqslant 1$ is projective-bounded, by the lemma. Suppose $u \in \alpha^*$ and define

$$| u | = \underset{\| x \| \leqslant 1}{l.u.b.} |(x, u)|.$$

This defines a norm on the vector space α^* such that

$$|(x, u)| \leqslant | u | \, \| x \| \quad \text{.................................(1)}$$

for all $x \in \alpha$, $u \in \alpha^*$. Suppose $x \in \alpha$ and define

$$| x | = \underset{| u | \leqslant 1}{l.u.b.} |(x, u)|.$$

This defines a norm on the vector space α such that

$$|(x, u)| \leqslant | u | | x |$$

for every $x \in \alpha$, $u \in \alpha^*$. It follows that the set $\mathscr{U}$ of vectors u belonging to α^* such that $| u | \leqslant 1$ is projective-bounded.

We prove first that α is a Köthe-Banach space under the norm $| x |$. Suppose that $x^{(n)}$ $(n=1, 2, \ldots)$ is a strongly projective convergent sequence in α; then the strong projective limit x exists and is in α ((2), p. 308). If u belongs to $\mathscr{U}$ and $\epsilon > 0$ is given, then there is an $N(\epsilon)$ such that $|(x^{(p)} - x, u)| < \epsilon$ for all $p > N$ and all $u \in \mathscr{U}$. Thus

$$| x^{(p)} - x | = \underset{u \in \mathscr{U}}{l.u.b.} |(x^{(p)} - x, u)| \leqslant \epsilon$$

for $p > N$. It follows that $\lim_{n \to \infty} | x^{(n)} - x | = 0$.

Conversely, suppose that the sequence $x^{(n)}$ $(n=1, 2, \ldots)$ is convergent under the metric $| x |$, and that $\mathscr{R}$ is a projective-bounded set in α^*. Then there is a k such that $|(x, u)| \leqslant k$ for all $u \in \mathscr{R}$ and all $x \in \alpha$ such that $\| x \| \leqslant 1$, since the set of all $x \in \alpha$ with $\| x \| \leqslant 1$ is projective-bounded, by (1) ((2), p. 296). Thus $| u | \leqslant k$ for all $u \in \mathscr{R}$. Hence, given $\epsilon > 0$, there is an $N(\epsilon, \mathscr{R})$ such that $|(x^{(p)} - x^{(q)}, u)| \leqslant | u | | x^{(p)} - x^{(q)} | \leqslant \epsilon$ for all $p, q > N$, and $u \in \mathscr{R}$. It follows that $x^{(n)}$ is strongly projective convergent and that its strong projective limit x exists and is in α. It follows that $\lim_{n \to \infty} | x^{(n)} - x | = 0$. Thus α is a Köthe-Banach space under the norm $| x |$.

Suppose that $\mathscr{X}$ is the set of all $x \in \alpha$ such that $| x | \leqslant 1$. Then $\mathscr{X}$ is bounded under the metric $| x |$ and is therefore projective-bounded in α ((4), p. 205). We have $\| x \| \leqslant m$, $x \in \mathscr{X}$, by the lemma. Hence, if $x \in \alpha$, $x \neq 0$, we have $\| x / | x | \| \leqslant m$ and $\| x \| \leqslant m | x |$. The set $\mathscr{Y}$ of all $x \in \alpha$ such that $\| x \| \leqslant 1$ is projective-bounded, by the lemma. Since α is a Köthe-Banach space under the norm $| x |$, it follows that $| x | \leqslant M$, $x \in \mathscr{Y}$. Hence, if $x \in \alpha$, we have $| x | \leqslant M \| x \|$. Thus

$$m^{-1} \| x \| \leqslant | x | \leqslant M \| x \|, \; x \in \alpha \quad \text{..........................(2)}$$

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If $x^{(n)}$ $(n=1, 2, \ldots)$ is strongly projective convergent in α and x is the strong projective limit, we have $\lim_{n\to\infty} | x^{(n)}-x | = 0$ and hence $\lim_{n\to\infty} \| x^{(n)}-x \| = 0$, by (2). Conversely, if $x^{(n)}$ is distance convergent under the *given* metric, then $x^{(n)}$ is distance convergent under the metric $| x |$, by (2), and hence $x^{(n)}$ is strongly projective convergent. If x is the strong projective limit, then $\lim_{n\to\infty} | x^{(n)}-x | = 0$ and hence $\lim_{n\to\infty} \| x^{(n)}-x \| = 0$. Hence α is a Köthe-Banach space under the given definition of distance.

It follows at once that projective convergence and strong projective convergence coincide in α.

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