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DEFORMATIONS WITH CONSTANT MILNOR NUMBER AND MULTIPLICITY OF COMPLEX HYPERSURFACES

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Abstract. We investigate the constancy of the Milnor number of one parameter deformations of holomorphic germs of functions $f : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ with isolated singularity, in terms of some Newton polyhedra associated to such germs.

When the Jacobian ideals $J(f_t) = \langle \partial f_t/\partial x_1 \dots, \partial f_t/\partial x_n \rangle$ of a deformation $f_t(x) = f(x) + \sum_{s=1}^{\ell} \delta_s(t) g_s(x)$ are non-degenerate on some fixed Newton polyhedron Γ_+, we show that this family has constant Milnor number for small values of t, if and only if all germs g_s have non-decreasing Γ-order with respect to f. As a consequence of these results we give a positive answer to Zariski's question for Milnor constant families satisfying a non-degeneracy condition on the Jacobian ideals.

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1. Introduction. The determination of conditions for a family of isolated singularity germs $f_t : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ to have constant Milnor number is one of the most interesting questions in singularity theory. Varchenko gives in [**11**] a complete answer to this question for the case of weighted homogeneous germs with isolated singularity.

THEOREM 1.1. [**11**]. *Let* $F(x, t) = f(x) + \sum_{s=1}^{\ell} \delta_s(t) g_s(x)$ *be a deformation of a weighted homogeneous polynomial germ* $f : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ *with isolated singularity at* 0*, where* $\delta_s : (\mathbb{C}, 0) \to (\mathbb{C}, 0)$ *and* $g_s : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ *are holomorphic germs of functions and* $\delta_s \neq 0$*. Then, for small values of t the family* $f_t(x) = F(x, t)$ *has constant Milnor number, if and only if all monomials of each germ* g_s *have weighted degree higher than or equal to the weighted degree of* f.

Another number associated to a germ of a function is its multiplicity, and Zariski asked in [**13**] the following question. *For a hypersurface singularity, is the multiplicity an invariant of the topological type?*

A positive answer to Zariski's question was previously known for the case of plane curves and for homogeneous surfaces. In the case of families of

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semi-quasi-homogeneous germs, a positive answer to Zariski's question was given by Greuel in [**4**] and independently by D. O'Shea in [**7**]. Both authors applied Theorem 1.1, but used different methods.

THEOREM 1.2. [**4**], [**7**]. *Let* $f_t(x) = f(x) + \sum_{s=1}^{\ell} \delta_s(t) g_s(x)$ *be a deformation of a weighted homogeneous polynomial germ* $f : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ *with isolated singularity at* 0. *If the Milnor number of each* f_t *is constant, then the multiplicity of each* f_t *is constant.*

In this article we investigate the relationship between these questions and some Newton polyhedra associated to the germ f.

The Newton polyhedron of a germ f was used by Kouchnirenko in [**5**] to give sufficient conditions for the constancy of the Milnor number. Yoshinaga in [**12**] and Damon-Gaffney in [**3**] also dealt with this Newton polyhedron to obtain sufficient conditions for topological triviality.

Here we first show necessary and sufficient conditions for the constancy of the Milnor number in terms of the Newton polyhedron defined by the Jacobian ideal of f. Then we give sufficient conditions for the constancy of the Milnor number of families of germs which are non-degenerate on some fixed Newton polyhedron Γ_+. We show that these families have constant Milnor number for small values of t if and only if all germs g_s have non-decreasing Γ-order with respect to f.

In the final section we show that families with constant Milnor number and satisfying the non-degeneracy condition of the Jacobian ideals also have constant multiplicity, giving a positive answer to the question of Zariski for this kind of germ.

2. μ-constant deformations and integral closure. We fix a system of local coordinates x of $\mathbb{C}^n$. Consider the ring $\mathcal{O}_n$ of holomorphic germs $f : (\mathbb{C}^n, 0) \to \mathbb{C}$ and denote by m_n its maximal ideal. Due to the identification between $\mathcal{O}_n$ and the ring of convergent power series $\mathbb{C}\{x_1, \dots, x_n\}$ we identify a germ $f \in \mathcal{O}_n$ with its power series $f(x) = \sum a_\alpha x^\alpha$, where $x^\alpha = x_1^{\alpha_1} \dots x_n^{\alpha_n}$.

The Milnor number of a germ f, denoted by $\mu(f)$, is algebraically defined as the $\dim_{\mathbb{C}} \mathcal{O}_n / J(f)$, where $J(f)$ denotes the ideal generated by the partial derivatives $\{\partial f/\partial x_1, \dots, \partial f/\partial x_n\}$. A deformation $F : (\mathbb{C}^n \times \mathbb{C}, 0) \to (\mathbb{C}, 0)$ of f is *μ-constant* if $\mu(f_t) = \mu(f)$ for small values of t. We denote by $J(F) = \langle \partial F/\partial x_1, \dots, \partial F/\partial x_n \rangle$, the ideal in $\mathcal{O}_{n+1}$ generated by the partial derivatives of F with respect to the variables $x_1, \dots, x_n$.

Greuel gives in [**4**] a characterization of μ-constant deformations of f in terms of the integral closure of the Jacobian ideal of $J(F)$.

The *integral closure* of an ideal I in a ring R, is the ideal $\overline{I}$, of the elements $h \in R$ that satisfy a relation $h^k + a_1 h^{k-1} + \dots + a_{k-1} h + a_k = 0$, with $a_i \in I^i$.

Teissier gave in [**9**, p. 288] the following characterization for the integral closure of an ideal in $\mathcal{O}_n$.

PROPOSITION 2.1. *If I is an ideal in $\mathcal{O}_n$, the following statements are equivalent.*

1. $h \in \overline{I}$.
2. *For each system of generators $h_1, \dots, h_r$ of I there exists a neighbourhood U of* 0 *and a constant $C > 0$ such that*

$$|h(x)| \le C \sup\{|h_1(x)|, \dots, |h_r(x)|\}, \quad \textit{for all } x \in U.$$

3. *For each analytic curve $\varphi : (\mathbb{C}, 0) \to (\mathbb{C}^n, 0)$, $h \circ \varphi$ lies in $(\varphi^*(I))\,\mathcal{O}_1$.*

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Item 3 of this proposition is called a *valuative criterion* since it is equivalent to the condition $\nu(h \circ \varphi) \geq \inf\{\nu(h_1 \circ \varphi), \ldots, \nu(h_r \circ \varphi)\}$, where ν denotes the usual valuation of a complex curve. In this case, the valuation is the multiplicity of the curve. See Section 5 for the definition of multiplicity.

THEOREM 2.2. [**4**, p. 161]. *Let $F : (\mathbb{C}^n \times \mathbb{C}, 0) \to (\mathbb{C}, 0)$ be a one parameter deformation of a holomorphic germ $f : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ with isolated singularity. The following statements are equivalent.*

1. *F is a μ-constant deformation of f.*
2. $\frac{\partial F}{\partial t} \in \overline{J(F)}$.
3. $\frac{\partial F}{\partial t} \in \sqrt{J(F)}$, *where* $\sqrt{J(F)}$ *denotes the radical of* $J(F)$.
4. *The polar curve of F with respect to $\{t = 0\}$ does not split; i.e.,*

$$\left\{(x, t) \in \mathbb{C}^n \times \mathbb{C} \;\middle|\; \frac{\partial F}{\partial x_i}(x, t) = 0, \forall\, i = 1, \ldots, n\right\} = \{0\} \times \mathbb{C} \text{ near } (0, 0).$$

3. μ-constant deformations and Newton polyhedra. We give here necessary and sufficient conditions for a deformation $F(x, t)$ to be μ-constant. These conditions will be given in terms of some suitable Newton polyhedra associated to the germ f.

For a germ $f(x) = \sum a_k x^k$, we define supp $f = \{k \in \mathbb{Z}^n : a_k \neq 0\}$. For an ideal I in $\mathcal{O}_n$, we call supp $I = \cup\{\text{supp } g : g \in I\}$.

DEFINITION 3.1. The *Newton polyhedron* of I, denoted by $\Gamma_+(I)$, is the convex hull in $\mathbb{R}^n_+$ of the set

$$\cup\left\{k + v : k \in \text{supp } I,\ v \in \mathbb{R}^n_+\right\}.$$

$\Gamma(I)$ denotes the union of all compact faces of $\Gamma_+(I)$.

A germ g is of *non-decreasing Newton order* with respect to $\Gamma_+(I)$ if $\Gamma_+(g) \subseteq \Gamma_+(I)$.

From the integral closure of the Jacobian ideal $J(f)$ we define the polyhedron $T(f)$, which is a key tool to study the μ-constancy.

DEFINITION 3.2. $T(f)$ is the convex hull in $\mathbb{R}^n_+$ of

$$\cup\{m + v : x^m \in \overline{J(f)} \text{ and } v \in \mathbb{R}^n_+\}.$$

In the next lemma we exhibit a necessary condition for the μ-constancy of families defined by first order deformations.

LEMMA 3.3. *Let $F(x, t) = f(x) + tg(x)$ be a first order deformation of a complex germ f with isolated singularity. A necessary condition for the μ-constancy of the family f_t is $\Gamma_+(g) \subseteq T(f)$.*

Proof. If $\Gamma_+(g) \not\subseteq T(f)$, it follows from the valuative criterion that there exists a holomorphic curve $\gamma : (\mathbb{C}, 0) \to (\mathbb{C}^n, 0)$, such that

$$\nu(g \circ \gamma) < \inf\left\{\nu\left(\frac{\partial f}{\partial x_i} \circ \gamma\right),\ \forall i = 1, \ldots, n\right\}.$$

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We define the curve $\psi : (\mathbb{C}, 0) \to (\mathbb{C}^n \times \mathbb{C}, 0)$ as $\psi = (\gamma, 0)$. Since $\frac{\partial F}{\partial x_i} = \frac{\partial f}{\partial x_i} + t\frac{\partial g}{\partial x_i}$, we obtain $\frac{\partial F}{\partial x_i} \circ \psi = \frac{\partial f}{\partial x_i} \circ \gamma$. The result follows from Item 3 of Proposition 2.1 and Theorem 2.2.

We describe sufficient conditions for the constancy of μ.

Yoshinaga in [**12**] gave conditions for the topological triviality of families of type $F(x, t) = f(x) + tg(x)$ in terms of the gradient polyhedron $\Lambda_+(f)$, defined as the convex hull of the set

$$\bigcup \left\{ m + v, : v \in \mathbb{R}^n_+ \text{ and } \left| x_1 \frac{\partial f}{\partial x_1} \right| + \cdots + \left| x_n \frac{\partial f}{\partial x_n} \right| \geq \mathcal{E} \left| x^m \right| \right\},$$

for a positive $\mathcal{E}(m)$ in a neighbourhood of the origin in $\mathbb{C}^n$. In Theorem 1.6 of [**12**] it is shown that if $\Gamma_+(g) \subset \Lambda_+(f)$, then $F(x, t) = f(x) + tg(x)$ is topologically trivial for sufficiently small values of t. Damon-Gaffney in [**3**] also gave similar results for the topological triviality for deformations of type $f_t(x) = f(x) + \sum_{s=1}^{\ell} \delta_s(t) g_s(x)$, of a germ $f : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ with isolated singularity at 0.

We remark that if a germ g satisfies the condition $\Gamma_+(g) \subset \Lambda_+(f)$, it is equivalent to say that g is in the integral closure of the ideal generated by $\{x_1 \frac{\partial f}{\partial x_1}, \ldots, x_n \frac{\partial f}{\partial x_n}\}$. Since this ideal is in the integral closure of the ideal $J(F)$, from the results of Yoshinaga, Damon-Gaffney and Theorem 2.2, we get the following result.

PROPOSITION 3.4. *Let* $F(x, t) = f(x) + \sum_{s=1}^{\ell} \delta_s(t) g_s(x)$ *be a deformation of* f *with isolated singularity at* 0. *If* $\Gamma_+(g_s) \subset \Lambda_+(f)$, *for all* $s = 1, \ldots, \ell$, *then* $F(x, t)$ *is* μ-*constant for sufficiently small values of* t.

Next we give a sufficient condition for the μ-constancy in terms of the polyhedron $T(f)$.

PROPOSITION 3.5. *Let* $F(x, t) = f(x) + \sum_{s=1}^{\ell} \delta_s(t) g_s(x)$ *be a deformation of a complex germ* f *with isolated singularity. If* $g_s \in m_n$ *and* $\Gamma_+(J(g_s)) \subseteq T(f)$ *for all* $s = 1, \ldots, \ell$, *then* $F(x, t)$ *is* μ-*constant for small values of* t.

Proof. Suppose that $\Gamma_+(J(g_s)) \subseteq T(f)$, we have from Item 2 of Proposition 2.1, that for each $i = 1, \ldots, n$ and $s = 1, \ldots, \ell$ there exist a neighbourhood $U_{i,s}$ of 0 and a constant $C_{i,s} > 0$, such that

$$\left| \frac{\partial g_s}{\partial x_i} \right| \leq C_{i,s} \sup \left\{ \left| \frac{\partial f}{\partial x_1} \right|, \ldots, \left| \frac{\partial f}{\partial x_n} \right| \right\}.$$

Then

$$\sum_{s=1}^{\ell} |\delta_s(t)| \left| \frac{\partial g_s}{\partial x_i} \right| \leq \sum_{s=1}^{\ell} |\delta_s(t)| \, C_{i,s} \, \sup \left\{ \left| \frac{\partial f}{\partial x_1} \right|, \ldots, \left| \frac{\partial f}{\partial x_n} \right| \right\}.$$

Therefore, for all x in a neighbourhood $U \subset U_{i,s}$, for all $s = 1, \ldots, \ell$ and $i = 1, \ldots, n$, we have

$$\begin{aligned}\sup_i\left|\frac{\partial F}{\partial x_i}\right| = \sup_i\left|\frac{\partial f}{\partial x_i} + \sum_{s=1}^{\ell}\delta_s(t)\frac{\partial g_s}{\partial x_i}\right| &\geq \sup_i\left|\frac{\partial f}{\partial x_i}\right| - \sup_i\sum_{s=1}^{\ell}|\delta_s(t)|\left|\frac{\partial g_s}{\partial x_i}\right| \\ &\geq \sup_i\left|\frac{\partial f}{\partial x_i}\right| - \sum_{s=1}^{\ell}|\delta_s(t)|C_{i,s}\sup_i\left\{\left|\frac{\partial f}{\partial x_i}\right|\right\} \\ &\geq \left(1-\sum_{s=1}^{\ell}|\delta_s(t)|C_{i,s}\right)\sup_i\left|\frac{\partial f}{\partial x_i}\right| \\ &\geq (1-\alpha)\sup_i\left|\frac{\partial f}{\partial x_i}\right|,\end{aligned}$$

for some $0 < \alpha < 1$ with $\sum_{s=1}^{l}|\delta_s(t)C_{i,s}| \leq \alpha$ for all $i = 1, \ldots, n$ and $s = 1, \ldots, \ell$.

This inequality implies that $\overline{\langle \frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n}\rangle} \subset \overline{\langle \frac{\partial F}{\partial x_1}, \ldots, \frac{\partial F}{\partial x_n}\rangle}$.

Now we show that for each analytic curve $\psi : (\mathbb{C}, 0) \to (\mathbb{C}^{n+1}, 0)$,

$$\nu\left(\frac{\partial F}{\partial t}\circ\psi\right) \geq \min_i\left\{\nu\left(\frac{\partial F}{\partial x_i}\circ\psi\right)\right\}.$$

We write $\psi = (\varphi, \lambda)$, with $\varphi : (\mathbb{C}, 0) \to (\mathbb{C}^n, 0)$ and $\lambda : (\mathbb{C}, 0) \to (\mathbb{C}, 0)$. Hence

$$\nu\left(\frac{\partial F}{\partial t}\circ\psi\right) \geq \min_s\{\nu(\delta_s'\circ\lambda) + \nu(g_s\circ\varphi)\} \geq \min_s\{\nu(g_s\circ\varphi)\}.$$

From the hypothesis that $g_s \in m_n$ and $\Gamma_+(J(g_s)) \subseteq T(f)$, we obtain that $\Gamma_+(g_s) \subseteq T(f)$; hence $g_s \in \overline{\langle \frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n}\rangle}$.

Therefore $\nu(g_s\circ\varphi) \geq \min_i\{\nu(\frac{\partial f}{\partial x_i}\circ\varphi)\} \geq \min_i\{\nu(\frac{\partial F}{\partial x_i}\circ\psi)\}$,

$$\nu\left(\frac{\partial F}{\partial t}\circ\psi\right) \geq \min_s\{\nu(g_s\circ\varphi)\} \geq \min_i\left\{\nu\left(\frac{\partial F}{\partial x_i}\circ\psi\right)\right\},$$

and the result follows from Theorem 2.2.

We see in the example below that the conditions given in Propositions 3.4 and 3.5 give rise to different classes of μ-constant deformations.

EXAMPLE 3.6. Let $f(x, y) = y^7 + x^4y + x^9$.

From Theorem 1.3 of Yoshinaga (see Proposition 4.5), we know that $\Lambda_+(f) = \Gamma_+(f)$ is the polygon with vertices (0, 7), (4, 1) and (9, 0). From Proposition 4.4 we see that the polygon $T(f)$ has vertices (0, 6), (3, 1) and (4, 0).

If we consider $g(x, y) = x^2y^4$, $\Gamma_+(g) \subset \Lambda_+(f)$, and so the deformation $F(x, y, t) = y^7 + x^4y + x^9 + tx^2y^4$ is μ-constant from Proposition 3.4, but $\Gamma_+(J(g)) \not\subset T(f)$.

We also see in this example that the condition $\Gamma_+(J(g)) \subset T(f)$ does not imply that $\Gamma_+(g) \subset \Lambda_+(f)$. To see this, consider $g(x, y) = x^6$. The deformation $F(x, y, t) = y^7 + x^4y + x^9 + tx^6$ is μ-constant from Proposition 3.5, and $\Gamma_+(g) \not\subset \Lambda_+(f)$.

4. The non-degenerate case. In this section we describe how to obtain μ-constant deformations in terms of non-degeneracy conditions given by some Newton polyhedra associated to the germ f.

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Let I be an ideal of finite codimension in $\mathcal{O}_n$; i.e, $\dim_{\mathbb{C}} \mathcal{O}_n/I < \infty$. For each compact face $\Delta \subseteq \Gamma(I)$ we denote by $C(\Delta)$ the cone given by the union of half-rays emanating from the origin and passing through Δ. We call $\mathcal{A}_\Delta$ the *subring with unity of* $\mathcal{O}_n$ given by $\mathcal{A}_\Delta = \{g \in \mathcal{O}_n : \operatorname{supp} g \subseteq C(\Delta) \cap \mathbb{Z}^k\}$.

If D is a fixed subset of $\Gamma_+(I)$ and $g = \sum_k a_k x^k$, we set $g_D = \sum_{k \in D} a_k x^k$.

DEFINITION 4.1. The ideal I is *Newton non-degenerate* on a face $\Delta \subset \Gamma(I)$ if the ideal I_Δ generated by $g_{1_\Delta}, g_{2_\Delta}, \dots, g_{s_\Delta}$ has finite codimension in $\mathcal{A}_\Delta$.

By Theorem 6.2 of [**5**], the ideal I_Δ generated by $g_{1_\Delta}, \dots, g_{s_\Delta}$ has finite codimension in $\mathcal{A}_\Delta$ if and only if, for each compact face Δ_1 of Δ, the equations $g_{1\Delta_1} = \dots = g_{s\Delta_1} = 0$ have no common solution in $(\mathbb{C} \setminus \{0\})^n$.

DEFINITION 4.2. An ideal I is *Newton non-degenerate* if there exists a system of generators $\{g_1, \dots, g_s\}$ of I that is Newton non-degenerate on each compact face $\Delta \subseteq \Gamma(I)$.

We denote by $C(\overline{I})$ the convex hull in $\mathbb{R}^n_+$ of the set $\cup\{m : x^m \in \overline{I}\}$.

In the case of Newton non-degenerate ideals, the set $C(\overline{I})$ is a key tool to compute the integral closure of the ideal.

THEOREM 4.3. [**8**]. *Let I be an ideal with finite codimension in $\mathcal{O}_n$. Then $C(\overline{I}) \subseteq \Gamma_+(I)$ and equality holds if and only if I is Newton non-degenerate.*

When the Jacobian ideal $J(f)$ is Newton non-degenerate we have the following result.

PROPOSITION 4.4. *Suppose that the system $\{\partial f/\partial x_1, \dots, \partial f/\partial x_n\}$ of generators of the Jacobian ideal $J(f)$ is Newton non-degenerate. If $g_s \in m_n$ and $\Gamma_+(J(g_s)) \subseteq \Gamma_+(J(f))$ for all s. Then $F(x, t)$ is μ-constant for small values of t.*

A germ $g = \sum_k a_k x^k$ is *Newton non-degenerate*, if the ideal generated by the system $\{x_1 \partial g/\partial x_1, \dots, x_n \partial g/\partial x_n\}$ is Newton non-degenerate and of finite codimension in $\mathcal{O}_n$. For Newton non-degenerate germs we also have the following result.

PROPOSITION 4.5. [**12**], [**3**]. *Suppose that $f : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ is Newton non-degenerate. If each g_s has non-decreasing Newton order with respect to $\Gamma_+(f)$, the deformation $f(x) + \sum_{s=1}^{\ell} \delta_s(t) g_s(x)$ is μ-constant for sufficiently small values of t.*

The proof of these results is a direct consequence of Propositions 3.5 (or 3.4) and Theorem 4.3 applied to the ideals $J(f) = \langle \frac{\partial f}{\partial x_1}, \dots, \frac{\partial f}{\partial x_n} \rangle$ and $I = \langle x_1 \frac{\partial f}{\partial x_1}, \dots, x_n \frac{\partial f}{\partial x_n} \rangle$. Here $C(\overline{I}) = \Lambda_+(f)$ and $T(f) = C(\overline{J(f)})$.

The next example shows that these two conditions are not enough to give all μ-constant deformations of any germ with isolated singularity.

EXAMPLE 4.6. Briançon and Speder showed in [**1**] that the family of hypersurfaces $X_t \in \mathbb{C}^3$ defined by the equations $f_t(x, y, z) = z^5 + y^7 x + x^{15} + t y^6 z = 0$ has constant topological type, but we cannot apply Proposition 4.4 since the monomial $y^6 z$ does not satisfy its condition. We cannot apply Proposition 4.5 either, since f is not Newton non-degenerate because the ideal generated by the system $\{x \partial f/\partial x, y \partial f/\partial y, z \partial f/\partial z\}$ does not have finite codimension in $\mathcal{O}_n$.

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The germ $f(x, y, z) = z^5 + y^7 x + x^{15}$ has isolated singularity at 0 and is weighted homogeneous with weights (1, 2, 3) and weighted degree 15. Hence the μ-constancy of this family follows from Theorem 1.1.

In order to generalize these results to a bigger class of germs that includes the Newton non-degenerate germs and also the class of semi-weighted homogeneous germs, we apply the results of Bivia-Fukui-Saia, given in [**2**], to give a necessary and sufficient condition for the μ-constancy of families defined by germs which are non-degenerate on some Newton polyhedron. We repeat the basic results for this definition.

A subset $\Gamma_+ \subseteq \mathbb{R}^n_+$ is a *Newton polyhedron* if there exist some $k_1, \ldots, k_r \in \mathbb{Q}^n_+$ such that Γ_+ is the convex hull in $\mathbb{R}^n_+$ of the set

$$\{k_i + v : v \in \mathbb{R}^n_+,\ i = 1, \ldots, r\}$$

and Γ_+ intersects all the coordinate axes. We call Γ the union of the compact faces of Γ_+, Γ_- (the set $\mathbb{R}^n - \Gamma_+$) and $\mathrm{V}_n(\Gamma_-)$ denotes the n-dimensional volume of Γ_-. From the boundary of a Newton polyhedron $\Gamma_+ \subseteq \mathbb{R}^n$ we construct a piecewise-linear function $\phi_\Gamma : \mathbb{R}^n \to \mathbb{R}$ with the following properties:

(i) ϕ_Γ is linear on each cone $C(\Delta)$, where Δ is a compact face of Γ;
(ii) ϕ_Γ takes positive integer values on the lattice points of $\mathbb{R}^n_+ - \{0\}$;
(iii) there exists a positive integer M such that $\phi_\Gamma(k) = M$, for all $k \in \Gamma$.

This map ϕ_Γ induces a filtration in $\mathcal{O}_n$:

Definition 4.7. The *Newton filtration* of $\mathcal{O}_n = \mathcal{A}_0 \supseteq \mathcal{A}_1 \supseteq \mathcal{A}_2 \supseteq \ldots$ is defined by the ideals $\mathcal{A}_q = \{g \in \mathcal{O}_n : \operatorname{supp} g \subseteq \phi_\Gamma^{-1}(q + \mathbb{N})\}$, for all $q \in \mathbb{N}$. See [**5**] and Proposition 2.1.

The Γ-*order* of a germ $g = \sum_k a_k x^k$, denoted by $\mathrm{d}(g)$ is the number

$$\mathrm{d}(g) = \min\{\phi_\Gamma(k) : k \in \operatorname{supp} g\} = \max\{q : g \in \mathcal{A}_q\}.$$

The *principal part of g* is the polynomial $\mathrm{in}(g) = \sum a_k x^k$ such that $\phi_\Gamma(k) = \mathrm{d}(g)$.

For any compact face Δ of Γ, this filtration induces a filtration on $\mathcal{A}_\Delta$ in a natural way. *The principal part of g over* Δ, denoted by $\mathrm{in}_\Delta(g)$, is the polynomial

$$\mathrm{in}_\Delta(g) = \sum\{a_k x^k : k \in \operatorname{supp} g \cap C(\Delta) \text{ and } \phi_\Gamma(k) = \mathrm{d}(g)\}.$$

Definition 4.8. A system of generators $g_1, \ldots, g_s$ of an ideal I is *non-degenerate on* Γ_+ if, for each compact face $\Delta \subseteq \Gamma$, the ideal of $\mathcal{A}_\Delta$ generated by $\mathrm{in}_\Delta(g_1), \ldots, \mathrm{in}_\Delta(g_s)$ has finite codimension in $\mathcal{A}_\Delta$. When the system $g_1, \ldots, g_s$ does not satisfy the above definition, we say that this system is *degenerate* on Γ_+.

We remark that this definition depends on the system of generators, while the definition of Newton non-degeneracy does not depend on the system of generators of the ideal.

The next theorem is essential to prove the main results of this section.

Theorem 4.9. ([**2**], Theorem 3.3.). *Let $g_1, \ldots, g_n$ be a system of generators of an ideal I with finite codimension in $\mathcal{O}_n$ and $\Gamma_+ \subseteq \mathbb{R}^n$ be a Newton polyhedron. If M is the value on Γ of the filtration induced by Γ_+ and $d_1 = \mathrm{d}(g_1), \ldots, d_n = \mathrm{d}(g_n)$ are the Γ-orders of the given set of generators of I, then*

1. $\dim_{\mathbb{C}} \mathcal{O}_n/I \geq \frac{d_1 \ldots d_n}{M^n} n!\, \mathrm{V}_n(\Gamma_-)$;
2. *equality holds if and only if the system $g_1, \ldots, g_n$ is non-degenerate on Γ_+.*

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In the sequel we consider a deformation $f_t(x) = f(x) + \sum_{s=1}^{\ell} \delta_s(t) g_s(x)$, of a germ $f : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ with isolated singularity at 0 and fix a Newton polyhedron Γ_+.

We denote by D_i, D_i^s, D_{ti}, the Γ-order of the partial derivatives $\frac{\partial f}{\partial x_i}$, $\frac{\partial g_s}{\partial x_i}$, $\frac{\partial f_t}{\partial x_i}$, respectively.

THEOREM 4.10. *Suppose that the system of generators* $\{\partial f/\partial x_1, \dots, \partial f/\partial x_n\}$ *of the Jacobian ideal* $J(f)$ *is non-degenerate on* Γ_+. *If* $g_s \in m_n$ *and* $D_i^s \geq D_i$ *for all* $i = 1, \dots, n$ *and all* $s = 1, \dots, \ell$, *then, for small values of* t, *the system of generators* $\{\partial f_t/\partial x_1, \dots, \partial f_t/\partial x_n\}$ *of each Jacobian ideal* $J(f_t)$ *is non-degenerate on* Γ_+. *Moreover,* $\mu(f_t)$ *is constant.*

Proof. Suppose first that $D_i^s > D_i$ for all $i = 1, \dots, n$ and all $s = 1, \dots, \ell$. Hence, $D_{ti} = D_i$, for all $i = 1, \dots, n$, and the principal part of each $\partial f_t/\partial x_i$ is equal to the principal part of $\partial f/\partial x_i$. From the non-degeneracy of $J(f)$ on Γ_+, we get the non-degeneracy of each Jacobian ideal $J(f_t)$, and the equality $\mu(f_t) = \mu(f)$ follows from Item 2 of Theorem 4.9.

If there exist $D_i^s = D_i$, we conclude that each Jacobian ideal $J(f_t)$ is non-degenerate, for small values of t, from the fact that the set of non-degenerate ideals on some Newton polyhedron is open for the Zariski topology. Hence the equality $\mu(f_t) = \mu(f)$ follows from Item 2 of Theorem 4.9.

The following example illustrates this result.

EXAMPLE 4.11. Let $f(x, y) = x^{12} + y^8$. Fix the Newton polyhedron $\Gamma_+ = \Gamma_+(f)$, since f is weighted homogeneous with respect to the weight $w = (2, 3)$, Γ_+ has only one compact face with vertices (12, 0) and (0, 8). The Jacobian ideal $J(f) = \langle f_x, f_y \rangle$ is non-degenerate on Γ_+, with $\mathrm{d}(f_x) = 22$ and $\mathrm{d}(f_y) = 21$.

Consider the family $f_t(x, y) = x^{12} + y^8 - 2tx^6y^4$. Here $\mathrm{d}(f_{tx}) = 22 = \mathrm{d}(f_x)$ and $\mathrm{d}(f_{ty}) = 21 = \mathrm{d}(f_y)$. The principal part of the generators $\{f_{tx}, f_{ty}\}$ of the Jacobian ideal $J(f_t)$ is different from the principal part of the generators $\{f_x, f_y\}$ of the Jacobian ideal $J(f)$, but for each $0 < t < 1$, the system of generators $\{f_{tx}, f_{ty}\}$ is also non-degenerate. Hence $\mu(f) = \mu(f_t)$, for all $0 \leq t < 1$. For $t = 1$, we see that the germ $f_1 = (x^6 - y^4)^2$ does not have isolated singularity, and so $\mu(f_1) = \infty$.

We see in the example below that it is possible to find families f_t such that the germ f is non-degenerate, there exists a $t \neq 0$ with f_t degenerate, but the family f_t is also μ-constant.

EXAMPLE 4.12. Let $f(x, y, z) = x^2 + y^2 + xz + z^2$. The germ f is homogeneous and Newton non-degenerate. Consider the family $f_t = x^2 + y^2 + xz + z^2 + 2txy$. From Theorem 4.10 we see that for $0 < t < 1$, $\mu(f_t) = \mu(f)$. When $t = 1$, the Jacobian ideal $J(f_1)$ is not Newton non-degenerate. (See [**5**, p. 8]) However $\mu(f_1) = \mu(f)$.

On the other hand, we have the following result.

THEOREM 4.13. *Suppose that the system of generators* $\{\partial f/\partial x_1, \dots, \partial f/\partial x_n\}$ *of the Jacobian ideal* $J(f)$ *is non-degenerate on* Γ_+. *If* $\mu(f_t)$ *is constant for small values of* t *and the system of generators* $\{\partial f_t/\partial x_1, \dots, \partial f_t/\partial x_n\}$ *is non-degenerate on* Γ_+, *then* $D_i^s \geq D_i$, *for all* $i = 1, \dots, n$ *and all* $s = 1, \dots, \ell$.

Proof. If $\mu(f_t) = \mu(f)$ then, since $J(f_t)$ is non-degenerate on Γ_+, it follows from Item 2 of Theorem 4.9 that

$$\mu(f_t) = \frac{D_{t1} \dots D_{tn}}{M^n} n!\, \mathrm{V}_n(\Gamma_-) \quad \text{and} \quad \mu(f) = \frac{D_1 \dots D_n}{M^n} n!\, \mathrm{V}_n(\Gamma_-).$$

From the hypothesis we have

$$\frac{D_{t1} \dots D_{tn}}{M^n} n!\, \mathrm{V}_n(\Gamma_-) = \frac{D_1 \dots D_n}{M^n} n!\, \mathrm{V}_n(\Gamma_-).$$

Hence we conclude that $D_{t1} \dots D_{tn} = D_1 \dots D_n$, but $D_{ti} = \min\{D_i, D_i^s\}$ for small values of t and all $i = 1 \dots, n$, $s = 1, \dots, \ell$. Therefore $D_{ti} = D_i$ for all $i = 1 \dots, n$ and $D_i^s \geq D_i$ for all $i = 1 \dots, n$, $s = 1, \dots, \ell$.

EXAMPLE 4.14. Here we exhibit a generalized Briançon-Speder example. We consider families of type $f_t(x, y, z) = z^5 + y^7 x + x^{15} + t x^a y^b z^c$ for a fixed monomial $x^a y^b z^c$.

Since the Jacobian ideal $J(f) = \langle 15x^{14} + y^7, 7y^6 x, 5z^4 \rangle$ is Newton non-degenerate, $T(f) = \Gamma_+(J(f))$. To use Lemma 3.3 we consider monomials $x^a y^b z^c$ with $(a, b, c) \in \Gamma_+(J(f))$.

For instance, for the monomial $y^6 z$, we fix the Newton polyhedron Γ_+ with vertices $(3, 0, 0)$, $(0, 2, 0)$ and $(0, 0, 1)$, that has one 2-dimensional compact face associate to the weights $(1, 2, 3)$. $J(f)$ is non-degenerate on Γ_+ and the Γ-orders of the partial derivatives of f are given by

$$\mathrm{d}(15x^{14} + y^7) = 14, \quad \mathrm{d}(7y^6 x) = 13, \quad \mathrm{d}(5z^4) = 12.$$

On the other hand, the Γ-order of the partial derivatives of the monomial $y^6 z$ are $\mathrm{d}(6y^5 z) = 13$ and $dN(y^6) = 12$. We apply Theorem 4.10 to conclude that the system of generators $\{15x^{14} + y^7, 7y^6 x + t6y^5 z, 5z^4 + ty^6\}$ is non-degenerate on Γ_+ and the family f_t is μ-constant for small values of t. An analogous argument can be applied to any family of type $f_t(x) = f(x) + \sum_{s=1}^{\ell} \delta_s(t) g_s(x)$ satisfying the conditions of Theorem 4.10 for this Newton polyhedron.

But there are some monomials $x^a y^b z^c$ satisfying the condition $(a, b, c) \in \Gamma_+(J(f))$ that do not satisfy the conditions of Theorem 4.10 for this Newton polyhedron. For example, consider the monomial yz^4. The Γ-order of $\mathrm{d}(\partial(yz^4)/\partial y) = 11$ and we cannot apply Theorem 4.10.

On the other hand, we apply Theorem 4.13 to show that the family $f_t(x, y, z) = z^5 + y^7 x + x^{15} + tyz^4$ is not μ-constant. For this we fix the Newton polyhedron $\Gamma_+ = \Gamma_+(J(f_t))$. Here $J(f_t)$ is Newton non-degenerate and $J(f)$ is degenerate on this polyhedron. Hence $\mu(f_t) < \mu(f)$ for all $t \neq 0$.

We can apply an analogous argument for any family of type $f_t(x) = f(x) + \sum_{s=1}^{\ell} \delta_s(t) g_s(x)$ such that $J(f_t)$ is Newton non-degenerate and the germs g_s satisfy the condition $\Gamma_+(g_s) \subseteq \Gamma_+(J(f))$, but do not satisfy the condition of Theorem 4.10.

5. μ-constant deformations and the multiplicity. The *multiplicity* of a germ $f(x) = \sum a_k x^k$, is defined as the lowest degree in the power series of $f(x)$.

Zariski proposed in [**13**] the following question.

For a hypersurface singularity, is the multiplicity an invariant of the topological type?

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It is well known that this question has a positive answer in the case of plane curves and for families of semi-quasi homogeneous complex hypersurfaces, as shown by Greuel in [**4**].

For the case of first order deformations $F(x, t) = f(x) + tg(x)$ of a complex germ f with isolated singularity, Trotman gives in [**10**] a positive answer to Zariski's question.

From the results given in Section 4 we obtain a positive answer to the question of Zariski for families $F(x, t) = f(x) + \sum_{s=1}^{\ell} \delta_s(t) g_s(x)$ of germs for which the Jacobian ideals $J(f_t)$ are non-degenerate on some Newton polyhedron Γ_+.

COROLLARY 5.1. *Let* $F(x, t) = f(x) + \sum_{s=1}^{\ell} \delta_s(t) g_s(x)$ *be a deformation of a germ* f *with isolated singularity. Suppose that, for small values of* t*, the system of generators* $\{\frac{\partial f_t}{\partial x_1}, \dots, \frac{\partial f_t}{\partial x_n}\}$ *of each Jacobian ideal* $J(f_t)$ *is non-degenerate on* Γ_+ *and* $g_s \in m_n$ *for all* s*. If* $\mu(f_t)$ *is constant, then the multiplicity of each* $f_t(x)$ *is constant.*

Proof. From the hypothesis and Theorem 4.13 we conclude that $D_i^s \geq D_i$, for all $i = 1, \dots, n$ and all $s = 1, \dots, \ell$. Therefore $\Gamma_+(g_s) \subset \Gamma_+(f)$, for all $s = 1, \dots, \ell$. Hence the equality $m(f_t) = m(f)$ follows.

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