COMPOSITES OF TRANSLATIONS AND ODD RATIONAL POWERS ACT FREELY

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Dedicated to Gilbert Baumslag belatedly on his 60th birthday
with our appreciation and respect.

It is shown that no non-trivial composition of translations \( x \mapsto x + a \) and odd rational powers \( x \mapsto x^{p/q} \), where \( p, q \) are odd co-prime integers, positive or negative with \( p/q \neq \pm 1 \), acts like the identity on a field of characteristic zero. This extends a theorem of Adeleke, Glass, and Morley in which only odd positive rational powers were considered. Moreover, the nature of the proof itself (by field theory) is a simplification and natural refinement of previous proofs. It has applications in other settings.

1. INTRODUCTION

Let \( L \) be a field of characteristic zero (such as \( \mathbb{R} \) or \( \mathbb{C} \)). Denote by \( T_L \) the Abelian group (under composition) of translations \( T_L = \{ t_a : a \in L \} \), where \( xt_a = x + a \), and by \( P^+_0 \) that of odd positive rational power maps

\[
P^+_0 = \{ e_{p/r} q : p, q \text{ odd co-prime positive integers} \},
\]

where \( xe_p = x^p \) and \( x^{1/q} \) and it is assumed that the action \( x \mapsto x^{p/q} \) is always effected by \( e_{p/r} q \) in that order.

Let \( w \) be a non-empty (reduced) word in the (formal) free product \( P^+_0 \ast T_L \); \( w \) is a string of elements (not the identity) alternately from \( P^+_0 \) and \( T_L \). Then \( w \) may be considered to act on an arbitrary \( \alpha \in L \) to produce an element in its algebraic closure \( \overline{L} \), although, in general, any action of \( e_{p/r} q \) (with \( q > 1 \)) has to prescribe which \( q \)th root is extracted. It was shown by Adeleke, Glass and Morley [1] that \( w \) cannot act as the identity on \( L \) even if there is complete freedom in the selection of roots. Of course, when \( L = \mathbb{R} \), \( w \) can be regarded naturally as an element of \( \text{Sym}(\mathbb{R}) \), the group

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of bijections of $\mathbb{R}$ into itself, and their theorem implies that the subgroup of $\text{Sym}(\mathbb{R})$ generated by $T_\mathbb{R}$ and $P_0^+$ is isomorphic to the free product $P_0^+ * T_\mathbb{R}$.

This result incorporates the pioneering work of White [5] who proved that the subgroup of $\text{Sym}(\mathbb{R})$ generated by $T_\mathbb{R}$ and $e_p$ for a fixed odd prime $p$ is their free product. Later, Cohen [2], overcame major technical obstacles to probe the analogue of the theorem of Adeleke, Glass and Morley for the free product $P^+ * T_L$, where $P^+$ is the group of all positive rational powers, that is,

$$P^+ = \{e_p r_q : p, q \text{ co-prime positive integers}\}.$$ 

It would be natural to seek to extend the above results to the free products $P_0 * T_L$ and $P * T_L$, where $P_0$, $P$ are the groups of all odd rational powers (positive and negative) and all non-zero rational powers, respectively, that is,

$$P_0 = \{e_p r_q : p, q \text{ co-prime odd integers}\},$$

$$P = \{e_p r_q : p, q \text{ non-zero co-prime integers}\}.$$ 

For these we adopt the conventions that the action of any element of $P$ on zero is undefined and that two words in $P * T_L$ can be supposed to have the same action on $L$ if they agree whenever both are defined. Whereas, however, the exact set of $a \in \mathbb{C}$ with $|a| < 2$ for which $t_a$ and $e_{-1}$ do not generate a free product $\mathbb{Z} * (\mathbb{Z} / 2\mathbb{Z})$ is unknown, it is certainly non-empty (see [4]); in particular $t_1$ and $e_{-1}$ themselves do not generate such a free product because, for example $t_1 e_{-1} t_1^{-1} e_{-1} t_1 e_{-1}$ has order 2, yet is not conjugate to $e_{-1}$, as can easily be seen. It is therefore pointless to investigate these free products in their entirety. So let $S_0(L)$ be the subset of $P_0 * T_L$ comprising those non-empty words in whose reduced form the power $e_{-1} t_1$ (corresponding to $x \mapsto 1/x$) does not appear and $S(L)$ be the corresponding subset of $P * T_L$. Then $S_0(L)$ and $S(L)$ are closed as regards the taking of inverses. We shall show that no member of $S_0(L)$ has the same action as the identity whenever it is defined. We believe that a similar result prevails for $S(L)$ but have not undertaken the details of a proof. Our dual aim is to present the extended result and to display the nature of the proof which is a considerable refinement of those of [5] and [1] distilled from [2] but freed from the technicalities of [2]. Indeed, the proof given here is far more perspicuous than that of [1].

For $w$ in $S_0(L)$ let $Q_w$ be the field (finitely) generated over $\mathbb{Q}$ by $\{a \in L : t_a \text{ occurs in the expansion of } w\}$. Evidently, for any $\alpha$ in $L$, $\alpha w$ is undefined only on a finite subset of $\overline{Q_w}$, the algebraic closure of $Q_w$.

**Theorem 1.** Let $L$ be a field of characteristic zero and $w$ a word in $S_0(L)$. Then for every $\alpha$ in $L$ not in a certain subset of $L \cap \overline{Q_w}$, $\alpha w$ is defined and $\alpha w \neq \alpha$, no matter how the roots are extracted at any stage.
Of course in Theorem 1 we can replace \( L \) by its algebraic closure. Further, given \( w \), define \( K = \overline{Q_w} \) which we may assume to be a subfield of \( C \). The bulk of the proof is associated with proving that \( \zeta w \neq \zeta \) for any element \( \zeta \) transcendental over \( K \); we may adjoin \( \zeta \) to \( L \) if necessary. It is then easy to deduce the result for \( \alpha \) in \( K \), see Section 6. So until then we suppose \( \zeta \) is a given transcendental.

In fact we deduce Theorem 1 from a stronger result which is the subject of the next section.

2. HYPOTHESIS H

We use notation and conventions developed from [5], [1] and [2].

Any word \( w \) in \( S_0(L) \) can be expressed (essentially uniquely) as a string of symbols \( w = v_1 \ldots v_n \) that allow no cancellation. Here \( n \) is the length of \( w \). Specifically, each \( v_j \) (\( 1 \leq j \leq n \)) is either \( t_a \) (\( a(\neq 0) \in L \)), \( e_p \) (\( p \in \mathbb{Z}, |p| > 1 \)) or \( r_q \) (\( q \in \mathbb{Z}, |q| > 1 \)). In particular, any \( e_p \) or \( r_q \) with \( p = \pm 1 \) or \( q = \pm 1 \) have been absorbed into neighbouring symbols. Moreover, \( r_q \) must be followed by a translation (unless it is at the end of \( w \)). If \( v_1 = t_a \), then \( w \) will be called a translation word. If a consecutive pair \( e_p r_q \) has \( p/q \) positive we can assume both \( p \) and \( q \) are positive whereas, if \( p/q \) is negative we permit the (harmless) ambiguity about which of the pair \( p, q \) is positive. Given \( \zeta = \zeta_1 \), we define the transcendental chain for \( w \) to be \( \{\zeta_1, \ldots, \zeta_{n+1}\} \), where \( \zeta_{j+1} = \zeta v_j \), \( j = 1, \ldots, n \) and, when \( v_j = r_q \), some choice of root is made.

There is also a syllable form for \( w \). To this end, call a word \( f \) none of whose symbols is a root a rational word because \( C, f \) is a rational function in \( K(\zeta) \). Associated with its action is a rational function \( f(x) \) which is either \( x + a \) (\( a \neq 0 \)) or

\[
(2.1) \quad f(x) = ((x + a_1)^{p_1} + a_2)^{p_2} + \cdots + a_\ell)^{p_\ell} + a_{\ell+1} (\ell \geq 1),
\]

where \( |p_j| > 1 \), \( 1 \leq j \leq \ell \) and \( a_j \neq 0 \), \( 2 \leq j \leq \ell \), though \( a_1 \) or \( a_{\ell+1} \) may be zero. From this, \( w \) has an expression (essentially unique) as \( w = s_1 \ldots s_k \) (\( k \geq 1 \)), where for each \( j = 1, \ldots, k - 1 \), the syllable \( s_j \) has the shape \( s_j = f_j r_{q_j} \), with \( f_j \) a rational word that, for \( j > 1 \), is necessarily a translation word. When \( j = k \) there need not be a concluding root \( r_{q_k} \) though it is sometimes convenient to interpret \( q_k \) as 1 in the latter situation. Associated with the syllable form is the syllable transcendental chain \( \{\mu_1, \ldots, \mu_{k+1}\} \), where

\[
\mu_1 = \zeta_1 = \zeta, \mu_{j+1} = \mu_j s_j, 1 \leq j \leq k.
\]

This is a sub-chain of \( \{\zeta_1, \ldots, \zeta_{n+1}\} \). In association with either chain we sometimes use notation such as \( (\mu_i, \mu_j) \) (\( i < j \)) as shorthand for a sub-word \( s_i \ldots s_{j-1} \) of \( w \) whose action sends \( \mu_i \) to \( \mu_j \).
When \( k = 1 \) and \( w \) is a rational word (represented by (2.1)) we can dispose of Theorem 1 by the following argument. By an easy induction on \( \ell \), \( f \) is a quotient of co-prime polynomials \( f_1/f_2 \) with \( \max(\deg f_1, \deg f_2) = |p_1 \ldots p_\ell| \) and the result is immediate.

When \( f \) is not rational word, for each \( j = 1, \ldots, k + 1 \), define \( K_j = K(\mu_1, \mu_j) \), where each such field is evidently an algebraic extension of \( K_1 \). We shall show that, in fact, \( K_{k+1} \neq K_1 \) and hence \( (\zeta_{n+1} = \mu_{k+1} =) \zeta w \neq \zeta(= \zeta_1 = \mu_1) \), which implies Theorem 1 for \( \zeta \). This assertion is incorporated in the main result we shall prove which we label Hypothesis H for comparison with [1] and [2]. (Recall that a field \( F \) is a pure extension of a field \( E \) if \( F = E(b^{1/m}) \) for some \( b \in E \) and positive integer \( m \).)

**Theorem 2.** (Hypothesis H) Let \( w = v_1 \ldots v_n = s_1 \ldots s_k \) be a word in \( S_0(L) \) and \( \zeta \) be transcendental over \( K \). Then

\[
H_1: K(\zeta_1) \subseteq K(\zeta_1, \zeta_2) \subseteq \ldots \subseteq K(\zeta_1, \zeta_{n+1});
\]
\[
H_2: K_1 \subseteq K_2 \subseteq \ldots \subseteq K_{k+1}, \text{ where the inclusions are strict}
\]
\[
\quad \text{(except the final one if } q_k = 1).\]
\[
H_3: \text{ if } F \text{ is a pure extension of } K_1 \text{ contained in } K_{n+1}, \text{ then } F \subseteq K_2.
\]

Note that \( H_1 \) implies that \( K_1 \subseteq K_2 \subseteq \ldots \subseteq K_{k+1} \) and that the substance of \( H_2 \) is that generally these containments are strict. We also note the following immediate consequence of Theorem 2 (specifically of \( H_2 \)).

**Corollary 3.** For \( w, \zeta \) as in Theorem 2, \( [K_{k+1}: K_1] = |q_1 \ldots q_k| \).

The truth of Theorem 2 for words of length not exceeding \( n \) will be labelled \( H(n) \) and that of each part \( H_j(n), j = 1, 2, 3 \), as appropriate. \( H(n) \) is established by induction on \( n \). \( H(1) \) is simple and the induction step proceeds in stages according to the scheme

\[
H(n) \Rightarrow H_1(n + 1) \Rightarrow H_2(n + 1) \Rightarrow H_3(n + 1).
\]

Since we shall always assume \( H(n) \) and be investigating \( H(n + 1) \), throughout we shall suppose that \( \omega = v_1 \ldots v_{n+1} \) (with associated transcendental chain \( \{\zeta_1, \ldots, \zeta_{n+2}\} \)). Nevertheless we shall continue to suppose \( w = s_1 \ldots s_k \) has \( k \) syllables and use the notation of this section. The theorem is easy if \( k = 1 \) so we assume \( k \geq 2 \).

We observe that induction always takes care (easily) of words that begin or end with a translation so we may assume this is not the case. Moreover, as far as Theorem 2 is concerned, we may replace \( \zeta \) by \( \zeta_1^{-1} \) and/or \( \zeta_{n+2} \) by \( \zeta_{n+2}^{-1} \), if necessary, and assume that \( w \) begins and ends with a positive power \( e_p \) (\( p > 1 \)) or positive root \( r_q \) (\( q > 1 \)).
3. Proof of $H_1(n + 1)$

By $H_1(n)$ (applied to $v_1 \ldots v_n$ and $v_2 \ldots v_{n+1}$)

(3.1) \[ K(\zeta_1) \subseteq K(\zeta_1, \zeta_2) \subseteq \ldots \subseteq K(\zeta_1, \zeta_{n+1}) \]

and

(3.2) \[ K(\zeta_2) \subseteq K(\zeta_2, \zeta_3) \subseteq \ldots \subseteq K(\zeta_2, \zeta_{n+2}). \]

Suppose, however, that $K(\zeta_1, \zeta_{n+2})$ does not contain $K(\zeta_1, \zeta_{n+1})$. Then obviously $v_{n+1}$ is a power (and $s_k$ does not end in a root). Trivially, $\zeta_{n+2} = \zeta_{n+1} v_{n+1} \in K(\zeta_{n+1})$ and hence $K(\zeta_1, \zeta_{n+2})$ is strictly contained in $K(\zeta_1, \zeta_{n+1})$. Further, $v_1$ is a root because otherwise $\zeta_2 \in K(\zeta_1)$ and the inconsistent conclusion $K(\zeta_1, \zeta_{n+1}) \subseteq K(\zeta_1, \zeta_{n+2})$ is a consequence of adjoining $\zeta_1$ to the final two fields in the chain (3.2). Moreover, we may also assume that $K(\zeta_1, \zeta_{n+2}) \cap K(\zeta_1, \zeta_2) = K(\zeta_1)$; for this purpose, if $v_1 = r_q$ ($q > 1$) it may be necessary to replace $\zeta_1 = \zeta_2^q$ by $\zeta_2^m$, where $m$ (≠ $q$) is a positive divisor of $q$, and $v_1$ by $r_{q/m}$. Since $\zeta_{n+2} \in K(\zeta_{n+1})$ and $\zeta_1 \in K(\zeta_2)$ we deduce that

(3.3) \[ K(\zeta_1, \zeta_{n+1}) = K(\zeta_2, \zeta_{n+2}) = K(\zeta_2, \zeta_{n+1}), \]

this field strictly containing $K(\zeta_1, \zeta_{n+2})$.

In terms of syllables, (3.1)-(3.3) yield the following (for which we note that $\mu_1 = \zeta_1 = \mu_2^q$):

(3.4) \[ K(\mu_1, \mu_k) = K(\mu_2, \mu_{k+1}) = K(\mu_2, \mu_k), \]

a field which strictly contains $K_{k+1} = K(\mu_1, \mu_{k+1})$. Moreover, $K_{k+1} \cap K_2 = K_1$ and $K_k = K_{k+1}(\mu_2)$ is a pure extension of $K_{k+1}$ of degree $q$.

Suppose that $k = 2$. Then $w = r_q f$, where $f$ is a rational translation word. From the above, $K_3 \cap K_2 = K_1$ so that $\mu_3 = f(\mu_2) \in K(\mu_1) = K(\mu_2^q)$. Hence, identically

(3.5) \[ f(x) = g(x^q) \]

for some rational function $g$. This is easily seen to be impossible since $f$ is a translation word: in any case it is covered by Lemma 4 below.

Suppose therefore that $k > 2$. Now $K_k/K_{k+1}$ is a cyclic Galois extension of degree $q$ (since $K$, being algebraically closed, contains all $q$th roots of unity). We apply to $K_k$ a generating automorphism $\tau$ of its Galois group. Thus $\tau$ fixes $K_{k+1}$ (element-wise) and sends $\mu_2$ to $\omega \mu_2$, where $\omega$ is a primitive $q$th root of unity. Set
\[ \mu_3 = \tau(\mu_3) \in K_k \] and let the second syllable \( s_2 \) be \( fr_d \). An application of \( \tau \) to the expression \( \mu_3 = f(\mu_2) \) yields \( \mu_3^d = f(\omega\mu_2) \). Both \( K_3 = K(\mu_2, \mu_3) \) and \( K(\mu_2, \mu_3) \) are pure extensions of \( K_2 = K(\mu_2) \) of degree \( d \) contained in \( K_k \) and so, by \( H_3(n) \) applied to the word \( (\mu_2, \mu_3) \), we deduce that these two fields are identical. From the basic result on pure extensions (see Exercise 16.16 of [3]) it follows that for some \( t \) (prime to \( d \)) \( \mu_3\mu_2^d \in K(\mu_2) \). Hence, taking \( d \)th powers and, setting \( x = \mu_2 \), we have

\[ (3.6) \quad f(x)f^t(\omega x) = h^d(x), \]

identically for some rational function \( h(x) \). Evidently, (3.6) is impossible when \( f(x) = x + a \). For other cases it is timely to introduce a lemma adapted from [5], [1] and [2]. It disposes immediately of (3.5) and (3.6) and plays a similar role in the verification of \( H_2 \) and \( H_3 \). For other cases it is timely to introduce a lemma adapted from [5], [1] and [2].

**Lemma 4.** Suppose that \( p, q, d \) are odd integers of absolute value exceeding 1 and \( \omega \ (\neq 1) \) is a \( q \)-th root of unity. Suppose also that \( f, g, h \) are rational functions in \( K(x) \) with \( f(x) = f_0((x + a)^p) \), \( a \neq 0, f \neq f_1 \). Then, for no integer \( t \) is there an identity of the form

\[ (3.7) \quad f(x)f^t(\omega x)g(x^q) = h^d(x). \]

**Proof:** Easily we may assume that \( p, q \) and \( d \) are positive. Assuming (3.7), we may multiply it by \( (f_2(x)f_2^t(\omega x)g_2(x^q))^d \), where \( f_2 \) and \( g_2 \) are the denominators of \( f \) and \( g \), respectively, and obtain an analogous identity with \( f \) and \( g \) replaced by polynomials \( f_2^2f \) and \( g_2^2g \), respectively in which case the "new" \( h \) is also a polynomial. The result is then immediate from Lemma 10 of [1] or Lemma 9.1 of [2].

4. **Proof of \( H_2(n + 1) \)**

We can now assume \( H_1(n + 1) \) in addition to \( H(n) \). By \( H_2(n) \), it remains to prove that \( K_k \subset K_{k+1} \) when \( s_k \) ends in \( \tau_q \) (\( q \geq 3 \)). Assume that \( K_k = K_{k+1} \). This is unaffected when \( q \) is replaced by a prime divisor \( d \). If \( \mu_{k+1} \in K(\zeta_2, \mu_k) \), then \( K(\zeta_2, \mu_k) = K(\zeta_2, \mu_{k+1}) \) contradicting \( H_2(n) \) applied to \( (\zeta_2, \zeta_{n+2}) \). Hence \( \mu_{k+1} \not\in K(\zeta_2, \mu_k) \) and, in particular, \( w \) must begin with a power, \( v_1 = e_p \) (\( p \geq 3 \), say).

Now, by assumption and \( H_1(n + 1) \),

\[ (4.1) \quad K(\zeta_2, \mu_k)(\mu_{k+1}) = K(\zeta_2, \mu_{k+1}) \]

\[ \subset K(\zeta_1, \mu_{k+1}) = K(\zeta_1, \mu_k) = K(\zeta_2, \mu_k)(\zeta_1). \]

From (4.1) the field \( K(\zeta_2, \mu_{k+1}) \) intermediate between \( K(\zeta_2, \mu_k) \) and \( K(\zeta_1, \mu_k) \) has the form \( K(\zeta_1^s, \mu_k) \) for some proper divisor \( s \) of \( p \). Since \( d = [K(\zeta_2, \mu_{k+1}): K(\zeta_2, \mu_k)] \),
we have \( p = sd \). By replacing \( v_1 \) by \( e_{p/s} \) we can assume \( p = d \). Summarising, 
\( \zeta_1 \not\in K(\zeta_2, \mu_k) \), yet

\[
K(\zeta_2, \mu_{k+1}) = K(\zeta_1, \mu_k) = K(\mu_1, \mu_{k+1}).
\]  

For an analysis of (4.2) write

\[
w = \ldots e_m g^{-1} r_q f r_d,
\]

where only the latter section of \( w \) is displayed and \( f \) and \( g \) are rational translation words with \( g^{-1} \) denoting the inverse of \( g \). Also let \( u = (s_1 \ldots s_{k-1})^{-1} = (\mu_k, \zeta_1) \) have \( \{\nu_1 = \mu_k, \nu_2, \ldots\} \) as its associated syllable transcendental chain and put \( F = K(\mu_k, \mu_{k+1}) \). Since \( \mu_{k+1}^d = f(\mu_k) \), \( F \) is a pure extension of \( K(\nu_1) (= K(\mu_k)) \) of prime degree \( d \) contained in \( K(\mu_k, \zeta_1) \) but not \( K(\mu_k, \zeta_2) \). Apply \( H_3(n) \) to the word \( u \) with respect to \( K(\nu_1) \subseteq F \subseteq K(\mu_k, \zeta_1) \). Then \( F \subseteq K(\nu_1, \nu_2) \). Unless \( u \) is a monosyllable, by \( H_1(n) \) applied to \( u \), \( K(\nu_1, \nu_2) \subseteq K(\nu_1, \zeta_2) = K(\mu_k, \zeta_2) \) which yields the contradiction \( \mu_{k+1} \in K(\zeta_2, \mu_k) \). Thus \( u \) is indeed monosyllabic with \( K(\mu_k)(\zeta_1) = K(\mu_k)(\mu_{k+1}) = F \) and, necessarily, \( m = d \) and

\[
w = e_d g^{-1} r_q f r_d.
\]

Hence, for some \( t \) (prime to \( d \)), \( \mu_{k+1} \mu_t \in K(\mu_k) \). Raising this to the \( d \)-th power and setting \( x = \mu_k \) we obtain from (4.4)

\[
f(x)g^t(x^q) = h^d(x)
\]

for some rational function \( h \). This contradicts Lemma 4.

We remark that now that \( H_2(n + 1) \) has been established we may use Corollary 3.

5. PROOF OF \( H_3(n + 1) \)

We may assume \( H_1(n + 1), H_2(n + 1), H_3(n) \) and Corollary 3.

Let \( F \) be a pure extension of \( K_1 \) contained in \( K_{k+1} \) but not in \( K_2 \). By \( H_3(n) \) we can suppose \( s_k \) ends in a root \( r_q \) \((q > 1)\). To obtain a contradiction, it suffices to suppose that \( F/(K_2 \cap F) \) is a pure extension of prime degree \( d \). Again by \( H_3(n) \) we can suppose that \( F \not\subseteq K_k \). Hence \( F(\mu_k) (= F_1, \text{ say}) \), which clearly contains \( K_k \), must be a pure extension of \( K_k \) of degree \( d \) contained in \( K_{k+1} \). By Corollary 3 we may replace the final root \( r_q \) of \( w \) by \( r_d \) and assume that \( F_1 = K_{k+1} \).

Again write \( w \) as (4.3) (where \( q \) has a new meaning). When \( k \geq 3 \) let \( F_0 \) be the subfield \( F(\mu_{k-1}) \) of \( F_1 \). When \( k = 2 \), defer the possibility

\[
K_1 \subset K_2 \subset F = F_1 = K_3
\]
meantime, and otherwise set \( F_0 = F \). Then \( F_0 \) contains \( K_{k-1} \), yet \( F_0/K_{k-1} \) (\( F_0/(K_2 \cap F_0) \) when \( k = 2 \)) must be an extension of degree \( d \). By Corollary 3, \([K_{k+1}: K_{k-1}] = qd \) and so \( F_0 \neq F_1 \), whereas \( F_0(\mu_k) = K_{k+1} \); in particular, \( K_{k+1}/F_0 \) is a pure extension of degree dividing \( q \). Hence there is an \( F_0 \)-automorphism \( \tau \) of \( K_{k+1} \) which maps \( \mu_k \mapsto \omega \mu_k \), where \( \omega \) is a \( q \)th root of unity. Moreover, if \( k = 2 \) and (5.1) holds, then \( K_3/K_1 \) is a cyclic extension of degree \( dq \) and there is a \( K_1 \)-automorphism \( \tau \) of \( K_3 \) with a similar property. Set \( \mu_{k+1} = \tau(\mu_{k+1}) \in K_{k+1} \). Then, in either case, clearly \( K_3(\mu_{k+1}) = K_3(\mu_{k+1}) = K_{k+1} \), whence \( \mu_{k+1} \mu_{k+1}^t \in K_k \) for some integer \( t \) (indivisible by \( d \)). Further, \( K(\mu_k, \mu_{k+1} \mu_{k+1}^t) \subseteq K(\mu_k, \mu_1) \) yet
\[
(\mu_{k+1} \mu_{k+1}^t)^d = f(\mu_k) f^t(\mu_k) \in K(\mu_k).
\]

As in Section 4 (following (4.3)), by applying \( H_3(n) \) to \( u = (s_1 \ldots s_{k-1})^{-1} \) with syllable transcendental chain \( \{ \mu_k = \nu, \nu_2, \ldots \} \) we deduce that \( \mu_{k+1} \mu_{k+1}^t \in K(\nu_1, \nu_2) = K(\mu_k)(\nu_2) \). Hence \( d \mid m \) and, for some integer \( u \), divisible by \( m/d \),
\[
\mu_{k+1} \mu_{k+1}^t \in K(\mu_k).
\]
Taking \( d \)th powers and replacing \( \mu_k \) by \( y \) yields
\[
f(z)^f(\omega y) y^u(y^2) = h^d(z),
\]
for some rational function \( h \). This contradicts Lemma 4.

The proof of Theorem 2 is complete.

6. COMPLETION OF THE PROOF OF THEOREM 1

With \( \omega, \zeta \) as in Theorem 2 and Corollary 3 we can explicitly construct \( P(z, y) \), a monic irreducible polynomial in \( z \) of degree \( q_1 \ldots q_k \) with coefficients in \( K(y) \) such that \( P(\mu_{k+1}, \mu_1) = P(\zeta \omega, \zeta \omega) = 0 \). The same \( P \) is obtained no matter how we extract roots when we consider the action of \( \omega \).

Set \( P_{k+1}(z, y) = z - y \) and define \( P_j(z, y) \), \( j = k, \ldots, 1 \), as follows:
\[
\text{let } P_j(z, \mu_j) = \prod_{i=0}^{Q_j-1} P_{j+1}(z, \omega_{j+1}^i \mu_{j+1}), \quad j = k, \ldots, 1,
\]
where \( Q_j = |q_j| \) and \( \omega_{j+1}^i \) is a primitive \( Q_j \)th root of unity. Then \( P_j(z, \mu_j) \) is a polynomial in \( z \) whose coefficients are rational functions in \( K(\mu_j) \subseteq K(\zeta, \mu_j) \) with \( \mu_j \) transcendental over \( K \). To obtain \( P_j(z, y) \) simply replace these coefficients by the corresponding rational functions in an indeterminate \( y \) (transcendental over \( K(z) \)). Put \( P(z, y) = P_1(z, y) \) and our claim is justified (by Corollary 3).

It follows that \( P(z, \zeta) \) certainly cannot have \( z - \zeta \) as a factor. Specialising \( \zeta \mapsto \alpha \in K \) we conclude that \( P(z, \alpha) \) is undefined or has a factor \( a - \alpha \) for only finitely many values of \( \alpha \). For all other values of \( \alpha \) in \( K \), \( P(\alpha, \alpha) \neq 0 \) and so \( \omega a \neq \alpha \). This completes the proof.
REFERENCES


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