ON CHARACTERS IN THE PRINCIPAL 2-BLOCK, II

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(Received 30 August 1977; revised 18 December 1978)

Communicated by M. H. Newman

Abstract

Let $k$ be a non-zero complex number and let $u$ and $v$ be elements of a finite group $G$. Suppose that at most one of $u$ and $v$ belongs to $O(G)$, the maximal normal subgroup of $G$ of odd order. It is shown that $G$ satisfies $X(v) - X(u) = k$ for every complex nonprincipal irreducible character $X$ in the principal 2-block of $G$, if and only if $G/O(G)$ is isomorphic to one of the following groups: $C_2$, $PSL(2, 2^n)$ or $P\Sigma L(2, 5^{2a+1})$, where $n \geq 2$ and $a \geq 1$.

Subject classification (Amer. Math. Soc. (MOS) 1970): 20 C 20

1. Introduction

Let $G$ be a finite group. It was shown by Berger and Herzog (1978) that if $u \in G$ and $k \in \mathbb{C}$ satisfy:

$$X(1) - X(u) = k$$

for every complex non-principal irreducible character in the principal 2-block of $G$, then either $u \in O(G)$ or $G/O(G)$ is isomorphic to one of the following simple groups: $C_2$, $PSL(2, 2^n)$, $n \geq 2$. The converse also holds.

The aim of this paper is to consider the more general equality

$$X(v) - X(u) = k,$$

where $k$ is a non-zero complex number, $v, u \in G$ and (1) holds for every complex non-principal irreducible character in the principal 2-block of $G$. In this case we obtain new candidates for $G/O(G)$, namely $P\Sigma L(2, 5^{2a+1})$, $a \geq 1$, the extension of $PSL(2, 5^{2a+1})$ by the group of automorphisms of the Galois field with $5^{2a+1}$ elements.

The first author was supported by a grant from the Israel Commission for Basic Research.
Our main result is

**Theorem 1.** Let $G$ be a finite group, $u$ and $v$ be elements of $G$ and $k$ be a non-zero complex number. Suppose that (1) is satisfied by every complex non-principal irreducible character of $G$ belonging to $B$, the principal 2-block of $G$. Then, either 

\[ \{u, v\} \cap O(G) = \emptyset \text{ or } k = \pm 4 \text{ and } G/O(G) \text{ is isomorphic to } P\Sigma L(2, 5^{2a+1}), \ a \geq 1. \]

We also prove the following

**Proposition.** Let $G = P\Sigma L(2, 5^{2a+1})$, $u \in G$ be an involution and $v \in G$ be of order $2a + 1$ such that $G = \langle P\Sigma L(2, 5^{2a+1}), v \rangle$. Then (1) holds for every complex non-principal irreducible character in the principal 2-block of $G$, with $k = 4$.

The authors are grateful to the referee for providing the proof of the Proposition. Combining these results with the Theorem of Berger and Herzog (1978), we get

**Theorem 2.** $G$ satisfies the assumptions of Theorem 1, if and only if $G/O(G)$ is isomorphic to one of the following groups: $C_2$, $PSL(2, 2^n)$, $n \geq 2$ and $P\Sigma L(2, 5^{2a+1})$, $a \geq 1$.

In this paper $G$ denotes a finite group. The order of $G$ is $g$ and if $v \in G$, $o(v)$ denotes the order of $v$. The principal 2-block of $G$ is denoted by $B$, and the number of irreducible characters in $B$ is $b$. The letter $X$ will always denote an irreducible character in $B$ ($X \in B$). A fixed Sylow 2-subgroup of $G$ will be denoted by $S$. If $H$ is a subgroup of $G$ and $v \in G$, then $o(v \text{ mod } H)$ is the least positive integer $n$ satisfying $v^n \in H$, and $\exp H$ is the least positive integer $m$ satisfying: $h^m = 1$ for every $h \in H$. The group of outer automorphisms of $H$ will be denoted by Out $H$. We denote by $\Sigma$ or $\Sigma^*$ the summation over all $X \in B$ or $X \in B \setminus 1_G$, respectively. The expression 'the orthogonality relations in blocks' will be abbreviated by O.R.B. Finally, $C_2$ will denote the cyclic group of order 2.

### 2. Proof of Theorem 1

It is well known that $O(G) = \cap \{\ker X \mid X \in B\}$. As $k \neq 0$, it follows that not both $u$ and $v$ belong to $O(G)$. So assume that $u, v \notin O(G)$ and it suffices to prove the theorem under the assumption that $O(G) = 1$.

It is well known that if $y \in G$, then $\Sigma^* X(y)$ is a rational integer. Thus, by (1),

\[ (b - 1)k \in \mathbb{Z} \text{ and since } X(v) - X(u) \text{ is an algebraic integer, we conclude that} \]

\[ k \in \mathbb{Z} - \{0\}. \]
Suppose that \( y \in G \) does not belong to the 2-sections of either \( v \) or \( u \) in \( G \). Then, by (1) and the O.R.B.,

\[
0 = \sum X(y)(X(v) - X(u)) = k \sum \delta X(y)
\]
yielding

(3) \[ \sum \delta X(y) = 0. \]

It follows that \( y \neq 1 \) and consequently we may assume without loss of generality that

(4) \[ v \text{ has odd order, } \sigma(v) > 1. \]

Let \( w \) be a 2-element of \( G \) of maximal order, and let \( z \) be the involution in \( \langle w \rangle \). Then, by the O.R.B., \( \sum X(1)X(z) = 0 \), and since as in Berger and Herzog (1978)

(5) \[ X(w) \equiv X(z) \equiv X(1) \pmod{\mathcal{P}}, \]

where \( \mathcal{P} \) is the prime ideal lying over 2 in \( \mathcal{O} \), the integers in \( Q(\sqrt{1}) \), it follows that

\[ \sum \delta X(1) \equiv \sum \delta X^2(1) \equiv \sum \delta X(1)X(z) \equiv 1 \pmod{2}. \]

Hence

(6) \[ \sum \delta X(w) \equiv \sum \delta X(z) \equiv \sum \delta X(1) \equiv 1 \pmod{2}. \]

Thus, in view of (3) and (4), \( w = z \) and we get

(7) \[ \exp S = 2, \text{ where } S \text{ is a Sylow 2-subgroup of } G, \]

(8) \[ G \text{ has one class of involutions, and} \]

(9) \[ \sigma(u) = 2f, \text{ where } f \text{ is an odd integer.} \]

In particular, \( G \) has exactly two 2-sections.

Choose \( H \), a minimal normal subgroup in \( G \). As \( O(G) = 1 \), it follows by (8) that \( G/H \) is of odd order and as in Berger and Herzog (1978), either \( H = S \) or \( H \) is isomorphic to one of the following simple groups: \( PSL(2,q) \), \( q > 3, q \equiv 0, 3 \) or 5 \( \pmod{8} \), \( J \) (Janko's smallest group) or \( Re(q) \) (a group of Ree type). Since none of the above-mentioned groups satisfies (1) for a \( v \) satisfying (4), it follows that

(10) \[ G/H \text{ is a non-trivial soluble group of odd order.} \]

Let \( Y \) be a non-principal linear character of \( G/H \) and suppose that \( Y \in \mathcal{B} \). Clearly, by (1) and (2), \( k = \pm 1 \) or \( \pm 2 \). If \( k = \pm 2 \), then by (1) \( \{ Y(v), Y(u) \} = \{1, -1\} \), which is impossible since \( G/H \) is of odd order. If \( k = \pm 1 \), then by (1)

\[ \{ Y(v), Y(u) \} = \{ \exp(\frac{4}{3} \pi i), \exp(\frac{2}{3} \pi i) \} \text{ or } \{ \exp(\frac{5}{3} \pi i), \exp(\frac{5}{3} \pi i) \}, \]
again in contradiction to (10). Thus:

(11) No non-principal linear character of \( G/H \) belongs to \( B \).

Proceeding exactly as in Berger and Herzog (1978), we get

(12) \( G = C_G(S) H \),

(13) \( H \) is non-abelian simple,

(14) \( G/H \) is isomorphic to a subgroup of Out \( H \),

(15) \( H \neq J, PSL(2, 2^n), n \geq 2 \), and

(16) If \( Y \) is an irreducible character of \( G/H \) belonging to \( B \), then \( Y = 1 \).

Suppose that \( H \cong \text{Re}(q) \). As in Berger and Herzog (1978), \( B \) consists of 8 characters \( X_i, i = 1, \ldots, 8 \), such that \( X_i|_H = \xi_i, i = 1, \ldots, 8 \). We use here the notation of Ward (1966) for the irreducible characters and elements of \( H \). By the O.R.B., (1), (4) and (9) we get

\[
0 = \sum X(u) X(u) = 1 + k \sum X(u) + \sum |X(u)|^2
\]

whence

(17) \[ 0 = k \sum X(u) + \sum |X(u)|^2. \]

In addition, the O.R.B. yield:

(18) \[ 0 = \sum X(u) X(R) = X_1(u) + X_2(u) + X_3(u) + X_4(u) \]

and

(19) \[ 0 = \sum X(u) (3X(R) + X(S) + X(V) + X(W)) = 6X_1(u) + 6X_2(u). \]

As \( X_1(u) = 1 \), (18) and (19) yield:

(20) \[ X_2(u) = -X_1(u) = -1, \quad X_4(u) = -X_2(u). \]

The O.R.B. also yield:

\[
0 = \sum X(u) X(Y) = m(X_3(u) + X_6(u) + X_7(u) + X_8(u))
\]

whence

(21) \[ X_3(u) + X_6(u) + X_7(u) + X_8(u) = 0. \]

It follows from (17), (18) and (21) that

(22) \[ k = \sum |X(u)|^2. \]
Applying the O.R.B. to \( v \) we get

\[ 0 = \sum X(v) X(JR) = X_1(v) - X_2(v) + X_3(v) - X_4(v), \]

which implies in view of (1) and (20)

\[ X_3(u) = -X_4(u) = (k-2)/2. \]

Thus \( k \) is even, and by (20), (22) and (23):

\[ k \geq 1 + 1 + (k-2)^2/2. \]

It follows that one of the following holds:

\[ k = 4, \quad X_i(u) = 0 \quad \text{for} \quad i = 5, 6, 7, 8, \]

or

\[ k = 2, \quad X_i(u) = 0 \quad \text{for} \quad i = 3, 4, 5, 6, 7, 8. \]

Another application of the O.R.B. yields, in view of (1), (20) and (23),

\[ 0 = \sum X(v) X(JS) = 1 - (k-1) - (3k/2-1) + (k/2+1) \]

so that \( k = 2 \).

A final application of the O.R.B., together with (20), yields:

\[ 0 = \sum X(u) X(1) = 1 + (-1)(q^2-q+1) = q(1-q), \]

a contradiction.

Finally, suppose that \( H \cong PSL(2, q), q > 5 \) and \( q \equiv 3 \) or 5 (mod 8). As in Berger and Herzog (1978), \( B \) consists of 4 characters \( X_i, \quad i = 1, \ldots, 4 \), such that \( X_i|_H = \theta_i, \quad i = 1, \ldots, 4 \). We use here the notation of Ward (1966), pp. 62–65, for the irreducible characters and elements of \( H \). By the O.R.B. we have

\[ 0 = \sum X(u) X(R) = X_3(u) - eX_4(u), \]

where \( e = \pm 1 \) satisfying \( q \equiv 4 + e \) (mod 8), as defined in Ward’s paper. Hence,

\[ X_4(u) = e. \]

Thus, again by the O.R.B.,

\[ 0 = \sum X(u) X(1) = 1 + (q + e)(X_2(u) + X_3(u))/2 + eq \]

yielding

\[ X_2(u) + X_3(u) = -2e. \]

A final application of the O.R.B., together with (1), (24) and (25), yields

\[ 0 = \sum X(v) X(S_0^{(q-e)/4}) = 1 - 2ke + 2 + ek + 1, \]

whence \( k = 4e \) and \( X_4(v) = 5e \).
Now by (10) and (14)

(26) \[ PSL(2, q) \triangleleft G \leq P \sum L(2, q). \]

Thus \( G \) has a 2-transitive permutation representation of degree \( q+1 \), the restriction of which to \( H \) is also 2-transitive. Let \( Y \) be the irreducible character of \( G \) of degree \( q \) corresponding to this representation. Then \( Y|_H \) is irreducible, and since \( Y(1) = q \), \( Y|_H = \theta_4 = X_4|_H \) and consequently \( X_4 = Y \cdot \xi \), where \( \xi \) is a linear character of the cyclic group \( G/H \) (see Isaacs (1976), (6.17)). Thus \( 5e = X_4(v) = Y(v) \cdot \xi(v) \), where \( Y(v) \) is an integer \( \geq -1 \) and \( \xi(v) \) is an odd root of 1. We conclude that \( e = 1 \) and \( Y(v) = 5 \). So \( v \) fixes exactly \( 5+1 = 6 \) elements in the permutation representation of \( G \). Let \( q = p^e \) and let \( \alpha(v \text{ mod } H) \) be \( d \). Then \( d \) divides \( c \) and

\[ 6 = \text{fix}(v) = 1 + p^{e/d}. \]

Consequently \( p = 5 \) and \( d = c \); as \( 5^e = q \equiv 4 + e = 5 \pmod{8} \), \( c = 2a + 1 \) for some \( a \geq 1 \). Since \( \alpha(v \text{ mod } H) = c = 2a + 1 \), by (26) \( G = P \sum L(2, 5^{2a+1}) \), and the proof of Theorem 1 is complete.

3. Proof of the Proposition

Let \( 2a + 1 = r \), \( q = 5^r \) and let \( H \triangleleft G \), \( H \cong PSL(2, q) \). Since \( |G:H| = r \), then \( u \in H \). It follows from the arguments of Section 2 that the principal 2-block \( B \) of \( G \) consists of 4 irreducible characters: \( X_i \), \( i = 1, \ldots, 4 \), such that \( X_i|_H = \theta_i \), \( i = 1, \ldots, 4 \). For the irreducible characters and elements of \( H \) we use again the notation of Ward (1966), pp. 62–65.

As in Section 2, \( G \) has an irreducible character \( Y \) of degree \( q \) corresponding to the 2-transitive permutation representation of \( G \) of degree \( 1+q \) on \( \Omega \), and again \( Y|_H \) is irreducible, whence \( Y|_H = \theta_4 \). Being the unique extension of \( \theta_4 \) which is rational, \( Y \in B \) forcing \( Y = X_4 \). Moreover, \( Y(u) = 1 \) and \( Y(v) = 5 \) since \( v \) fixes exactly 6 elements of \( \Omega \). Thus \( X_4 \) satisfies (1) with \( k = 4 \), and it suffices to show that also \( X_2 \) and \( X_3 \) do so. By the O.R.B. we have:

\[ 0 = \sum X(u) X(v) = 1 - X_2(v) - X_3(v), \]

whence \( X_2(v) + X_3(v) = 6. \) As \( X_2(u) = X_3(u) = -1 \), it suffices to show that \( X_2(v) = 3 \). In particular, it suffices to show that \( \psi = \theta_4 \) has an extension \( \hat{\psi} \) to \( G \) with \( \hat{\psi}(v) = 3 \), since being the unique extension of \( \psi \) which is rational on \( v \), \( \hat{\psi} \in B \) whence \( \hat{\psi} = X_2 \).

Let \( R = \langle v \rangle \) and choose \( Q \in \text{Syl}_5(H) \) and a cyclic subgroup \( C \) of \( H \) of order \( (q-1)/2 \), such that \( N = N_H(Q) = QC \) and \( R \subseteq N_G(Q) \cap N_G(C) \). It follows from the character table of \( H \) that \( \psi|_N = \theta + \lambda \), where \( \theta \) is irreducible of degree \( (q-1)/2 \) and \( \lambda^2 = 1 \). Now \( \theta \) has a unique extension \( \hat{\theta} \) to \( NR \) such that \( \hat{\theta} \) is real (see Isaacs
(1976), Theorems 11.22 and 6.17, remembering that $N$ has 2 irreducible characters of degree $(q-1)/2$ and $|NR : N|$ is odd). It can be shown similarly, that $\psi$ has a unique extension $\hat{\psi}$ to $G$ such that $\hat{\psi}|_{NR}$ contains $\theta$ as a component. Thus $\hat{\psi}|_{NR} = \hat{\theta} + \lambda$, where $\lambda$ is an extension of $\lambda$. Since $\hat{\psi}$ is unique and $\hat{\theta}$ is real, so also $\hat{\psi}$ is real, forcing $\hat{\lambda}(v)$ to be real. Consequently $\lambda(v) = 1$ and it suffices to show that $\hat{\theta}(v) = 2$.

Since $\theta$ is a character of $N$ induced from $Q$, $\theta|_{C}$ is the regular character of $C$. Write the representation which affords $\hat{\theta}|_{CR}$ with reference to a basis consisting of eigenvectors for a generator $c$ of $C$. As the eigenvectors correspond to distinct eigenvalues, and as $v$ normalizes $C$, the matrix representing $v$ must be monomial, and has precisely two zero entries on the diagonal (namely, in the positions corresponding to the eigenvalues $+1$ and $-1$ of $c$; no other $\frac{1}{5}(q-1)$th root of $1$ is invariant under the fifth powering action of $v$). Thus $\hat{\theta}|_{CR} = \mu + \nu + \tau$, where $\mu$ and $\nu$ are linear characters and $\tau$ is a character vanishing on $v$ and on $uv$, where $u$ denotes an involution in $C$. Since $\hat{\theta}$ is real, $\mu + \nu$ is real on $v$ and on $uv$. Choose the notation so that $\mu(u) = 1 = -\nu(u)$. Then both $(\mu + \nu)(v) = \mu(v) + \nu(v)$ and $(\mu + \nu)(uv) = \mu(v) - \nu(v)$ are real, forcing $\mu(v)$ and $\nu(v)$ to be real. Consequently $\mu(v) = \nu(v) = 1$ and $\hat{\theta}(v) = 2$, as required.

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