Morley's theorem — once again

That Morley's theorem was published some two millennia after Euclid may possibly be linked with the problem of trisecting an angle. Recent interest has been shown in the Gazette articles by Barbara [1] and Clarke [2]. In this note, we start with an equilateral triangle and deduce the angle trisection property for a host triangle by means of the sine rule and angular arithmetic.

An equilateral triangle $PQR$ has triangles $PQB$ and $PRC$ attached to it externally such that

$\angle BQP = \angle CRP = \alpha + \frac{\pi}{3}, \quad \angle CPR = \beta + \frac{\pi}{3}, \quad \angle BPQ = \gamma + \frac{\pi}{3}, \quad (\alpha + \beta + \gamma = \pi)$

whence $\angle PBQ = \beta, \angle PCR = \gamma$. Note that $\angle CPB = \alpha + \frac{2\pi}{3} < \pi$.

Letting $\angle CBP = \theta$ and $\angle BCP = \phi$, we use the sine rule on triangles $PBC, PBQ$ and $PCR$ to obtain

$$\frac{\sin \theta}{\sin \beta} = \frac{\sin \phi}{\sin \gamma},$$

where $\theta + \phi = \gamma + \beta$. Since all four angles are acute,

$$\theta > \beta \Rightarrow \phi < \gamma \Rightarrow \frac{\sin \theta}{\sin \beta} > 1 > \frac{\sin \phi}{\sin \gamma}$$

with a similar contradiction for $\theta < \beta$. Thus $\theta = \beta$ and $\phi = \gamma$.

The attachment of a triangle $AQR$ with $\angle AQR = \beta + \frac{\pi}{3}, \angle ARQ = \gamma + \frac{\pi}{3}$ allows the aforesaid argument to be repeated, and the result is that the vertices of the equilateral triangle $PQR$ are the meets of the nearside angle trisectors of the triangle $ABC$ as required for Morley's theorem.

![Figure 1](https://www.cambridge.org/core/)

The reader may wish to verify the well-known expression

$$8 \sin \left(\frac{A}{3}\right) \sin \left(\frac{B}{3}\right) \sin \left(\frac{C}{3}\right)$$

for the ratio of the length of a side of the Morley triangle $PQR$ to the circumradius of the host triangle $ABC$.

It may also be shown that the relative area of the Morley triangle has a
maximum value given by the square of \( \sin(\pi/18)/\cos(\pi/9) \) or approximately \((2 + \sqrt{2})/100\) for an equilateral host triangle.

References

J. A. SCOTT

1 Shiptons Lane, Great Somerford, Chippenham SN15 5EJ

86.08 An areal view of Feuerbach’s theorem

This theorem (1822, [1]) states that the nine-points circle of a triangle \(ABC\) is tangential to the inscribed circle and to each of the three escribed circles. In this note we prove an equivalent theorem. Standard notation for the triangle is used as far as possible, and square brackets will be used for actual areal coordinates (sum normalised to unity) and round brackets for relative areal coordinates as in the Gazette note [2].

We first introduce the Nagel point of a triangle. This is the point of concurrence of the lines from the vertices to the corresponding points where the escribed circles touch the triangle (mid-perimeter points). In his Gazette note [3], Eperson uses vectors to show that the Nagel point (\(E\), say) is collinear with the centroid and the incentre. The actual areal coordinates of the Nagel point are easily found from Figure 2 in [3] to be \(\frac{1}{2} [s-a, s-b, s-c]\) where \(2s = a + b + c\).

The actual areal coordinates of the circumcentre \(O\) are \(R^2 [\alpha/b^2c^2, \beta/c^2a^2, \gamma/a^2b^2]\) where \(\alpha, \beta, \gamma\) represent the quantities \(b^2 + c^2 - a^2, c^2 + a^2 - b^2, a^2 + b^2 - c^2\), respectively. These coordinates may be derived from the distances of \(O\) to the sides of triangle \(ABC\), where for example the distance to \(BC\) is \(R \cos A = Ra/2bc\).

An expression for the distance \(OE\) is now obtained by means of the metric

\[
PP'^2 = -a^2(y - y')(z - z') - b^2(z - z')(x - x') - c^2(x - x')(y - y'),
\]

where \((P[x, y, z], P'[x', y', z'])\),

which was derived in [2]. With \(P = O\) and \(P' = E\), we write \(OE^2 = U + V + W\) where

\[
U = -a^2yz - b^2zx - c^2xy,
\]

\[
V = a^2(yz' + y'z) + b^2(zx' + z'x) + c^2(xy' + x'y),
\]

\[
W = -a^2y'z' - b^2z'x' - c^2x'y',
\]

and use the key formulae

\[
\beta y + \gamma a + \alpha \beta = -a^4 - b^4 - c^4 + 2(b^2c^2 + c^2a^2 + a^2b^2) = 16\Delta^2,
\]

\[
abc = 4\Delta R, \quad \Delta = rs.
\]