A ring \( R \) is called an \( H \)-ring if for every \( x \in R \) there exists an integer \( n = n(x) > 1 \) such that \( x^n - x \in C \), where \( C \) is the center of \( R \). I. N. Herstein proved that \( H \)-rings must be commutative [See 3 pp. 220—221]. We now introduce the following definition.

**DEFINITION.** \( R \) and \( R' \) are two rings, we say \( R \) is an \( H \)-extension of \( R' \) if \( R' \) is a subring of \( R \) and for any \( x \in R \), there exists an integer \( n > 1 \) (depending on \( x \)) such that \( x^n - x \in R' \).

In this paper we shall show how the Jacobson radical of \( R \) is related to that of \( R' \) (Theorem 1) and then we shall give some information about \( H \)-extension of a commutative one-sided ideal (Theorem 2). An example is also given at the end of section 2 to show in general we can not arrive at the sharper conclusion that an \( H \)-extension of commutative ideal is commutative.

1

In this section, we denote \( R \) as an \( H \)-extension of a subring \( R' \) and \( J(R) \), the Jacobson radical of the ring \( R \). It is well known \( J(R) \) can be characterized as the intersection of all primitive ideals of \( R \) or it is the set \( \{x \in R \mid xR \text{ is a right quasi-regular right ideal of } R\} \). We shall prove the theorem 1 as follows, the proof was patterned after the argument of the paper of Armendariz [1].

**LEMMA 1.1.** (1). For any \( x \in R \), there exists an arbitrarily high \( n \) such that \( x^n - x \in R' \).

(2). All nilpotent elements of \( R \) belong to \( R' \).

**Proof.** (1) If this is false we have an integer \( m \) which is the largest \( m \) such that \( x^m - x \in R' \). Let us choose another \( n > 1 \) which satisfies \((x^m)^n - x^m \in R'\), then \( x^{mn} - x = (x^m)^n - x^m + (x^m - x) \in R' \). This is contradictory to the maximality of \( m \). (2) Let \( x^m = 0 \). Choose \( N > m \) so that \( x^N - x \in R' \), since \( x^N = 0 \), and we have \( x \in R' \).

We now consider the \( n \)-square matrix ring \( \Gamma_n \) \((n > 1)\) over a ring \( \Gamma \) with unit element. If \( \Gamma_n \) is an \( H \)-extension of a subring \( B \), then by
Lemma 1.1 $B$ contains all nilpotent elements, in particular, the matrices $E_{ij}d(i \neq j, d \in \Gamma)$ and therefore the matrices $E_{i}d = E_{ii}dE_{ii}$. So we have:

**Lemma 1.2.** If the $n$-square matrix ring $\Gamma_n (n > 1)$ is an $H$-extension of a subring $B$. Then $\Gamma_n = B$.

**Lemma 1.3.** If $R$ is a division ring, then $R'$ is also a division ring.

**Proof.** Let $0 \neq x \in R'$, then there exists an integer $n > 1$ such that $b = (x^{-1})^n x^{-1} \in R'$. Multiplying $b$ by $x^n$ and $x^{n-1}$ respectively, we see that $1$ and $x^{-1}$ belong to $R'$. So $R'$ is a division ring.

Now let $R$ be a primitive ring and $R' \neq 0$. By the theorems appearing in [3] chapter II, $R$ can be considered as a dense subring of the ring of all linear transformations of a vector space $V$. If the dimension of $V$ is one, $R$ is a division ring. Then by Lemma 1.3 $R'$ is also a division ring. This proves $R'$ is a primitive ring. If the dimension of $V$ is larger than one, then considering $V$ as a right faithful module over $R'$ we shall prove it is an irreducible module as follows: Let $v_1$ be a non-zero fixed element of $V$ and $v_2$ any element of $V$. There exists a 2-dimensional vector subspace $V_2$ which contains $v_1$ and $v_2$. Let $U = \{x \in R|V_2x \subseteq V_2\}$, $K = \{x \in R|V_2x = (0)\}$, $U_1 = U \cap R'$. Because $R$ is dense, $U/K$ is isomorphic to the full ring of linear transformations of $V_2$. Moreover, it is clear that $U/K$ is still an $H$-extension of its subring $(U_1+K)/K$. So by Lemma 1.2 we have $U/K = (U_1+K)/K$. This assures there exists a linear transformation $x \in R'$ that sends $v_1$ to $v_2$. From this we see any element of $V$ is the form $v_1x$ for some $x \in R'$, in other words $V$ is a cyclic $R'$-module with every non-zero element as a generator. This proves $V$ is irreducible. So we have the following:

**Lemma 1.4.** If $R$ is a primitive ring and $R' \neq 0$, then $R'$ is a primitive ring.

**Remark.** Some one may wonder in Lemma 1.4 $R'$ is always equal to $R$. Here we give a primitive ring which is an $H$-extension of some proper subring. Let $Z_p$ be the prime field of characteristic $p$ and $R$ be a ring of linear transformations of an infinite dimensional vector space $M$ over $Z_p$. Here $R$ is so chosen that the matrices of its elements have the form

\[
\begin{bmatrix}
A \\
d & 0 \\
& d \\
&& d \\
& & & \ddots \\
0 & & & & & d \\
& & & & & 0 \\
& & & & & & \ddots \\
\end{bmatrix}
\]
and \( R' \) is the ring of the form

\[
\begin{bmatrix}
A & 0 \\
0 & 0 \\
0 & 0 \\
0 & \ddots
\end{bmatrix}
\]

where \( A \) is an arbitrary finite square matrix and \( d \) is any element of \( \mathbb{Z}_d \). Then for any \( a \in R \), we have \( a^2 - a \in R' \). Moreover \( R \) is a primitive ring [See 3 p. 36 example 3].

**Theorem 1.** If \( R \) is an \( H \)-extension of a subring \( R' \), then

\[
J(R) \cap R' = J(R').
\]

**Proof.** Let \( x \in J(R) \cap R' \). We want to prove that any \( y \in xR' \) has a right quasi inverse in \( R' \). Since \( y \in xR' \subseteq xR \), there is \( z \in R \) such that

\[
y + z - yz = 0.
\]

Now for some \( n = n(z) > 1 \), \( z^n - z \in R' \). Then \( y(z^n - z) = yz^n - z - y \in R' \). This implies \( yz^n - z \in R' \). Multiply (\( *) \) from right by \( z^{n-1} \) and we get

\[
yz^{n-1} = yz^n - z^n = yz^n - z - (z^n - z) \in R'.
\]

Again multiply \( y \) on the left and \( z^{n-2} \) on the right of \( y = yz - z \) and we get \( y^2 z^{n-2} \in R' \). Repeating the process \( n-1 \) times, we get

\[
y^{n-1} z \in R', \ z = yz - y = y(yz - y) - y = \cdots = y^{n-1} z - y^{n-1} - \cdots y \in R'.
\]

Consequently \( xR' \) is a right quasi-regular right ideal of \( R' \), so \( x \in J(R') \).

The opposite inclusion can be proved as follows: If \( P \) is a primitive ideal of \( R \), \( R/P \) is a primitive ring and an \( H \)-extension of \( (R' + P)/P \). By Lemma 1.4 \( (R' + P)/P \approx R'/(P \cap R') \) is a primitive ring, so \( P \cap R' \) is a primitive ideal of \( R' \). We have:

\[
J(R) \cap R' = \bigcap_{P: \text{primitive ideal of } R} (P \cap R') \cap R' = \bigcap (P \cap R') \supseteq J(R').
\]

**Corollary.** \( R \) is semi-simple if and only if \( R' \) is semi-simple.

W. S. Martindale III defined an \( \gamma \)-ring as a ring \( R \) in which \( \omega^{n(\omega)} - \omega \) belongs to the center \( C \) of \( R \) for every commutator \( \omega \) of \( R \) and proved in his paper [4] that every commutator of an \( \gamma \)-ring is contained in the center.
In this section we can obtain a parallel result about an $H$-extension of an one-sided ideal.

We first cite a theorem which is proved in Carl Faith’s [2 p. 47] as follows. Let $\phi[X]$ be the polynomial ring over the field $\phi$ and $[x_1, \ldots, x_r, X]$ denote the subring of $\phi[X]$ generated by $X$ and $r$ fixed non-zero elements $x_1, \ldots, x_r$ in the field $\phi$, and set:

$$(\ast) \quad N(x_1, \ldots, x_r) = \{X^n - X^{n+1}P(X)\mid P(X) \in [x_1, \ldots, x_r, X], n = 1, 2, \ldots\}.$$ 

**Theorem (Faith).** Let $D$ be a division algebra over the field $\phi$, and let $A$ be a subalgebra such that to each $d \in D$ there corresponds non-zero elements $x_1, \ldots, x_r \in D$ (depending on $d$) such that for each $a \in \phi(d)$ there exists $f_a(X) \in N_d$ satisfying $f_a(a) \in A$, where $N_d = N(x_1, \ldots, x_r)$ is a set of the type $(\ast)$. Then $D$ is a field.

If $R$ is a division ring and an $H$-extension of a commutative subring $R'$, by Lemma 1.2 $R'$ is a division subring. So $R'$ contains the prime field $\phi$ of $R$. We can consider $R$ as a division algebra over $\phi$ and $R'$ its subalgebra. Furthermore it is clear that every $x \in R$ satisfies the condition of the above theorem if we take all $x_i$ are 1. So $R$ is commutative.

**Lemma 2.1.** If $R$ is a semi-simple $H$-extension of a commutative subring $R'$, then $R$ is commutative.

**Proof.** It is sufficient to prove this for a primitive ring, because $R$ is a subdirect sum of primitive rings and the $H$-extension property is inherited by homomorphic images. In this case $R$ ought to be a division ring, otherwise, by [3 p. 33 proposition 3] it contains a subring $U$ which has a homomorphic image isomorphic to the complete matrix ring $\Gamma_n$ $(n > 1)$ over a division ring $\Gamma$. As $\Gamma_n$ is an $H$-extension of the homomorphic image $U'$ of $U \cap R'$, by Lemma 1.2, we have $\Gamma_n = U'$. But $U'$ is still commutative since it is the homomorphic image of the commutative ring $U \cap R'$. This is contradictory. So $R$ is a division ring. Now by Faith’s theorem we see $R$ is commutative.

**Lemma 2.2.** If $R$ is an $H$-extension of a commutative subring $R'$, then every commutator $w = xy - yx$ of $R$ belongs to $J(R)$.

**Proof.** $R/J(R)$ is an $H$-extension of its commutative subring $(R' + J(R))/J(R)$, where $(R' + J(R))/J(R)$ is isomorphic to $R'(J(R) \cap R')$. By Theorem 1 $R'(J(R) \cap R') = R'/J(R')$ which is semi-simple. So $R/J(R)$ is commutative by Lemma 2.1. The residue class of a commutator $w$ modulo $J(R)$ is zero. This implies $w = xy - yx \in J(R)$.

**Lemma 2.3.** If $R$ is an $H$-extension of a commutative right ideal $I$, then every commutator $w = xy - yx$ is nilpotent.
PROOF. By Zorn’s Lemma we can find a maximal commutative subring $R'$ of $R$, which contains $I$. Let $w = xy - yx$, $y \in R'$, there exists an integer $n = n(w) > 2$ such that $w^n - w \in I$, hence

$$(w^n - w)(xy - yx) = (w^n - w)xy - y(w^n - w)x = (w^n - w)xy - (w^n - w)xy = 0.$$ 

The quasi-regularity of $w^{n-1}$ (by Lemma 2.2) forces: $w(xy - yx) = 0$, in other words $w^2 = 0$. These kinds of $w$ belong to $I$ by Lemma 1.1.

Now $J(R)$ shall be proved commutative as follows: If $a \in J(R)$, there exists an integer $m > 2$ such that $a^m - a \in I$. Then for any $y \in R'$

$$(a^m - a)(xy - yx) = (xy - yx)(a^m - a) = 0.$$ 

The quasi-regularity of $a^{m-1}$ will yield $a(xy - yx) = 0$, $(xy - yx)a = 0$. Let $x = a$, then $a^2y = aya = ya^2$ for all $y \in R'$. Considering the subring $R''$ of $R$ generated by $R'$ and $a^2$ we get $R''$ is commutative containing $R'$. The maximal property of $R'$ forces $R'' = R'$. So we have $a^2 \in R'$. If $m$ is even, then $a^m - a \in R'$ implies $a \in R'$. If $m$ is odd, $a^{m-1} \in R'$. The quasi-regularity of $a^{m-1}$ and $a^m - a \in R'$ yield $a \in R'$. As a consequence we can see that $J(R)$ is contained in the commutative subring $R'$. So $J(R)$ is a commutative ideal.

Finally, by Lemma 2.2 $w$ is contained in $J(R)$, we can conclude that:

$$w^3 = w^2(xy - yx) = w^2xy - w(wy)x = w^2xy - (wy)(wx) = w^2xy - (wx)(wy) = w^2xy - (wx)w) = w^2xy - w^2xy = 0.$$ 

**Theorem 2.** If $R$ is an $H$-extension of a commutative one sided ideal $I$, then every commutator $w$ belongs to $I$.

**Proof.** By Lemma 2.3 and Lemma 1.1 we can see that $w$ belongs to $I$.

**Remark.** An example is given here to show that in general an $H$-extension of a commutative ideal is not necessarily commutative:

Let $Z_2$ be the prime field of characteristic 2 and $R$ be the algebra over $Z_2$ generated by $a, b$ satisfying

$$a^2 = a, ab = b^2 = 0, ba = b.$$ 

Then $R$ is a non-commutative $H$-extension of its commutative ideal $(o, b)$.

**References**


