POSTERIOR REGRET Γ-MINIMAX ESTIMATION OF INSURANCE PREMIUM IN COLLECTIVE RISK MODEL

BY

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ABSTRACT

The collective risk model for the insurance claims is considered. The objective is to estimate a premium which is defined as a functional H specified up to an unknown parameter θ (the expected number of claims). Four principles of calculating a premium are applied. The Bayesian methodology, which combines the prior knowledge about a parameter θ with the knowledge in the form of a random sample is adopted. Two loss functions (the square-error loss function and the asymmetric loss function LINEX) are considered. Some uncertainty about a prior is assumed by introducing classes of priors. Considering one of the concepts of robust procedures the posterior regret Γ-minimax premiums are calculated, as an optimal robust premiums. A numerical example is presented.

KEYWORDS

Bayesian model, classes of priors, posterior regret, square error loss, LINEX, insurance premium.

1. INTRODUCTION

We consider a Bayesian collective risk model. Our objective is to estimate a premium which is defined as a functional H that assigns to any risk S a real number $H(S)$, the premium for taking the risk S. In practical situations the premium $H(S)$ can be calculated if the distribution of the risk S is known. We shall consider the case in which the distribution of S or the premium $H(S)$ is specified up to an unknown parameter θ, thus the risk premium will be denoted $H(\theta)$. The premium $H(\theta)$ is calculating according to four principles (for definitions see Section 3). Next we ought to estimate $H(\theta)$. We will use the Bayesian methodology, which combines the prior knowledge about a parameter θ (defined by a prior distribution Π) with the knowledge in the form of a random sample $X = (X_1, X_2, \ldots, X_n)$, where the distribution of this random variable depends on θ.

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Given a value x of X and using the loss function approach the Bayesian premium is defined (Heilmann(1989)) to be a real number $\hat{H}^B_\Pi$ minimising the posterior expected loss (posterior risk). In our work we will consider two loss functions:

– the square-error loss function

$$L_s(\theta,d) = (H(\theta) - d)^2,$$

– the LINEX loss function with a fixed coefficient c

$$L_l(\theta,d) = \exp(c(H(\theta) - d)) - c(H(\theta) - d) - 1,$$

where $c \neq 0$, $H(\theta)$ is a functional of a premium and d is an estimate.

The square-error loss function has been considered in many papers about Bayesian analysis in the risk theory, see Makov et al. (1996), Klugman (1992), Klugman et al. (1998) for examples. Being symmetric, the square-error loss equally penalizes over- and under-estimation of the same magnitude. In insurance the penalty for underestimation does not necessarily have to be the same as for overestimation. The asymmetric loss function (LINEX loss) defined by Varian (1975) (for motivation see also Zellner, 1986) gives greater error for underestimation than for overestimation if $c > 0$ (for $c < 0$ it is the other way around). Without loss of generality we will assume $c > 0$. The LINEX loss is connected with the premium calculated according to the well known exponential principle (see e.g. Kaas et al., 2001), whereas the square loss gives the net premium.

The obtained Bayesian premium depends on the choice of a prior Π. In most Bayesian analysis the elicitation of a prior is quite difficult and can be uncertain. To model uncertainty of the prior information the robust Bayesian inference uses a class Γ of priors. It deals with the problem of measuring the range of a posterior quantity (for example: the range of a Bayes estimator, a posterior risk) while a prior distribution runs over the class Γ. Its aim is also to find robust, optimal procedures: conditional Γ-minimax, stable or posterior regret Γ-minimax. The general references on robust Bayesian methods are Berger (1994), Ríos Insua and Ruggeri (2000). In insurance models the range of a premium when priors run over a class Γ has been considered in Ríos Insua et al. (1999) and Gómez-Déniz et al. (1999), (2002a,b) among others. The posterior regret Γ-minimax estimators under the square-error loss were applied in Gómez-Déniz et al. (2006), for the asymmetric loss (LINEX loss) the optimal robust estimators have not been presented in insurance models so far.

In this paper we assume that the actuary is unable to specify a simple prior distribution of the expected number of claims and, considering one concept of robust procedures, we calculate the posterior regret Γ-minimax premium as an optimal, robust premium when a prior runs over a class Γ. As we see in Section 2 its value depends only on the bounds of a set of Bayes actions calculated with respect to the priors belonging to the class Γ. Thus computing a posterior regret Γ-minimax estimator is simple provided that we have procedures to compute the range of posterior expectations. For every value of a random sample X there exists a prior in the class Γ such that the posterior regret Γ-minimax estimator becomes Bayes with respect to this prior (for more motivation see Vidakovic (2000) and references there in). The formula of robust estimators using other concepts of optimality (the conditional Γ-minimax and stable estimators) is not so easy. For the parameter of the Poisson distribution with the classes Γ_1 and Γ_2 of priors (for definitions see Section 3) these estimators are presented in Boratyńska (2002b), for the class Γ_4 the solution has not been found so far.

The article is organized as follows. Section 2 presents a guide for finding posterior regret Γ-minimax estimators. Section 3 reviews the structure of the collective risk model, presents functionals of a premium. Section 4 defines classes of priors and presents posterior regret Γ-minimax estimators of a premium (we will use notation PRGM estimator or PRGM premium) and Section 5 contains a numerical example and concluding remarks.

2. Posterior regret Γ-minimax estimators

Generally let X be an observed random variable with a distribution P_θ indexed by a real parameter θ. Suppose θ has a prior distribution Π belonging to a family Γ. Let $X = x$ and $R_x(\Pi, \hat{g}(x))$ denote the posterior risk of an estimate $\hat{g}(x)$ of a real function $g(\theta)$ if the prior is Π. The posterior regret of an estimate $\hat{g}(x)$ is

$$r_x(\Pi, \hat{g}(x)) = R_x(\Pi, \hat{g}(x)) - R_x(\Pi, \hat{g}_\Pi^B(x)),$$

where $\hat{g}_\Pi^B$ is a Bayes estimator, if the prior is Π. In a sense, it measures the loss of optimality due to choosing d instead of the optimal Bayes estimate. The estimator $\hat{g}_\Gamma^{PR}$ is the posterior regret Γ-minimax estimator (PRGM estimator) if for every value x of X

$$\inf_{d \in \mathcal{R}} \sup_{\Pi \in \Gamma} r_x(\Pi, d) = \sup_{\Pi \in \Gamma} r_x(\Pi, \hat{g}^{PR}(x)).$$

We will use the following theorems to calculate the PRGM premium.

Theorem 1. (Zen and DasGupta (1993)) *Let* $L_s(\theta, d) = (g(\theta) - d)^2$ *and* $X = x$. Let $\underline{d} = \underline{d}(x) = \inf_{\Pi \in \Gamma} \hat{g}_\Pi^B(x)$ and $\overline{d} = \overline{d}(x) = \sup_{\Pi \in \Gamma} \hat{g}_\Pi^B(x)$ *be finite and* $\underline{d} < \overline{d}$. *Then* $\hat{g}_\Gamma^{PR}(x) = \frac{1}{2}(\underline{d} + \overline{d})$. ■

Theorem 2. (Boratyńska (2002a)) *Let* $L_l(\theta, d) = \exp(c(g(\theta) - d)) - c(g(\theta) - d) - 1$ *and* $X = x$. Let $\underline{d} = \underline{d}(x) = \inf_{\Pi \in \Gamma} \hat{g}_\Pi^B(x)$ and $\overline{d} = \overline{d}(x) = \sup_{\Pi \in \Gamma} \hat{g}_\Pi^B(x)$ *be finite and* $\underline{d} < \overline{d}$. *Then*

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$$\hat{g}_{\Gamma}^{PR} = \frac{1}{c} \cdot \ln \frac{e^{c\overline{d}} - e^{c\underline{d}}}{c\left(\overline{d} - \underline{d}\right)} = \underline{d} + \frac{1}{c} \ln \frac{e^{c\left(\overline{d} - \underline{d}\right)} - 1}{c\left(\overline{d} - \underline{d}\right)}. \qquad \blacksquare$$

Note, that if the set $\{\hat{g}_{\Pi}^{B}(x) : \Pi \in \Gamma\}$ is connected for every value x of X, then for every x there exists $\Pi \in \Gamma$ such that $\hat{g}_{\Gamma}^{PR}(x) = \hat{g}_{\Pi}^{B}(x)$.

In the next section our aim will be to estimate a linear function $H(\theta) = u\theta$. So here we find PRGM estimators of a function $H(\theta) = u\theta$. Using the square-error loss function and properties of the expected value we obtain

$$\hat{H}_{\Pi}^{B}(x) = E_{\Pi}(H(\theta)|x) = u\hat{\theta}_{\Pi}^{B}(x) \quad \text{and} \quad \hat{H}_{\Gamma}^{PR}(x) = u\hat{\theta}_{\Gamma}^{PR}(x),$$

where $E_{\Pi}(g(\theta)|x)$ denotes the expected value of a function $g(\theta)$ when θ has the posterior distribution while $\hat{\theta}_{\Pi}^{B}$ and $\hat{\theta}_{\Gamma}^{PR}$ are the Bayes estimator and the PRGM estimator of a parameter θ under the square-error loss function.

Similarly, under the LINEX loss function with a coefficient c

$$\hat{H}_{\Pi}^{B,c}(x) = \frac{1}{c} \ln E_{\Pi}\left(e^{cH(\theta)}\middle| x\right) = u\frac{1}{cu} \ln E_{\Pi}\left(e^{cu\theta}\middle| x\right) = u\hat{\theta}_{\Pi}^{B,cu},$$

where $\hat{\theta}_{\Pi}^{B,cu}$ is a Bayes estimator of θ under the LINEX loss function with a coefficient cu. Hence, applying Theorem 2, the PRGM estimator of a function H is equal

$$\hat{H}_{\Gamma}^{PR,c}(x) = u\hat{\theta}_{\Gamma}^{PR,cu}(x),$$

where $\hat{\theta}_{\Gamma}^{PR,cu}$ is the PRGM estimator of θ under the LINEX loss function with a coefficient cu.

3. The collective risk model and premium calculations

In the collective risk model a risk is a sequence of independent random variables $N, Y_1, Y_2, \ldots$. A random variable N describes the number of claims of a given contract or a portfolio of contracts. Assume it has a Poisson distribution P_θ, where $\theta > 0$ is unknown. Random variables $Y_1, Y_2, \ldots$ are i.i.d. and they describe sizes of claims. Assume the distribution of Y_i is known. Let $S = \sum_{i=1}^{N} Y_i$ and $S = 0$ if $N = 0$. Consider the following premium principles:

I) the net premium

$$H_1(\theta) = E_\theta S = E_\theta N \cdot EY_1 = \theta\mu,$$

where $\mu = EY_1$,

II) the variance principle premium with a coefficient $\eta > 0$

$$H_2(\theta) = E_\theta S + \eta Var_\theta S = \theta(\mu + \eta(\sigma^2 + \mu^2)),$$

where $\sigma^2 = VarY_1$ and $\mu = EY_1$,

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III) to calculate the Esscher premium with a coefficient $v > 0$ use the equalities

$$E_\theta(e^{vS}) = E_\theta[(Ee^{vY_1})^N] = e^{\theta(M_{Y_1}(v)-1)}$$

and

$$E_\theta(Se^{vS}) = E(Y_1 e^{vY_1})\, E_\theta(N \cdot (Ee^{vY_1})^{N-1}) = E(Y_1 e^{vY_1})\, \theta e^{\theta(M_{Y_1}(v)-1)},$$

where $M_Y(v)$ denotes the moment generating function of a random variable Y at a point v; hence the Esscher premium is

$$H_3(\theta) = \frac{E_\theta\left(Se^{vS}\right)}{E_\theta\left(e^{vS}\right)} = E\left(Y_1 e^{vY_1}\right)\theta,$$

IV) the exponential premium with a coefficient $\zeta > 0$ is

$$H_4(\theta) = \frac{1}{\zeta} \ln E_\theta e^{\zeta S} = \frac{1}{\zeta}\theta(M_{Y_1}(\zeta) - 1),$$

where M_{Y_1} is the moment generating function of the random variable Y_1.

In all cases we obtain the premium which is a linear function of an unknown parameter θ.

4. PRGM estimators of the premium in Poisson-Gamma model

Let $X = (X_1, X_2, \ldots, X_n)$ be an observed random sample, where X_i has the Poisson distribution P_θ with a parameter $\theta > 0$ (like a random variable N). Our aim is to estimate premiums $H_i(\theta)$, $i = 1, 2, 3, 4$, which are linear functions of an unknown parameter θ, so generally we would like to estimate $H(\theta) = u\theta$.

One of the most useful compound collective risk models involves assuming a Gamma prior distribution over θ. Let $\Pi_{\alpha,\beta} = Gamma(\alpha,\beta)$ be a Gamma distribution with a density function

$$\pi_{\alpha,\beta}(\theta) = \frac{\beta^\alpha}{\Gamma(\alpha)}\theta^{\alpha-1}e^{\beta\theta} \quad \text{for } \theta > 0,$$

where $\alpha, \beta > 0$ are the parameters. Suppose that the prior distribution of a parameter θ is not exactly specified and consider four classes of priors

$$\Gamma_1 = \{Gamma(\alpha, \beta_0) : \alpha \in [\alpha_1, \alpha_2]\},$$

where $\beta_0 > 0,\ 0 < \alpha_1 < \alpha_2$ are fixed,

$$\Gamma_2 = \{Gamma(\alpha_0, \beta) : \beta \in [\beta_1, \beta_2]\},$$

where $\alpha_0 > 0,\ 0 < \beta_1 < \beta_2$ are fixed,

$$\Gamma_3 = \{Gamma(\alpha,\beta) : \alpha \in [\alpha_1,\alpha_2], \beta \in [\beta_1,\beta_2]\},$$

where $0<\alpha_1<\alpha_2$ and $0<\beta_1<\beta_2$ are fixed,

$$\Gamma_4 = \{\Pi_Q = (1-\varepsilon)\,Gamma(\alpha_0,\beta_0) + \varepsilon Q : Q \in \mathcal{P}\},$$

where $\mathcal{P}$ is the family of all probability distributions on $(0,+\infty)$ and $\alpha_0>0$, $\beta_0>0$ $\varepsilon \in (0,1)$ are fixed.

If $\alpha_0 \in (\alpha_1,\alpha_2)$ and $\beta_0 \in (\beta_1,\beta_2)$ then families Γ_1, Γ_2 and Γ_3 express three types of uncertainty about parameters of the elicitated prior $Gamma(\alpha_0,\beta_0)$. Family Γ_4 is the ε-contamination class, where $Gamma(\alpha_0,\beta_0)$ is the initially specified prior distribution and ε is the amount of uncertainty.

To obtain PRGM premiums it is enough to find PRGM estimators of a parameter θ in the Poisson model. If a prior is $Gamma(\alpha,\beta)$ and $X = x = (x_1,x_2,\ldots,x_n)$, then the posterior distribution is $Gamma(\alpha+T,\beta+n)$, where $T=\sum_{i=1}^n x_i$. The Bayes estimators of θ under the square-error loss and under the LINEX loss function with a coefficient c are then respectively equal

$$\hat{\theta}^{B}_{\alpha,\beta}(x) = \frac{\alpha+T}{\beta+n} \text{ and } \hat{\theta}^{B,c}_{\alpha,\beta}(x) = \frac{\alpha+T}{c}\ln\frac{\beta+n}{\beta+n-c}.$$

They are increasing functions of α and decreasing functions of β. Table 1 presents the PRGM estimators of θ if the classes of priors are Γ_i, $i=1,2,3$. Under the square-error loss function they were obtained in Ríos Insua, Ruggeri and Vidakovic (1995).

TABLE 1

THE PRGM ESTIMATORS FOR θ, IF THE CLASSES OF PRIORS ARE EQUAL Γ_i, $i=1,2,3$.

Γ	$\hat{\theta}^{PR}_{\Gamma}$ under the square loss L_s	$\hat{\theta}^{PR}_{\Gamma}$ under LINEX loss L_l
Γ_1	$\dfrac{\frac{\alpha_1+\alpha_2}{2}+T}{\beta_0+n}$	$\dfrac{1}{c}\ln\dfrac{z^{\alpha_2+T}-z^{\alpha_1+T}}{(\alpha_2-\alpha_1)\ln z}$ i $z=\dfrac{\beta_0+n}{\beta_0+n-c}$
Γ_2	$\dfrac{\alpha_0+T}{\beta^*+n}$ i $\beta^*=\dfrac{2\beta_1\beta_2+(\beta_1+\beta_2)n}{\beta_1+\beta_2+2n}$	$\dfrac{1}{c}\ln\left[\dfrac{z_1^{\alpha_0+T}-z_2^{\alpha_0+T}}{(\alpha_0+T)(\ln z_1-\ln z_2)}\right]$ $z_i=\dfrac{\beta_i+n}{\beta_i+n-c}$, $i=1,2$
Γ_3	$\dfrac{\alpha^{**}+T}{\beta^{**}+n}$ $\alpha^{**}=\dfrac{\alpha_1(\beta_1+S)+\alpha_2(\beta_2+n)}{\beta_1+\beta_2+2n}$ i $\beta^{**}=\beta^*$	$\dfrac{1}{c}\ln\dfrac{z_1^{\alpha_2+T}-z_2^{\alpha_1+T}}{(\alpha_2+T)\ln z_1-(\alpha_1+T)\ln z_2}$

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Now consider the class Γ_4. Each distribution $Q \in \mathcal{P}$ is a mixture of one point priors Q_ϑ, where $Q_\vartheta(\vartheta) = 1$ and $\vartheta > 0$. Let

$$\Gamma_4' = \{\Pi_Q = (1-\varepsilon)\, Gamma(\alpha_0, \beta_0) + \varepsilon Q_\vartheta : \vartheta > 0\}.$$

Thus the bounds of the Bayes estimators under the class Γ_4 is equal the bounds of Bayes estimators under the class Γ_4' (see Sivaganesan and Berger (1989)). If a prior is $\Pi = (1-\varepsilon)\, Gamma(\alpha_0, \beta_0) + \varepsilon Q_\vartheta$, then under the square-error loss function the Bayes estimator is

$$\hat{\theta}_\Pi^B = \rho(\vartheta) = \frac{A\hat{\theta}^B_{\alpha_0,\beta_0} + \vartheta^{T+1} e^{-n\vartheta}}{A + \vartheta^T e^{-n\vartheta}},$$

where

$$A = \frac{1-\varepsilon}{\varepsilon} \frac{\beta_0^{\alpha_0} \Gamma(\alpha_0 + T)}{\Gamma(\alpha_0)(\beta_0 + n)^{\alpha_0 + T}},$$

and it reaches its bounds (as a function of ϑ) for ϑ_1 and ϑ_2 which are solutions of the equation

$$\vartheta^{T+1} e^{-n\vartheta} = A\left(n\vartheta^2 - n\vartheta\left(\frac{T+1}{n} + \hat{\theta}^B_{\alpha_0,\beta_0}\right) + T\hat{\theta}^B_{\alpha_0,\beta_0}\right) \tag{1}$$

(see Sivaganesan (1988)). Hence, the PRGM estimator of $H(\theta) = u\theta$ is

$$\hat{H}^{PR}_{\Gamma_4} = \frac{1}{2} u(\rho(\vartheta_1) + \rho(\vartheta_2)).$$

To find the PRGM estimator of θ under the LINEX loss function with a coefficient c it is enough to find $\sup_{\Pi \in \Gamma_4'} E_\Pi(e^{c\theta}|x)$ and $\inf_{\Pi \in \Gamma_4'} E_\Pi(e^{c\theta}|x)$. We will use the linearization techniques (see Lavine, Wasserman, Wolpert (1991)), which implies that

$$\sup_{\Pi \in \Gamma_4'} E_\Pi(e^{c\theta}|x) = \inf\{q : \sup_{\Pi \in \Gamma_4'} h(q, \Pi) < 0\}$$

and

$$\inf_{\Pi \in \Gamma_4'} E_\Pi(e^{c\theta}|x) = \sup\{q : \inf_{\Pi \in \Gamma_4'} h(q, \Pi) > 0\},$$

where

$$\begin{aligned} h(q, \Pi) &= \int_\Theta \left(e^{c\theta} - q\right) e^{-n\theta} \theta^T \,\Pi(d\theta) \\ &= (1-\varepsilon)\, B\left[\left(\frac{\beta_0 + n}{\beta_0 + n - c}\right)^{\alpha_0 + T} - q\right] + \varepsilon\left(e^{c\vartheta} - q\right) e^{-n\vartheta} \vartheta^T \end{aligned}$$

and

$$B = \frac{\beta_0^{\alpha_0}\Gamma(\alpha_0 + T)}{\Gamma(\alpha_0)(\beta_0 + n)^{\alpha_0 + T}}.$$

The function $s(\vartheta) = (e^{c\vartheta} - q)e^{-n\vartheta}\vartheta^T$ tends to 0 if ϑ tends to 0 or $+\infty$ (assume $c < n$) and its derivative is equal

$$s'(\vartheta) = e^{-n\vartheta}\vartheta^{T-1}[e^{c\vartheta}(T-(n-c)\vartheta) + q(n\vartheta - T)].$$

For $q > 1$ (only $q > 1$ is interesting because $E_\Pi(e^{c\theta}|x) > 1$) the equation

$$s'(\vartheta) = 0, \tag{2}$$

equivalent to the equation

$$e^{c\vartheta} = \frac{qn}{n-c} + \frac{qTc}{(n-c)^2\left(\vartheta - \frac{T}{n-c}\right)},$$

has two possitive roots, denote them $\vartheta_1(q)$ and $\vartheta_2(q)$. Hence, the bounds of $E_\Pi(e^{c\theta}|x)$, when Π runs over Γ_4, are equal q_1, q_2 which are solutions of the equation

$$(1-\varepsilon)B\left[\left(\frac{\beta_0 + n}{\beta_0 + n - c}\right)^{\alpha_0 + T} - q\right] + \varepsilon\left(e^{c\vartheta_i(q)} - q\right)e^{-n\vartheta_i(q)}\vartheta_i(q)^T = 0, \quad i = 1,2. \tag{3}$$

The oscillation of the Bayes estimator of θ is equal $\left|\frac{1}{c}(\ln q_2 - \ln q_1)\right|$ and the PRGM estimator is

$$\hat{\theta}_\Gamma^{PR} = \frac{1}{c}\ln\frac{q_2 - q_1}{\ln q_2 - \ln q_1}.$$

5. Example

Assume that Y_i has the exponential distribution and the expected value of a claim is $\mu = 100$. Then the premiums are equal

I) the net premium

$$H_1(\mu,\theta) = \mu\theta \quad \text{and} \quad H_1(100,\theta) = 100\theta;$$

II) the variance principle premium with a coefficient $\eta = 0.0001$

$$H_2(\mu,\theta) = \theta(\mu + 2\eta\mu^2) \quad \text{and} \quad H_2(100,\theta) = 102\theta;$$

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III) the Esscher premium with a coefficient $v = 0.00004$ (see Freifelder (1974))

$$H_3(\mu,\theta) = \frac{\mu}{(1 - v\mu)^2}\theta \quad \text{and} \quad H_3(100,\theta) = 108.5\theta;$$

IV) the exponential premium with a coefficient $\zeta < 0.01$ (we will assume $\zeta = 0.0001$)

$$H_4(\mu,\theta) = \frac{\mu}{1 - \zeta\mu}\theta \quad \text{and} \quad H_4(100,\theta) = 101.01\theta.$$

Assume that the base prior is $Gamma(1.6049, 15.8778)$ (Gómez et al. (2002a) calculated this distribution using data from Lemaire (1979) and, considering ε-contamination classes of priors, found the oscillation of a premium in bonus-malus system). To illustrate the obtained results consider two classes of priors

$$\Gamma_3 = \{Gamma(\alpha,\beta) : \alpha \in [1,2] \wedge \beta \in [15,17]\},$$

$$\Gamma_4 = \{(1 - \varepsilon)\, Gamma(1.6049, 15.8778) + \varepsilon Q : Q \in \mathcal{P}\}.$$

Assume that in n periods T claims are observed. Table 2 presents values of Bayes premiums $\hat{H}_i^B$, $i = 1,2,3,4$, if the prior is $Gamma(1.6049, 15.8778)$, under square-error and LINEX loss functions with choosen values of the coefficient c. Table 3 presents the oscillation $r_i = \sup_{\Pi \in \Gamma} \hat{H}_i^B - \inf_{\Pi \in \Gamma} \hat{H}_i^B$ of these estimators, if the prior runs over class Γ_3, and values of the PRGM premiums. Tables 4 and 5 presents similar results for the class Γ_4. The equation (1) and the system of equations (2) and (3) have been solved with the aid of the Mathematica package.

If the ε-contamination class is considered, the oscillation is an increasing function of ε and of the distance between the sample mean $\frac{T}{n}$ and the expected value of θ, if the prior is $Gamma(1.6049, 15.8778)$, equal 0.101078. If $\frac{T}{n}$ is near 1.101078 the oscillation of the Bayes premium, if priors run over Γ_4, is less than if they run over Γ_3. However values of the PRGM premiums are greater for the class Γ_4 than for Γ_3. The Bayes and the PRGM premium are increasing functions of c. The calculated premiums under LINEX loss with $c = 0.0001$ and the square-error loss function are not significantly different. But the diffrence is an increasing function of $|c|$.

Remarks

1. Under the square-error loss function and classes Γ_i, $i = 1,2,3$, the PRGM premiums have the form of credibility formula. They are Bayes estimators with respect to the priors $Gamma(\bar{\alpha}_i, \bar{\beta}_i)$ belonging to the considered classes and with parameters not depending on observations.
2. Assume the premium is defined as a predictor $\hat{S}$ of the random variable S depending on $X = (X_1, X_2, \ldots, X_n)$ where X_i, $i = 1,2,\ldots,n$ are numbers of claims in previous periods and they have the Poisson distribution with an unknown parameter θ. If we use the square-error loss function, the Bayes

TABLE 2

THE BAYES PREMIUM WITH RESPECT TO THE PRIOR *Gamma*(1.6049, 15.8778).

n	T	$\hat{H}_1^B$	$\hat{H}_2^B$	$\hat{H}_3^B$	$\hat{H}_4^B$
The square loss function					
2	1	14.57	14.86	15.81	14.72
3	2	19.10	19.48	20.72	19.29
5	1	12.48	12.73	13.54	12.60
5	2	17.27	17.61	18.73	17.44
10	1	10.07	10.27	10.92	10.17
10	2	13.93	14.21	15.11	14.07
20	2	10.05	10.25	10.90	10.15
20	4	15.62	15.93	16.95	15.78
The LINEX loss function $c = 0.0001$					
2	1	14.57	14.87	15.81	14.72
3	2	19.10	19.48	20.73	19.29
5	1	12.48	12.73	13.54	12.61
5	2	17.27	17.62	18.74	17.45
10	1	10.07	10.27	10.92	10.17
10	2	13.93	14.21	15.12	14.07
20	2	10.05	10.25	10.90	10.15
20	4	15.62	15.94	16.95	15.78
The LINEX loss function $c = 0.001$					
2	1	14.61	14.90	15.86	14.76
3	2	19.15	19.53	20.78	19.34
5	1	12.51	12.76	13.57	12.63
5	2	17.31	17.66	18.78	17.48
10	1	10.09	10.29	10.94	10.19
10	2	13.96	14.24	15.15	14.10
20	2	10.06	10.26	10.92	10.16
20	4	15.64	15.96	16.98	15.80
The LINEX loss function $c = 0.01$					
2	1	14.99	15.30	16.31	15.15
3	2	19.62	20.02	21.34	19.82
5	1	12.79	13.05	13.90	12.92
5	2	17.69	18.06	19.24	17.88
10	1	10.27	10.48	11.16	10.37
10	2	14.21	14.50	15.44	14.35
20	2	10.19	10.40	11.07	10.29
20	4	15.84	16.17	17.21	16.01

TABLE 3

THE OSCILLATION OF THE BAYES PREMIUM AND THE PRGM PREMIUM, WHEN PRIORS RUN OVER Γ_3.

		square loss		LINEX					
				$c = 0.0001$		$c = 0.001$		$c = 0.01$	
n	T	r	$\hat{H}^{PR}$	r	$\hat{H}^{PR}$	r	$\hat{H}^{PR}$	r	$\hat{H}^{PR}$
Net premium									
2	1	7.12	14.09	7.12	14.09	7.15	14.13	7.37	14.52
3	2	7.22	18.61	7.22	18.62	7.25	18.66	7.48	19.15
5	1	5.91	12.05	5.91	12.05	5.93	12.08	6.08	12.36
5	2	6.36	16.82	6.37	16.82	6.38	16.86	6.56	17.25
10	1	4.59	9.70	4.59	9.71	4.60	9.72	4.70	9.91
10	2	4.89	13.56	4.89	13.56	4.90	13.58	5.01	13.84
20	2	3.32	9.77	3.32	9.77	3.33	9.78	3.38	9.91
20	4	3.63	15.33	3.63	15.33	3.64	15.35	3.69	15.55
Variance principle premium									
2	1	7.26	14.37	7.27	14.37	7.29	14.41	7.53	14.82
3	2	7.37	18.98	7.37	18.99	7.39	19.04	7.63	19.54
5	1	6.03	12.29	6.03	12.29	6.04	12.32	6.21	12.62
5	2	6.49	17.15	6.49	17.16	6.51	17.20	6.70	17.61
10	1	4.68	9.90	4.69	9.90	4.70	9.92	4.79	10.11
10	2	4.99	13.83	4.99	13.83	5.00	13.86	5.11	14.12
20	2	3.39	9.96	3.39	9.97	3.39	9.98	3.44	10.11
20	4	3.70	15.63	3.70	15.64	3.71	15.66	3.77	15.87
Esscher premium									
2	1	7.73	15.28	7.73	15.29	7.75	15.33	8.03	15.80
3	2	7.84	20.19	7.84	20.20	7.86	20.25	8.14	20.83
5	1	6.41	13.07	6.41	13.07	6.43	13.11	6.62	13.44
5	2	6.90	18.25	6.91	18.25	6.93	18.30	7.14	18.76
10	1	4.98	10.53	4.98	10.53	5.00	10.55	5.11	10.77
10	2	5.30	14.71	5.31	14.71	5.32	14.74	5.44	15.04
20	2	3.60	10.60	3.60	10.60	3.61	10.62	3.67	10.77
20	4	3.94	16.63	3.94	16.63	3.95	16.66	4.01	16.89
Exponential premium									
2	1	7.19	14.23	7.20	14.23	7.22	14.27	7.45	14.67
3	2	7.30	18.80	7.30	18.80	7.32	18.85	7.55	19.35
5	1	5.97	12.17	5.97	12.17	5.99	12.20	6.15	12.49
5	2	6.43	16.99	6.43	16.99	6.45	17.03	6.63	17.43
10	1	4.64	9.80	4.64	9.80	4.65	9.82	4.75	10.01
10	2	4.94	13.69	4.94	13.70	4.95	13.72	5.06	13.98
20	2	3.35	9.87	3.35	9.87	3.36	9.88	3.41	10.01
20	4	3.67	15.48	3.67	15.49	3.67	15.51	3.73	15.71

TABLE 4

THE OSCILLATION OF THE BAYES PREMIUM AND THE PRGM PREMIUM, WHEN PRIORS RUN OVER Γ_4 WITH $\varepsilon = 0.1$.

		square loss		LINEX					
				$c = 0.0001$		$c = 0.001$		$c = 0.01$	
n	T	r	$\hat{H}^{PR}$	r	$\hat{H}^{PR}$	r	$\hat{H}^{PR}$	r	$\hat{H}^{PR}$
Net premium									
2	1	15.42	21.65	15.57	21.67	16.10	22.03	26.17	27.72
3	2	34.14	35.37	34.28	35.43	35.35	36.08	50.35	45.03
5	1	3.76	13.76	3.77	13.76	3.82	13.82	4.37	14.38
5	2	11.83	22.46	11.77	22.47	12.07	22.63	14.76	24.43
10	1	1.71	10.34	1.71	10.34	1.71	10.36	1.79	10.58
10	2	3.30	14.97	3.30	14.98	3.32	15.01	3.58	15.39
20	2	1.37	10.21	1.40	10.19	1.38	10.22	1.41	10.37
20	4	3.05	16.49	3.05	16.49	3.07	16.52	3.20	16.78
Variance principle premium									
2	1	15.73	22.08	15.86	22.12	16.43	22.48	27.07	28.48
3	2	34.82	36.08	34.97	36.17	36.10	36.82	51.81	46.20
5	1	3.84	14.03	3.85	14.04	3.89	14.09	4.47	14.68
5	2	12.06	22.91	12.07	22.92	12.31	23.09	15.13	24.97
10	1	1.74	10.55	1.75	10.55	1.75	10.57	1.83	10.80
10	2	3.36	15.27	3.36	15.28	3.40	15.32	3.66	15.71
20	2	1.40	10.41	1.40	10.39	1.41	10.43	1.44	10.58
20	4	3.11	16.82	3.11	16.82	3.13	16.85	3.27	17.12
Esscher premium									
2	1	16.74	23.49	16.86	23.52	17.53	23.94	30.16	31.07
3	2	37.04	38.38	37.16	38.46	38.48	39.21	56.70	50.13
5	1	4.08	14.93	4.09	14.94	4.15	14.99	4.81	15.66
5	2	12.83	24.37	12.87	24.42	13.11	24.57	16.35	26.73
10	1	1.85	11.22	1.90	11.24	1.86	11.25	1.95	11.50
10	2	3.58	16.25	3.59	16.29	3.60	16.30	3.92	16.74
20	2	1.49	11.08	1.50	11.04	1.49	11.09	1.53	11.26
20	4	3.31	17.89	3.29	17.93	3.33	17.92	3.42	18.26
Exponential premium									
2	1	15.58	21.87	15.67	21.92	16.26	22.26	26.62	28.10
3	2	34.48	35.73	34.58	35.78	35.72	36.45	51.08	45.62
5	1	3.80	13.90	3.81	13.90	3.86	13.96	4.42	14.53
5	2	11.94	22.69	11.97	22.67	12.20	22.86	14.95	24.70
10	1	1.72	10.44	1.73	10.44	1.73	10.47	1.81	10.69
10	2	3.33	15.13	3.33	15.13	3.36	15.17	3.62	15.55
20	2	1.39	10.31	1.40	10.29	1.39	10.33	1.42	10.47
20	4	3.08	16.66	3.08	16.66	3.10	16.68	3.23	16.95

TABLE 5

THE OSCILLATION OF THE BAYES PREMIUM AND THE PRGM PREMIUM, WHEN PRIORS RUN OVER Γ_4 WITH $\varepsilon = 0.05$.

		square loss		LINEX					
				$c = 0.0001$		$c = 0.001$		$c = 0.01$	
n	T	r	$\hat{H}^{PR}$	r	$\hat{H}^{PR}$	r	$\hat{H}^{PR}$	r	$\hat{H}^{PR}$
Net premium									
2	1	7.91	18.21	7.89	18.23	8.27	18.43	13.80	21.65
3	2	19.50	28.45	19.64	28.51	20.27	28.89	30.18	34.68
5	1	1.87	13.11	1.87	13.12	1.89	13.15	2.18	13.58
5	2	6.26	20.04	6.27	20.05	6.40	20.15	7.93	21.32
10	1	0.84	10.20	0.84	10.20	0.84	10.22	0.88	10.42
10	2	1.66	14.46	1.66	14.46	1.68	14.49	1.81	14.81
20	2	0.68	10.13	0.68	10.13	0.68	10.14	0.70	10.28
20	4	1.57	16.07	1.57	16.07	1.57	16.10	1.64	16.33
Variance principle premium									
2	1	8.07	18.58	8.08	18.63	8.44	18.81	14.28	22.21
3	2	19.89	29.02	19.94	29.05	20.68	29.48	31.09	35.55
5	1	1.91	13.38	1.91	13.38	1.93	13.42	2.23	13.86
5	2	6.38	20.44	6.39	20.48	6.52	20.55	8.13	21.78
10	1	0.86	10.40	0.86	10.40	0.86	10.42	0.90	10.63
10	2	1.69	14.75	1.69	14.75	1.70	14.78	1.85	15.11
20	2	0.70	10.33	0.70	10.33	0.70	10.34	0.71	10.48
20	4	1.60	16.39	1.60	16.40	1.60	16.42	1.68	16.66
Esscher premium									
2	1	8.58	19.76	8.58	19.78	9.00	20.02	15.98	24.06
3	2	21.16	30.87	21.23	30.90	22.07	31.40	34.20	38.48
5	1	2.03	14.23	2.10	14.24	2.06	14.28	2.39	14.78
5	2	6.79	21.74	6.79	21.78	6.95	21.87	8.80	23.27
10	1	0.91	11.06	0.90	11.04	0.92	11.09	0.96	11.32
10	2	1.80	15.69	1.80	15.69	1.81	15.73	1.98	16.10
20	2	0.74	10.99	0.80	10.99	0.74	11.00	0.76	11.16
20	4	1.70	17.44	1.70	17.43	1.71	17.47	1.79	17.74
Exponential premium									
2	1	7.99	18.40	8.03	18.42	8.35	18.62	14.04	21.93
3	2	19.70	28.74	19.74	28.76	20.47	29.19	30.64	35.12
5	1	1.89	13.25	1.89	13.25	1.91	13.29	2.20	13.72
5	2	6.32	20.24	6.34	20.25	6.46	20.35	8.03	21.55
10	1	0.85	10.30	0.85	10.30	0.85	10.33	0.89	10.53
10	2	1.67	14.60	1.68	14.61	1.69	14.64	1.83	14.96
20	2	0.69	10.23	0.69	10.23	0.69	10.24	0.71	10.38
20	4	1.58	16.23	1.58	16.24	1.59	16.26	1.66	16.50

predictor (i.e. the predictor minimizing the function $E(S-\hat{S}(X))^2$, where the operator E emphasizes the expectation with respect to the joint probability distribution of all random variables S, X, θ) is equal to the Bayes estimator of the net premium, thus the PRGM predictor is equal to the PRGM estimator of the net premium too.

Under LINEX loss with a coefficient c the Bayes predictor of a random variable S is equal

$$\hat{S}_{\Pi}^{B} = \frac{1}{c} \ln E_{\Pi}\left(e^{\theta\,(M_Y\,(c)-1)}\middle|\, x\right)$$

and it is equal to the Bayes estimator of the exponential premium with the coefficent $\zeta = c$. Hence, the PRGM predictor of S and the PRGM estimator of the exponential premium with $\zeta = c$ are equal.

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