GROUP ALGEBRA MODULES. II

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1. Introduction. The present paper began as a natural outgrowth of our first paper, where we characterized the module homomorphisms from group algebras into a fairly restrictive class of group algebra modules. We now investigate module homomorphisms from group algebras into a more general class of group algebra modules. Although the two papers are thus related, they can be read quite independently.

Section 2 contains our extension, Theorem 2.1, of P. J. Cohen's theorem on factorization in Banach algebras **(1)**. Our extension is to Banach modules over Banach algebras equipped with an approximate identity. We should mention first that J.-K. Wang observed the existence of such a generalization, and secondly, that our proof requires no ideas different from those in Cohen's proof. Nevertheless, we include a proof that condenses the original proof considerably.

Next, let X be a locally compact, Hausdorff space and let Γ be not only a locally compact group, but also a locally compact group of homeomorphisms on X, with the property that the map $\pi : \Gamma \times X \to X$, defined by $\pi(\sigma, x) = \sigma x$, $\sigma \in \Gamma, x \in X$, is jointly continuous. If m_X is a Radon measure on X and if $Y \subseteq X$ is such that $m_X(Y) = 0$ implies that $m_X(\sigma Y) = 0$ for all $\sigma \in \Gamma$, then we say that m_X is quasi-invariant (with respect to Γ). From now on we assume that X has a non-zero, positive Radon measure m_X which is quasi-invariant. We can then define, and we devote § 3 to, a generalized measure algebra convolution $*$, which renders $L_1(X)$ and $M(X)$ as left $L_1(\Gamma)$-modules. It is no accident that we describe only left $L_1(\Gamma)$-modules. For Haar measure on Γ is left translation-invariant and not in general right translation-invariant. Among our results on the $L_1(\Gamma)$-module $M(X)$ we note that $L_1(\Gamma) * M(X)$ is usually a proper subset of $M(X)$. In fact, $\mu \in L_1(\Gamma) * M(X)$ if and only if the left shift of μ by elements of Γ is continuous with respect to Γ, and any measure in $M(X)$ absolutely continuous with respect to m_X has this property.

With any locally compact group Γ there is associated a modular function that depends upon the group structure of Γ. For general locally compact spaces there is no such associated function. Section 4 contains the definition of the generalized modular function J for (Γ, X). We show that J acts like the classical Jacobian and satisfies useful functional equations. As a consequence of our definitions and Theorem 2.1, we are able to show that $L_p(X)$

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($p \in (1, \infty]$) and $C_\infty(X)$ are also left $L_1(\Gamma)$-modules, and that

$$L_1(\Gamma) * L_p(X) = L_p(X), \quad p \in [1, \infty), \quad \textit{and} \quad L_1(\Gamma) * C_\infty(X) = C_\infty(X).$$

In § 5 we apply the results of the earlier sections to the problem of original interest to us. The theorems concerning homomorphisms are two. First, if $p \in (1, \infty]$, then as a Banach space the collection of $(L_1(\Gamma), L_p(X))$-homomorphisms corresponds isometrically to $L_p(X)$. Secondly, the space of $(L_1(\Gamma), L_1(X))$-homomorphisms corresponds isometrically to the subspaces N of $M(X)$, where

$$N = \{\mu \in M(X) : f * \mu \ll m_X \quad \text{for each } f \in L_1(\Gamma)\}.$$

The paper concludes with an analysis of N. We discuss the case $N = M(X)$. Finally we prove that if $N = M(X)$ and if Γ acts transitively on X, then X is homeomorphic to a factor space of Γ.

Throughout the paper, X will denote an arbitrary locally compact, Hausdorff space with elements $x, y, z, \ldots$. If $Y \subseteq X$, then $X \backslash Y$ is the complement of Y in X, and if $Y, Z \subseteq X$, then $Y \triangle Z = Y \backslash Z \cup Z \backslash Y$. Let $C_c(X)$ denote the normed space of continuous, complex-valued functions on X with compact support, endowed with the supremum norm.

Next, let X be a measure space with measure μ. The characteristic function of a measurable set Y in X is ξ_Y. A subset Y of X is locally null if for each compact set $K \subseteq X$, $\mu(K \cap Y) = 0$. If $p \in (1, \infty)$, the conjugate space (dual) of $L_p(X)$ is $L_q(X)$, where $1/p + 1/q = 1$. The dual of $L_1(X)$ is $L_\infty(X)$, where the functions are taken to be identical if they are locally identical. We think of $L_1(X)$ as a subspace of $L_1^{**}(X)$. Frequently we write $\int f(x)dx$ for $\int f(x)dm_X(x)$. We define the point mass δ_x at $x \in X$ by $\delta_x(f) = f(x)$, for all $f \in C_\infty(X)$.

Let Γ be an arbitrary locally compact group. Its elements are denoted by σ, τ, ϕ, and its subsets by Φ, Ψ, Ω. The left (Haar) measure corresponding to Γ is represented by m_Γ. The modular function associated with m_Γ is denoted by Δ. Under convolution, $L_1(\Gamma)$ becomes a Banach algebra and has an approximate identity $(e_\iota)_{\iota \in I}$ of norm one, where I is the indexing set related to the cardinality of the neighbourhood system of the identity element 1 of Γ. Let $M(\Gamma)$ be the Banach algebra of bounded countably additive regular measures on Γ, under convolution. Then $L_1(\Gamma)$ is a two-sided ideal in $M(\Gamma)$. For $f \in L_1(\Gamma)$ we define the left shift by $\sigma \in \Gamma$ by $f_\sigma(\tau) = f(\sigma\tau)$, $\tau \in \Gamma$. Also, if $f \in L_1(\Gamma)$, let $f'(\sigma) = \Delta(\sigma^{-1})f(\sigma^{-1})$, for almost all $\sigma \in \Gamma$. Then the map $f \to f'$ is an isometry on $L_1(\Gamma)$ of period two (**2**, Lemma 2.4). For $\mu \in M(X)$ and $\sigma \in \Gamma$ we define the shift $\mu_\sigma \in M(X)$ of μ by $\mu_\sigma(Y) = \mu(\sigma^{-1} Y)$, $Y \subseteq X$ Borel.

2. Factorization in Banach algebra modules.

2.1. *Definition.* Let A be a Banach algebra with multiplication $*$. Then a Banach space K is a *Banach module* over A if there exists a bilinear map

$*: A \times K \to K$ having the following properties:

(a) $(f_1 * f_2) * k = f_1 * (f_2 * k), \quad f_1, f_2 \in A, \ k \in K.$
(b) $\|f * k\| \leqslant \|f\| \, \|k\|, \quad f \in A, \ k \in K.$

The following theorem, which is a generalization of (**1**, Theorem 1) and whose proof is based on the original proof, plays a significant role through the rest of the paper. Since it is different in character from our other theorems, we set it and its corollary separately in § 2.

2.2. Theorem. *Let A be a Banach algebra with (bounded) approximate identity $(e_\iota)_{\iota \in I}$. Let K be a Banach module over A such that* $\lim_\iota e_\iota * k = k, k \in K$. *Then every element in K can be factored, i.e., for $k \in K$, there are $f \in A$ and $h \in K$ such that $k = f * h$.*

Proof. First we adjoin an identity J to A, and call the resultant Banach algebra A^+. If we define $J * k = k$, $k \in K$, then K becomes an A^+-module. Assume that $c \geqslant 1$ and that $\|e_\iota\| \leqslant c$, for all $\iota \in I$. Let $b \in (0, 1)$ such that $bc/(1 - b) < 1$. Let $E_\iota = J - bJ + be_\iota \in A^+$, so that

$$E_\iota = (1 - b)(J + (be_\iota/(1 - b)).$$

Note that $\|be_\iota/(1 - b)\| < 1$, which means that $J + be_\iota/(1 - b)$ has an inverse. Hence E_ι has an inverse in A^+. Indeed,

$$E_\iota^{-1} = (1 - b)^{-1}\sum_{n=0}^{\infty} \left(\frac{-b}{1 - b}\right)^n e_\iota^n \quad \text{and} \quad \|E_\iota^{-1}\| \leqslant (1 - b)^{-1}\sum_{n=0}^{\infty} \left(\frac{bc}{1 - b}\right)^n.$$

Next we show that for each $k \in K$, $\lim_\iota E_\iota^{-1} * k = k$. First we remark that since $c \geqslant 1$, we have

$$\|e_\iota^m * k - k\| \leqslant \|e_\iota^m * k - e_\iota^{m-1} * k\| + \ldots + \|e_\iota * k - k\| \leqslant mc^m \|e_\iota * k - k\|.$$

Consequently,

$$\begin{aligned}\|E_\iota^{-1} * k - k\| &= \left\|(1 - b)^{-1}\sum_{n=0}^{\infty} \left(\frac{-b}{1 - b}\right)^n e_\iota^n * k - (1 - b)^{-1}\sum_{n=0}^{\infty} \left(\frac{-b}{1 - b}\right)^n k\right\| \\ &\leqslant (1 - b)^{-1}\sum_{n=1}^{\infty} \left(\frac{b}{1 - b}\right)^n \|e_\iota^n * k - k\| \\ &\leqslant (1 - b)^{-1}\sum_{n=1}^{\infty} \left(\frac{bc}{1 - b}\right)^n \|e_\iota * k - k\|,\end{aligned}$$

whence $\lim_\iota \|E_\iota^{-1} * k - k\| = 0$. In the same way we can show that $\lim_\iota \|E_\iota^{-1} * f - f\| = 0$, for all $f \in A$. Let $k \in K$. We shall define a sequence of invertible elements $(F_n)_{n=0}^{\infty} \subset A^+$ such that $\|F_n^{-1} * k - F_{n-1}^{-1} * k\| < 2^{-n}$, for $n \geqslant 1$, and that $\lim_n F_n = f_n \in A$. If we have chosen such a sequence, then $(F_n^{-1} * k)_{n=1}^{\infty}$ is a Cauchy sequence, so it converges to $h \in K$. This implies

$$f * h = \lim_n F_n * F_n^{-1} * k = k$$

and the proof is therefore completed.

It remains to construct the required sequence $(F_0, F_1, F_2, \ldots) \subset A^+$. We begin with $F_0 = J$. For each positive n, define

$$F_n = \sum_{m=1}^{n} b(1-b)^{m-1} e_m + (1-b)^n J, \tag{2.1}$$

where we have yet to choose the $e_m \in (e_\iota)_{\iota \in I}$ inductively. Assume that $e_1, \ldots, e_n \in (e_\iota)_{\iota \in I}$ and $F_0, F_1, \ldots, F_n$ have been chosen, and that the F's have the appropriate properties. We shall prescribe e_{n+1} and thereby define F_{n+1}. For each $\iota \in I$, define $F_{n+1}(e_\iota) \in A^+$ by

$$E_\iota^{-1} * F_{n+1}(e_\iota) = F_n - \sum_{m=1}^{n} b(1-b)^{m-1}(J - E_\iota^{-1}) * e_m. \tag{2.2}$$

By a calculation above, $\lim_\iota E_\iota^{-1} * e_m = e_m$ for each m, so that

$$\lim_\iota E_\iota^{-1} * F_{n+1}(e_\iota) = F_n.$$

But F_n and E_ι^{-1} are invertible, and the set of invertible elements in A^+ is open; so $F_{n+1}(e_\iota)$ is invertible for all large ι. Furthermore,

$$\lim_\iota [F_{n+1}(e_\iota)]^{-1} * E_\iota = F_n^{-1}$$

and $\lim_\iota ||E_\iota^{-1} * k - k|| = 0$; therefore

$$\begin{aligned} \lim_\iota ||[F_{n+1}(e_\iota)]^{-1} * k - F_n^{-1} * k|| \\ \leqslant \lim_\iota ||\{[F_{n+1}(e_\iota)]^{-1} * E_\iota - F_n^{-1}\} * (E_\iota^{-1} * k)|| \\ + ||F_n^{-1} * (E_\iota^{-1} * k - k)|| = 0, \end{aligned}$$

since the $(E_\iota^{-1})_{\iota \in I}$ are bounded. We can choose $\iota_{n+1} \in I$ so that

$$||[F_{n+1}(e_{\iota_{n+1}})]^{-1} * k - F_n^{-1} * k|| < 2^{-n-1}.$$

Let $e_{n+1} = e_{\iota_{n+1}}$. Then F_{n+1} is duly defined by (2.1). This equation makes it clear that $\lim_n F_n$ exists and is in A. Finally, if we compare (2.1) and (2.2), we find that $F_{n+1} = F_{n+1}(e_{n+1})$, and hence F_{n+1} is invertible.

2.3. COROLLARY. *Let A be a Banach algebra with (bounded) approximate identity $(e_\iota)_{\iota \in I}$, and let K be a Banach module over A. Then $A * K$ is a closed subspace of K*, and *$k \in K$ implies $k \in A * K$ if and only if* $\lim_\iota e_\iota * k = k$.

Proof. Let

$$K_1 = \{k \in K : \lim_\iota e_\iota * k = k\}.$$

Then K_1 is a linear space and is closed since the approximate identity is bounded. Next, for $f \in A$, $k \in K$, we have

$$\lim_\iota e_\iota * (f * k) = \lim_\iota (e_\iota * f) * k = f * k,$$

so that $K_1 \supseteq A * K$. Hence $K_1 \supseteq A * K_1$, and K_1 is a Banach module over A. Therefore the preceding theorem applies, and $K_1 = A * K_1 \subseteq A * K$; thus $K_1 = A * K$, as postulated.

3. The generalized measure algebra convolution. Here we introduce the concept of generalized convolution, based on convolutions given in **(3)**. Recollect that $\pi : \Gamma \times X \to X$ is defined by $\pi(\sigma, x) = \sigma x$, $\sigma \in \Gamma$, $x \in X$. For a function k on X, we have

$$k(\sigma x) = (k \circ \pi)(\sigma, x), \qquad \sigma \in \Gamma,\ x \in X.$$

If $k \in C_\infty(X)$, then $k \circ \pi$ is a bounded, continuous function on $\Gamma \times X$. Therefore if $\mu \in M(\Gamma)$ and $\nu \in M(X)$, then the integral $(\mu \times \nu)(k \circ \pi)$ is well defined, and we can define a convolution $* : M(\Gamma) \times M(X) \to M(X)$ by

$$(3.1) \quad (\mu * \nu)(k) = (\mu \times \nu)(k \circ \pi) = \int_{\Gamma \times X} k(\sigma x) d(\mu \times \nu)(\sigma, x),$$
$$\mu \in M(\Gamma),\ \nu \in M(X), \text{ and } k \in C_\infty(X).$$

Then $\mu * \nu$ is an element of $M(X)$ and satisfies $\|\mu * \nu\| \leqslant \|\mu\|\,\|\nu\|$. Furthermore, if $\mu, \mu_1 \in M(\Gamma)$ and $\nu \in M(X)$, then $(\mu * \mu_1) * \nu = \mu * (\mu_1 * \nu)$. To prove this, let $k \in C_\infty(X)$ and remark that

$$\begin{aligned}(\mu * (\mu_1 * \nu))(k) &= \int_\Gamma \int_X k(\sigma x) d(\mu_1 * \nu)(x) d\mu(\sigma)\\ &= \int_\Gamma \int_X \int_\Gamma k(\sigma\tau x) d\mu_1(\tau) d\nu(x) d\mu(\sigma)\\ &= \int_X \int_\Gamma k(\phi x) d(\mu * \mu_1)(\phi) d\nu(x)\\ &= ((\mu * \mu_1) * \nu)(k).\end{aligned}$$

In analogy with **(3**, Theorem 19.11**)**, we can show that for each Borel set $Y \subseteq X$,

$$(3.1)' \quad (\mu * \nu)(Y) = (\mu \times \nu)(\pi^{-1} Y) = \int_X \int_\Gamma \xi_Y(\sigma x) d\mu(\sigma) d\nu(x)$$
$$= \int_\Gamma \nu(\sigma^{-1} Y) d\mu(\sigma).$$

By choosing $\mu \ll m_\Gamma$ and by replacing μ with the $L_1(\Gamma)$-function f it determines, we obtain a convolution $* : L_1(\Gamma) \times M(X) \to M(X)$ given by

$$(3.2) \quad (f * \nu)(k) = \int_{\Gamma \times X} k(\sigma x) f(\sigma) d(m_\Gamma \times \nu)(\sigma, x), \qquad k \in C_\infty(X),$$

$$(3.3) \quad (f * \nu)(Y) = \int_{\Gamma \times X} \xi_Y(\sigma x) f(\sigma) d(m_\Gamma \times \nu)(\sigma, x)$$
$$= \int_\Gamma f(\sigma) \nu(\sigma^{-1} Y) d\sigma, \qquad Y \subseteq X \text{ Borel}.$$

If in (3.1)$'$, $\nu \ll m_X$ and $m_X(Y) = 0$, then $m_X(\sigma^{-1} Y) = 0$ for every $\sigma \in \Gamma$ so that $\nu(\sigma^{-1} Y) = 0$ and $(\mu * \nu)(Y) = 0$. Hence $\mu * \nu \ll m_X$. Let ν correspond to $j \in L_1(X)$. This induces a convolution $* : M(\Gamma) \times L_1(X) \to L_1(X)$ through

$$\int_Y (\mu * j)(x) dx = \int_{\Gamma \times X} \xi_Y(\sigma x) j(x) d(\mu \times m_X)(\sigma, x), \qquad Y \subseteq X \text{ Borel}.$$

Furthermore, if we choose $\mu \ll m_\Gamma$ and replace μ by the $L_1(\Gamma)$-function f it determines, we obtain

$$(3.4) \qquad \int_Y (f * j)(x) dx = \int_{\pi^{-1} Y} f(\sigma) j(x) d(\sigma, x),$$

where $Y \subseteq X$ is Borel, and $f \in L_1(\Gamma)$ and $j \in L_1(X)$.

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We observe that usually $L_1(\Gamma) * M(X) \not\subseteq L_1(X)$. This will be the subject of § 5. We summarize our present comments in

3.1. THEOREM. *Both $M(X)$ and $L_1(X)$ are Banach modules over $M(\Gamma)$ and over $L_1(\Gamma)$.*

From the fact that $M(\Gamma)$ has a unit element and from the very definition of convolution, we see that $M(\Gamma) * M(X). = M(X)$ and $M(\Gamma) * L_1(X) = L_1X)$, without referring to Theorem 2.2. It is less trivial that always

$$L_1(\Gamma) * L_1(X) = L_1(X),$$

and this result we shall prove in § 4. On the other hand, if Γ is not discrete and if $X = \Gamma$, then $L_1(\Gamma) * M(X) = L_1(X) \neq M(X)$. Hence it is interesting to study the subspace $L_1(\Gamma) * M(X)$, which we know by Corollary 2.3 to be closed in $M(X)$. We have the following characterization of $L_1(\Gamma) * M(X)$:

3.2. THEOREM. *The following conditions on $\nu \in M(X)$ are equivalent:*
(i) *$\nu \in L_1(\Gamma) * M(X)$.*
(ii) *$\nu_\sigma \in M(X)$ is continuous as a function of $\sigma \in \Gamma$.*

Proof. Let $f \in L_1(\Gamma)$ and $\mu \in M(X)$, and let $\nu = f * \mu$. A simple computation shows that $\nu_\sigma = (f * \mu)_\sigma = (f_{\sigma^{-1}}) * \mu$. But $f_{\sigma^{-1}}$ is a continuous function of $\sigma \in \Gamma$ (**3**, Theorem 20.4), and $M(X)$ is an $L_1(\Gamma)$-module, and hence by Theorem 3.1, (i) implies (ii). Conversely, assume (ii) and let $\eta > 0$. By hypothesis there exists a neighbourhood Ω of $1 \in \Gamma$ such that $||\nu_\sigma - \nu|| < \eta$, $\sigma \in \Omega$. Then $||\nu(\sigma^{-1} Y) - \nu(Y)|| = |\nu_\sigma(Y) - \nu(Y)| < \eta$, $\sigma \in \Omega$, for all Borel $Y \subseteq X$. Without loss of generality, we may assume that $m_\Gamma(\Omega) < \infty$. Let

$$f = [m_\Gamma(\Omega)]^{-1}\xi_\Omega.$$

Then $f * \nu \in L_1(\Gamma) * M(X)$, and by the definition of convolution,

$$|(f * \nu)(Y) - \nu(Y)| \leqslant \eta$$

for each Borel Y. Therefore $||f * \nu - \nu|| \leqslant \eta$, so that ν is in the closure of $L_1(\Gamma) * M(X)$, which by Corollary 2.3 is already closed.

In fact, we can say more about $L_1(\Gamma) * M(X)$.

3.3. THEOREM. *The following conditions on $\mu \in M(X)$ are equivalent:*
(i) *$\mu \ll \nu$ for some $\nu \in L_1(\Gamma) * M(X)$.*
(ii) *For every Borel set $Y \subseteq X$ whose closure is compact,*

$$\lim_{\sigma\to 1} |\mu| \; (Y \triangle \sigma Y) = 0,$$

where $Y \triangle \sigma Y = (Y \cup \sigma Y)\backslash(Y \cap \sigma Y)$.

(iii) *For every compact $K \subseteq X$, $\mu(\sigma K)$ depends continuously on $\sigma \in \Gamma$.*

Proof. We first prove that (i) implies (ii). To that end, let $\mu \ll \nu$ for some $\nu \in L_1(\Gamma) * M(X)$. Note that by hypothesis, $\nu = f * \nu_1$ for some $f \in L_1(\Gamma)$,

$\nu_1 \in M(X)$, and thus $\nu \ll |f| * |\nu_1| \in L_1(\Gamma) * M(X)$. Therefore, we may assume that $\nu \geqslant 0$. For any Borel set Y such that $\bar{Y}$ is compact, and for any $\eta > 0$, there exist an open set U and a compact set K such that $K \subseteq Y \subseteq U$ and $\nu(U \backslash K) < \eta$. Since π is continuous, $\sigma K \subseteq U$ and $K \subseteq \sigma U$ for σ sufficiently near to 1. For such σ,

$$\nu(Y \triangle \sigma Y) = \nu(Y \backslash \sigma Y) + \nu(\sigma Y \backslash Y) \leqslant \nu(U) - \nu(\sigma K) + \nu(\sigma U) - \nu(K),$$

so that the continuity of ν_σ established in Theorem 3.2 yields

$$\limsup_{\sigma \to 1} \nu(Y \triangle \sigma Y) \leqslant 2\eta.$$

Because this is true for every positive η, $\lim_{\sigma \to 1} \nu(Y \triangle \sigma Y) = 0$, whence $\lim_{\sigma \to 1} |\mu| (Y \triangle \sigma Y) = 0$. That (ii) implies (iii) follows from the inequality $|\mu(\tau K) - \mu(\sigma K)| \leqslant |\mu| (\tau K \triangle \sigma\tau^{-1}(\tau K))$, for all $\sigma, \tau \in \Gamma$. Finally, we show that (iii) implies (i). Let $\mu(\sigma K)$ depend continuously on $\sigma \in \Gamma$. Let $f \in C_c(\Gamma)$, $f \geqslant 0$ with $f(1) > 0$. Define $\nu = f * |\mu|$. Then $\nu \in L_1(\Gamma) * M(X)$ and $\nu \geqslant 0$. Since μ is regular, in order to prove that $\mu \ll \nu$, it is enough to show that if $K \subseteq X$ is compact and $\nu(K) = 0$, then $\mu(K) = 0$. Now we note that if $\nu(K) = 0$, then

$$0 = \int_\Gamma f(o) \, |\mu| \, (\sigma^{-1} K) d\sigma,$$

so that $|\mu| \, (\sigma^{-1} K) = 0$ for almost all σ in the support of f. Then *a fortiori* $\mu(\sigma^{-1} K) = 0$ for these σ. But $\mu(\sigma^{-1} K)$ is a continuous function of σ, so $\mu(\sigma^{-1} K) = 0$ for all σ in the support of f. In particular, $\mu(K) = 0$.

3.4. COROLLARY. *If $\nu \in M(X)$ is such that $\nu(\sigma K)$ is a continuous function of σ for every compact set K, and if $\nu_1 \ll \nu$, then $\nu_1(\sigma K)$ is a continuous function of σ for each compact K.*

Proof. Apply Theorem 3.3 directly.

3.5. THEOREM. *Every element of $M(X)$ which is absolutely continuous with respect to m_X possesses the three properties of Theorem* 3.3.

Proof. It is easy to see that the measures in $M(X)$ which have property (iii) of Theorem 3.3 form a closed subspace of $M(X)$, and that each element of $M(X)$ is a limit of members with compact support. Thus we need only show that if $\nu \ll m_X$ and if the support of ν is the compact set K, then ν has property (i).

Assume we have such a ν, and let the support of $f \in C_c(\Gamma)$ be Φ, where $f \geqslant 0$ and $f(1) > 0$. If $L = \{\sigma^{-1} x : \sigma \in \Phi, x \in K\}$, then L is compact, since π is continuous. Let m_L be the restriction of m_X to L. Then

$$f * m_L \in L_1(\Gamma) * M(X).$$

We shall show that $\nu \ll f * m_L$. Let $(f * m_L)(K_1) = 0$ where we assume without loss of generality that $K_1 \subseteq K$. We shall show that $\nu(K_1) = 0$. Now

$$0 = \int_\Gamma f(\sigma) m_L(\sigma^{-1} K_1) d\sigma,$$

so that $m_L(\sigma^{-1}K_1) = 0$ for almost every σ in Φ. But if $\sigma \in \Phi$, then $\sigma^{-1}K_1 \subseteq L$, so that $m_L(\sigma^{-1}K_1) = m_X(\sigma^{-1}K_1)$; hence $m_X(\sigma^{-1}K_1) = 0$ for almost all $\sigma \in \Phi$. Since m_X is quasi-invariant, it follows that $m_X(K_1) = 0$, and hence $\nu(K_1) = 0$.

3.6. Corollary. *If $Y \subseteq X$ and $\bar{Y}$ is compact, then* $\lim_{\sigma\to 1} m_X(Y \triangle \sigma Y) = 0$, *and $m_X(\sigma Y)$ is a continuous function of $\sigma \in \Gamma$.*

Proof. To prove the first assertion, let Φ be a compact neighbourhood of $1 \in \Gamma$ and let $L = \Phi\bar{Y}$, so that L is also compact. From Theorem 3.5, the restriction m_L of m_X to L fulfils condition (ii) of Theorem 3.3 so that

$$\lim_{\sigma\to 1} m_X(Y \triangle \sigma Y) = \lim_{\substack{\sigma\to 1\\ \sigma\in\Phi}} m_X(Y \triangle \sigma Y) = \lim_{\sigma\to 1} m_L(Y \triangle \sigma Y) = 0.$$

The second assertion results from the following:

$$\lim_{\phi\to\tau} |m_X(\tau Y) - m_X(\phi Y)| = \lim_{\sigma\to 1} |m_X(\tau Y) - m_X(\sigma\tau Y)| \leqslant \lim_{\sigma\to 1} m_X(\tau Y \triangle \sigma\tau Y) = 0.$$

It is easy to generalize this corollary and to show that $\mu(\sigma Y)$ depends continuously upon σ for every Radon measure μ (bounded or not) which is absolutely continuous with respect to m_X.

4. The generalized modular function and generalized convolution.

In this section we introduce a generalized modular function, which we use to define a convolution in $L_1(\Gamma) \times L_p(X)$, where $p \in [1, \infty]$. We are then able to use Theorem 2.2 to prove that $L_1(\Gamma) * L_p(X) = L_p(X)$ if $p \in [1, \infty)$ and that $L_1(\Gamma) * C_\infty(X) = C_\infty(X)$. We begin, however, with the modular function J, which we assert exists.

4.1. Theorem. *There exists a positive locally integrable function J, defined on $\Gamma \times X$, such that*

(i) $\int_{\Gamma\times X} F(\sigma, x)d(\sigma, x) = \int_{\Gamma\times X} F(\sigma, \sigma x)J(\sigma, x)d(\sigma, x), \qquad F \in L_1(\Gamma \times X).$

We also have

(ii) $\int_{\Gamma\times X} F(\sigma, \sigma x)d(\sigma, x) = \int_{\Gamma\times X} F(\sigma, x)J(\sigma^{-1}, x)f(d\sigma, x),$

whenever $\int_{\Gamma\times X} F(\sigma, \sigma x)d(\sigma, x)$ exists.

Proof. Define $T : \Gamma \times X \to \Gamma \times X$ by $T(\sigma, x) = (\sigma, \sigma x)$. Then T is a homeomorphism. To prove the existence of J which satisfies the first equation above, it suffices (**3**, Theorem 12.17) to show that if $\mathfrak{A}$ is compact in $\Gamma \times X$, then $(m_\Gamma \times m_X)(\mathfrak{A}) = 0$ if and only if $(m_\Gamma \times m_X)(T^{-1}\mathfrak{A}) = 0$. Now assume that $\mathfrak{A} \subseteq \Gamma \times X$ and $\mathfrak{A}$ is compact and that $(m_\Gamma \times m_X)(\mathfrak{A}) = 0$. Then $\int_X \xi_{\mathfrak{A}}(\sigma, x)dx = 0$ for almost all $\sigma \in \Gamma$, so that the quasi-invariance of m_X implies $\int_X \xi_{\mathfrak{A}}(\sigma, \sigma x)dx = 0$ for almost all $\sigma \in \Gamma$. Thus

$$(m_\Gamma \times m_X)(T^{-1}\mathfrak{A}) = \int_{\Gamma\times X} \xi_{\mathfrak{A}}(\sigma, \sigma x)d(\sigma, x) = \int_\Gamma \int_X \xi_{\mathfrak{A}}(\sigma, \sigma x)dx\, d\sigma = 0.$$

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The proof that $(m_\Gamma \times m_X)(T^{-1}\mathfrak{A}) = 0$ implies that $(m_\Gamma \times m_X)(\mathfrak{A}) = 0$ is similar, so we omit it.

Next we prove the second statement of the theorem. Let F be a measurable function on $\Gamma \times X$ and assume that

$$\int_{\Gamma\times X} F(\sigma, \sigma x)d(\sigma, x)$$

exists. By applying the first half of our theorem, we obtain

$$\begin{aligned}\int_{\Gamma\times X} F(\sigma, \sigma x)d(\sigma, x) &= \int_X\int_\Gamma F(\sigma, \sigma x)d\sigma\, dx = \int_X\int_\Gamma F(\sigma^{-1}, \sigma^{-1} x)\Delta(\sigma^{-1})d\sigma\, dx\\ &= \int_{\Gamma\times X} F(\sigma^{-1}, \sigma^{-1} x)\Delta(\sigma^{-1})d(\sigma, x)\\ &= \int_X\int_\Gamma F(\sigma^{-1}, x)\Delta(\sigma^{-1})J(\sigma, x)d\sigma\, dx\\ &= \int_X\int_\Gamma F(\sigma, x)J(\sigma^{-1}, x)d\sigma\, dx\\ &= \int_{\Gamma\times X} F(\sigma, x)J(\sigma^{-1}, x)d(\sigma, x).\end{aligned}$$

The proof of Theorem 4.1 shows that for a locally null set $Y \subseteq X$ the set $\pi^{-1}(Y) = T^{-1}(\Gamma \times Y)$ is locally null in $\Gamma \times X$. Hence, if k is a measurable function on X, then $k \circ \pi$ is measurable on $\Gamma \times X$. Hence if

$$F(\sigma, x) \in L_1(\Gamma \times X)$$

and if $k \in L_1(X)$, then $F(\sigma, x)k(\sigma x)$ is measurable with respect to the product measure, and we can apply Fubini's theorem, as we shall frequently do in the future.

Examples: If Γ is measure-preserving on X, then $J(\sigma, x) = 1$, for almost all (σ, x), so that we can define $J(\sigma, x) = 1$ for all $\sigma \in \Gamma$, and all $x \in X$. If $X = \Gamma$ and if m_X is the right Haar measure, then $J(\sigma, x) = \Delta(\sigma^{-1})$, for all (σ, x). We mention here that A. M. Macbeath and S. Świerczkowski **(4)** consider the Jacobian which arises when X is a factor space of Γ. They prove that J satisfies the functional equation

$$J(\sigma\tau, x) = J(\sigma, \tau x)J(\tau, x), \qquad \sigma, \tau \in \Gamma,\ x \in X.$$

We derive a slightly weaker form (Theorem 4.14) of this relation for our more general investigation. A particular case of this third example is the following. Let X be an n-dimensional complex linear space and let Γ be the general linear group acting on X. Then $J(\sigma, x) = \det(\sigma)^{-1}$.

4.2. THEOREM. *For $f \in L_1(\Gamma)$ and $k \in L_1(X)$,*

$$(f * k)(x) = \int_\Gamma f(\sigma)k(\sigma^{-1} x)J(\sigma^{-1}, x)d\sigma,$$

for locally almost all $x \in X$.

Proof. For compact $Y \subseteq K$, Theorem 4.1 implies that

$$\begin{aligned}\int_X \xi_Y(x)[(f * k)(x)]dx &= \int_{\Gamma\times X} \xi_Y(\sigma x)f(\sigma)k(x)d(\sigma, x)\\ &= \int_{\Gamma\times X} \xi_Y(x)f(\sigma)k(\sigma^{-1} x)J(\sigma^{-1}, x)d(\sigma, x)\\ &= \int_X\int_\Gamma \xi_Y(x)f(\sigma)k(\sigma^{-1} x)J(\sigma^{-1}, x)d\sigma\, dx,\end{aligned}$$

and the result follows.

4.3. THEOREM. *If $k \in C_c(X)$, then*

$$\int_X k(\sigma x)dx = \int_X k(x)J(\sigma^{-1}, x)dx$$

for locally almost all $\sigma \in \Gamma$. If $k \in L_1(X)$, then

$$\int_X k(x)dx = \int_X k(\sigma x)J(\sigma, x)dx$$

for locally almost all $\sigma \in \Gamma$.

Proof. In either case define $F(\sigma, x) = \xi_\Phi(\sigma)k(x)$, where $\Phi \subseteq \Gamma$ is any open set of finite measure, and apply Theorem 4.1.

After we have more machinery (Theorem 4.11), we can change the values of J on a locally null set and prove Theorem 4.3 without the "locally almost." In any case, Theorem 4.1 readily yields a functional equation for J. Indeed, for any $F \in L_1(\Gamma \times X)$ we first use (i) and then (ii) to obtain

$$\int_{\Gamma\times X} F(\sigma, x)d(\sigma, x) = \int_{\Gamma\times X} F(\sigma, \sigma x)J(\sigma, x)d(\sigma, x)$$
$$= \int_{\Gamma\times X} F(\sigma, x)J(\sigma, \sigma^{-1} x)J(\sigma^{-1}, x)d(\sigma, x).$$

Thus we have

4.4. THEOREM. *For locally almost all $(\sigma, x) \in \Gamma \times X$,*

$$J(\sigma, \sigma^{-1} x)J(\sigma^{-1}, x) = 1.$$

In Theorem 4.3, $J(\sigma, \circ)$ plays the role of a "Jacobian" for the map $x \to \sigma x$. Accordingly, we have a chain rule:

4.5. THEOREM. *For locally almost all triples $(\sigma, \tau, x) \in \Gamma \times \Gamma \times X$, $J(\sigma\tau, x) = J(\sigma, \tau x)J(\tau, x)$.*

Proof. Take $F \in L_1(\Gamma \times \Gamma \times X)$. Then

$$\int_{\Gamma\times\Gamma\times X} F(\sigma, \tau, x)d(\sigma, \tau, x) = \int_\Gamma \int_{\Gamma\times X} F(\sigma, \tau, \sigma x)J(\sigma, x)d(\sigma, x)d\tau$$
$$= \int_\Gamma \int_{\Gamma\times X} F(\sigma, \tau, \sigma\tau x)J(\sigma, \tau x)J(\tau, x)d(\tau, x)d\sigma$$
$$= \int_{\Gamma\times\Gamma\times X} F(\sigma, \tau, \sigma\tau x)J(\sigma, \tau x)J(\tau, x)d(\sigma, \tau, x).$$

On the other hand,

$$\int_{\Gamma\times\Gamma\times X} F(\sigma, \tau, x)d(\sigma, \tau, x) = \int_{\Gamma\times X} \int_\Gamma F(\sigma, \sigma^{-1} \tau, x)d\tau d(\sigma, x)$$
$$= \int_\Gamma \int_{\Gamma\times X} F(\sigma, \sigma^{-1} \tau, x)d(\tau, x)d\sigma$$
$$= \int_\Gamma \int_{\Gamma\times X} F(\sigma, \sigma^{-1} \tau, \tau x)J(\tau, x)d(\tau. x)d\sigma$$
$$= \int_{\Gamma\times X} \int_\Gamma F(\sigma, \sigma^{-1} \tau, \tau x)J(\tau, x)d\tau d(\sigma, x)$$
$$= \int_{\Gamma\times X} \int_\Gamma F(\sigma, \tau, \sigma\tau x)J(\sigma\tau, x)d\tau d(\sigma, x)$$
$$= \int_{\Gamma\times\Gamma\times X} \mathrm{F}(\sigma, \tau, \sigma\tau x)J(\sigma\tau, x)d(\sigma, \tau, x)$$

and the theorem follows.

Now we can extend the concept of convolution to $L_1(\Gamma) \times L_p(X)$ where $p \in [1, \infty]$. Recall that $f'(\sigma) = f(\sigma^{-1})\Delta(\sigma^{-1})$, for almost all $\sigma \in \Gamma$, when $f \in L_1(\Gamma)$. Define a convolution $* : L_1(\Gamma) \times L_\infty(X) \to L_\infty(X)$ by

$$(f * k)j = k(f' * j),\ f \in L_1(\Gamma),\ k \in L_\infty(X),\ j \in L_1(X).$$

Clearly for such f and k, $f * k \in L_\infty(X)$ and $||f * k|| \leqslant ||f||\,||k||$, and $L_\infty(X)$ is rendered an $L_1(\Gamma)$-module. In addition,

$$(f * k)j = \int_X k(x)(f' * j)(x)dx = \int_X k(x) \int_\Gamma f(\sigma)j(\sigma x)J(\sigma, x)d\sigma dx$$
$$= \int_{\Gamma\times X} k(\sigma^{-1} x)f(\sigma)j(x)d(\sigma, x).$$

Hence

$$(f * k)j = \int_X j(x) \int_\Gamma f(\sigma)k(\sigma^{-1} x)d\sigma dx. \tag{4.1}$$

Now let $p \in 1, \infty)$. Then for any $\sigma \in \Gamma$, Theorem 4.3 and Hölder's inequality imply that

$$\int_X |k(\sigma^{-1} x)J(\sigma^{-1}, x)^{p^{-1}} j(x)|\, dx \leqslant [\int_X |k(\sigma^{-1} x)J(\sigma^{-1}, x)^{p^{-1}}|^p\, dx]^{p^{-1}}.$$
$$\times [\int_X |j(x)|^q\, dx]^{q^{-1}} = [\int_X |k(\sigma^{-1} x)|^p\, J(\sigma^{-1}, x)dx]^{p^{-1}} \times ||j||_q = ||k||_p\, ||j||_q.$$

Therefore, if for $f \in L_1(\Gamma)$, $k \in L_p(X)$, and $j \in L_q(X)$, we define $(f * k)(j)$ by

$$(f * k)(j) = \int_{\Gamma\times X} f(\sigma)k(\sigma^{-1} x)j(x)J(\sigma^{-1}, x)^{p^{-1}}\, d(\sigma, x), \tag{4.2}$$

then by the inequalities just above and by an application of Fubini's theorem, equation (4.2) defines $f * k \in L_q^*(X) = L_p(X)$, and $||f * k|| \leqslant ||f||\,||k||$. In addition, if we make the convention that $J(\sigma, x)^0 = 1$ for every $\sigma \in \Gamma$, $x \in X$, then (4.2) coincides with (4.1) when $p = \infty$. Also, if $p = 1$, then (4.2) agrees with the formula arising from Theorem 4.2. Thus we may define $f * k$ by means of (4.2); here $f \in L_1(\Gamma)$ and $k \in L_p(X)$, for any $p \in [1, \infty]$. We note that the convolution has a simplified form: for $p \in [1, \infty]$, if $f \in L_1(\Gamma)$ and $k \in L_p(X)$, (4.2) yields

$$(f * k)(x) = \int_\Gamma f(\sigma)k(\sigma^{-1} x)J(\sigma^{-1}, x)^{p^{-1}}\, d\sigma \text{ for locally almost all } x \in X. \tag{4.3}$$

Next, let $f, g \in L_1(\Gamma)$, and let $k \in L_p(X)$, where $p \in [1, \infty]$. Then

$$[(f * g) * k](j) = \int_X \{[\int_\Gamma \int_\Gamma f(\sigma)g(\sigma^{-1} \tau)d\sigma]k(\tau^{-1} x)J(\tau^{-1}, x)^{p^{-1}}\, d\tau\}j(x)dx$$
$$= \int_X\{\int_\Gamma f(\sigma)[\int_\Gamma g(\tau)k(\tau^{-1} \sigma^{-1} x)J(\tau^{-1} \sigma^{-1}, x)^{p^{-1}} d(\tau)]d(\sigma)\}j(x)dx$$
$$= \int_{\Gamma\times\Gamma\times X} f(\sigma)g(\tau)k(\tau^{-1} \sigma^{-1} x)j(x)J(\tau^{-1} \sigma^{-1}, x)^{p^{-1}}\, d(\sigma, \tau, x)$$
$$= \int_X\{\int_\Gamma f(\sigma)[\int_\Gamma g(\tau)k(\tau^{-1} \sigma^{-1} x)J(\tau^{-1}, \sigma^{-1} x)^{p^{-1}} d\tau]J(\sigma^{-1}, x)^{p^{-1}} d\sigma\}j(x)dx$$
$$= [f * (g * k)](j).$$

We recapitulate in

4.6. THEOREM. *Under convolution, $L_p(X)$ is an $L_1(\Gamma)$-module, where $p \in [1, \infty]$.*

We can now utilize Theorem 2.2, as we shall presently do.

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4.7. LEMMA. *If $f \in L_1(\Gamma)$, $k \in L_p(X)$, and $j \in L_q(X)$, then*

$$(f * k)j = k(f' * j).$$

Proof. By Theorems 4.1 (i) and 4.4,

$$\begin{aligned}(f * k)j &= \int_{\Gamma\times X} f(\sigma)k(\sigma^{-1} x)J(\sigma^{-1}, x)^{p^{-1}} j(x)d(\sigma, x)\\ &= \int_{\Gamma\times X} f(\sigma)k(x)J(\sigma^{-1}, \sigma x)^{p^{-1}} j(\sigma x)J(\sigma, x)d(\sigma, x)\\ &= \int_{\Gamma\times X} k(x)f'(\sigma)j(\sigma^{-1} x)J(\sigma, \sigma^{-1} x)^{p^{-1}} J(\sigma^{-1}, x)d(\sigma, x)\\ &= \int_{\Gamma\times X} k(x)f'(\sigma)j(\sigma^{-1} x)J(\sigma^{-1}, x)^{q^{-1}} d(\sigma, x) = k(f' * j).\end{aligned}$$

4.8. LEMMA. *The linear hull of the set $\{f * k : f \in L_1(\Gamma), k \in L_p(X)\}$ is dense in $L_p(X)$, for $p \in [1, \infty)$.*

Proof. Let $j \in L_q(X)$. Assume $j(f * g) = 0$ for all $f \in L_1(\Gamma)$, $k \in L_p(X)$. Then

$$0 = j(f' * k) = k(f * j) = \int_{\Gamma\times X} k(x)f(\sigma)j(\sigma^{-1} x)J(\sigma^{-1}, x)^{q^{-1}} d(\sigma, x),$$

by Lemma 4.7. However, $J(\sigma^{-1}, x) > 0$ locally almost everywhere in $\Gamma \times X$, so that $j(\sigma^{-1} x) = 0$ for locally almost all $(\sigma, x) \in \Gamma \times X$. However, m_X is quasi-invariant. Thus $\{x \in X : j(x) = 0\}$ is locally null and hence null, which means that $j = 0$. The Hahn–Banach theorem then yields the result.

The lemma is not true for $p = \infty$, even if $X = \Gamma$. However, considering $C_\infty(X)$ as a subspace of $L_\infty(X)$, we can prove

4.9. LEMMA. *The space $L_1(\Gamma) * C_\infty(X)$ is contained in $C_\infty(X)$.*

Proof. In Theorem 4.6 we proved that the convolution is jointly continuous. Therefore it suffices to prove that $C_c(\Gamma) * C_c(X) \subseteq C_c(X)$. If $f \in C_c(\Gamma)$ and $k \in C_c(X)$, then

$$(f * k)(x) = \int_\Gamma f(\sigma)k(\sigma^{-1} x)d\sigma \qquad \text{for locally almost all } x \in X.$$

However, the integral exists for all x. Therefore we may assume that the value of the integral is in fact $(f * k)(x)$, for all $x \in X$. There exist compact sets $\Phi \subseteq \Gamma$ and $K \subseteq X$ such that $f(\sigma) = 0$, $\sigma \notin \Phi$, and $k(x) = 0$, $x \notin K$. Since π is jointly continuous and k is continuous, we find that for any $\epsilon > 0$, and any given $x \in K$, there is a neighbourhood U of x such that

$$|k(\sigma^{-1} x) - k(\sigma^{-1} y)| < \epsilon, \qquad \sigma \in \Phi,\ y \in U.$$

But then $y \in U$ implies that $|(f * k)(x) - (f * k)(y)| < \epsilon\,||f||$, whence $f * k$ is continuous at x. Furthermore, $(f * k)(x) = 0$ if $x \notin \Phi K$, and ΦK is, again by the continuity of π, compact. Thus $f * k \in C_c(X)$.

4.10. LEMMA. *The linear hull of the set $\{f * k : f \in L_1(\Gamma), k \in C_\infty(X)\}$ is dense in $C_\infty(X)$.*

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Proof. By Lemma 4.9, $\{f * k : f \in L_1(\Gamma), k \in C_\infty(X)\} \subseteq C_\infty(X)$. It suffices to show that if $\mu \in M(X)$ satisfies $\mu(f * k) = 0$ for all $f \in L_1(\Gamma)$, $k \in C_c(X)$, then $\mu = 0$. For $k \in C_c(X)$ and $\sigma \in \Gamma$, we shall let $k_\sigma \in C_c(X)$ be defined by $k_\sigma(x) = k(\sigma x)$, $x \in X$. Then for $f \in L_1(\Gamma)$,

$$0 = \mu(f' * k) = \int_X \int_\Gamma f'(\sigma) k(\sigma^{-1} x) d\sigma d\mu(x) = \int_\Gamma \int_X f(\sigma) k(\sigma x) d\mu(x) d\sigma$$
$$= \int_\Gamma f(\sigma) \mu(k_\sigma) d\sigma.$$

Therefore $\mu(k_\sigma) = 0$ for locally almost every $\sigma \in \Gamma$. However, π is a continuous function, so that $k_\sigma \in C_c(X)$ depends continuously on σ. Thus $\mu(k_\sigma) = 0$ for each $\sigma \in \Gamma$, so that in particular $\mu(k) = 0$; therefore μ vanishes on $C_c(X)$, and hence $\mu = 0$.

At last we come to the theorem referred to after Theorem 4.6.

4.11. THEOREM. *Every element k of $L_p(X)$, $p \in [1, \infty)$ $(C_\infty(X))$ can be factored, i.e., $k = f * h$, for some $f \in L_1(\Gamma)$, $h \in L_p(X)$ $(h \in C_\infty(X))$. Moreover, if Γ_0 is an open subgroup of Γ and if Y is a subset of X outside of which k vanishes, then we can choose $f \in L_1(\Gamma)$ and $h \in L_p(X)$ $(C_\infty(X))$ such that $f(\sigma) = 0$ for $\sigma \notin \Gamma_0$, and $h(x) = 0$ for $x \notin \Gamma_0 Y$.*

Proof. By virtue of Lemma 4.8 and 4.10, the first statement follows from Corollary 2.3. Actually, we can say more. Let $A = L_1(\Gamma)$. We may take the approximate identity in A to have support in Γ_0, so that in the proof of Theorem 2.2, the portions in A of F_n, F_n^{-1} have support confined to Γ_0; hence $\lim_n F_n = f$ vanishes outside Γ_0 and $\lim_n F_n^{-1} * k = h$ vanishes outside $\Gamma_0 Y$.

We mentioned after Theorem 4.3 that Theorem 4.11 would enable us to improve the generalized modular function J. That is what we shall presently do. In fact, we shall change the values of J on a locally null set so that we obtain Theorem 4.3 without the "locally almost" occurring.

First, we note that Γ contains an open, sigma-compact subgroup Γ_0, as is well known. By (**3**, Theorem 11.39), there is a family $\{X_\iota : \iota \in I\}$ of disjoint compact subsets of X such that $X \setminus \bigcup X_\iota$ is locally null, while an open set of finite measure in X can intersect only countably many of the X_ι. For each $\iota \in I$ let

$$\Phi_\iota = \{\sigma \in \Gamma : J(\tau^{-1}\sigma, x) = J(\tau^{-1}, \sigma x) J(\sigma, x), \text{for almost all } (\tau, x) \in \Gamma_0 \times \Gamma_0 X_\iota\}.$$

By Theorem 4.5, $\Gamma \setminus \Phi_\iota$ is locally null, since $\Gamma_0 \times \Gamma_0 X_\iota$ is sigma-compact.

4.12. LEMMA. *Let $\iota \in I$ be fixed. Let $k \in L_1(X)$ with $k(x) = 0$ for $x \notin X_\iota$. Then*

$$\int_X k(x) dx = \int_X k(\sigma x) J(\sigma, x) dx \qquad \textit{for all } \sigma \in \Phi_\iota \cap \Gamma_0.$$

Proof. By Theorem 4.11, there exist $f \in L_1(\Gamma)$ and $h \in L_1(X)$ such that $k = f * h$ and $f(\tau) = 0$, $\tau \notin \Gamma_0$, and $h(x) = 0$, $x \notin \Gamma_0 X_\iota$. Let $\sigma \in \Phi_\iota \cap \Gamma_0$.

Now we use Theorems 4.1 (i) and 4.2 and the definition of Φ_ι to provide us with

$$\begin{aligned}\int_X k(x)dx &= \int_{\Gamma\times X} f(\tau)h(\tau^{-1}x)J(\tau^{-1},x)d(\tau,x) = \int_{\Gamma\times X} f(\tau)h(x)d(\tau,x)\\ &= \int_{\Gamma\times X} f(\sigma\tau)h(\mathrm{x})d(\tau,x) = \int_{\Gamma\times X} f(\sigma\tau)h(\tau^{-1}x)J(\tau^{-1},x)d(\tau,x)\\ &= \int_{\Gamma\times X} f(\tau)h(\tau^{-1}\sigma x)J(\tau^{-1}\sigma,x)d(\tau,x)\\ &= \int_{\Gamma_0\times\Gamma_0 X_\iota} f(\tau)h(\tau^{-1}\sigma x)J(\tau^{-1}\sigma,x)d(\tau,x)\\ &= \int_{\Gamma_0\times\Gamma_0 X_\iota} f(\tau)h(\tau^{-1}\sigma x)J(\tau^{-1},\sigma x)J(\sigma,x)d(\tau,x)\\ &= \int_X[\int_\Gamma f(\tau)h(\tau^{-1}\sigma x)J(\tau^{-1},\sigma x)\mathrm{d}\tau]J(\sigma,x)dx\\ &= \int_X k(\sigma x)J(\sigma,x)dx.\end{aligned}$$

This concludes the proof.

The J' which we now construct, and which differs from J by a locally null set, is created in four steps. We first define a subset $\mathfrak{A}$ of $\Gamma\times X$ where J behaves badly, and show that $\mathfrak{A}$ is locally null. Then secondly, we define J' on $\Gamma_0\times X$ so that

$$(1)\qquad \int_X k(x)dx = \int_X k(\sigma x)J'(\sigma,x)dx;$$

for all $k\in L_1(X)$ and all $\sigma\in\Gamma_0$; the J' so defined equals J on $(\Gamma_0\times X)\backslash\mathfrak{A}$. Thirdly, we extend J' to $\Gamma\times X$ so that (1) holds true for all $k\in L_1(X)$ and all $\sigma\in\Gamma$. Finally, we note that J' is measurable, and that $J'=J$ locally almost everywhere.

In the first place, we let $\mathfrak{A}=\bigcup_{\iota\in I}[(\Gamma_0\backslash\Phi_\iota)\times X_\iota]$. Then $\mathfrak{A}$ is locally null. For if $K\subseteq X$ is a compact set of finite m_X-measure, then by our remarks above, K intersects only countably many X_ι, say $X_1, X_2, \ldots$. This means that

$$\mathfrak{A}\cap(\Gamma\times K)\subseteq\bigcup_{n=1}^{\infty}[(\Gamma_0\backslash\Phi_n)\times X_n].$$

Thus $\mathfrak{A}$ is locally null.

In the second place, the quasi-invariance of m_X and (**3**, Theorem 12.17) guarantee the existence of a function J_1 on $\Gamma\times X$, such that J_1 is a locally integrable function on X for each $\sigma\in\Gamma$ and furthermore,

$$\int_X k(x)dx = \int_X k(\sigma x)J_1(\sigma,x)dx,\qquad \sigma\in\Gamma,\ k\in L_1(X).$$

Define J' on $\Gamma_0\times X$ by

$$J'(\sigma,x)=\begin{cases}J_1(\sigma,x) & \text{if } (\sigma,x)\in\mathfrak{A},\\ J(\sigma,x) & \text{if } (\sigma,x)\in(\Gamma_0\times X)\backslash\mathfrak{A}.\end{cases}$$

For a fixed $\iota\in I$, let $k\in L_1(X)$ with $k(x)=0$, $x\notin X_\iota$. Then

$$\int_X k(x)dx = \begin{cases}\int_X k(\sigma x)J_1(\sigma,x)dz = \int_X k(\sigma x)J'(\sigma,x)dx & \text{if } \sigma\in\Gamma_0\backslash\Phi_\iota,\\ \int_X k(\sigma x)J(\sigma,x)dx = \int_X k(\sigma x)J'(\sigma,x)dx & \text{if } \sigma\in\Gamma_0\cap\Phi_\iota,\end{cases}$$

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so that (1) holds for any $\sigma \in \Gamma_0$. However, any element of $L_1(X)$ is a sum of countably many functions each of which vanishes outside some particular X_ι. Hence (1) will hold for all $\sigma \in \Gamma_0$, $k \in L_1(X)$.

In the third place, let $\Phi \subseteq \Gamma$ be a set that intersects every coset of Γ_0 in exactly one point. Assume that $\Phi \cap \Gamma_0 = \{1\}$. Then for $\sigma \in \Phi$ and $\tau \in \Gamma_0$ define $J'(\sigma\tau, x)$ for locally almost every $x \in X$ by

$$J'(\sigma\tau, x) = J_1(\sigma, \tau x)J'(\tau, x)$$

so that J' is now defined on the whole of $\Gamma \times X$. For $\tau \in \Gamma_0$, $\sigma \in \Phi$, and $k \in L_1(X)$,

$$\int_X k(x)dx = \int_X k(\sigma x)J_1(\sigma, x)dx = \int_X k(\sigma\tau x)J_1(\sigma, \tau x)J'(\tau, x)dx$$
$$= \int_X k(\sigma\tau x)J'(\sigma\tau, x)dx.$$

Therefore (1) will now hold true for all $\sigma \in \Gamma$, $k \in L_1(X)$.

Now to show that J' is measurable we need only mention that $\Gamma_0 \times X$ is open in $\Gamma \times X$, and $J_1(\sigma, \circ)$ is measurable on X, for all $\sigma \in \Gamma$, so that J' is measurable on $\Gamma \times X$. Furthermore, for $F \in L_1(\Gamma \times X)$, we have

$$\int_{\Gamma\times X} F(\sigma, x)d(\sigma, x) = \int_\Gamma\int_X F(\sigma, x)dxd\sigma = \int_\Gamma\int_X F(\sigma, \sigma x)J'(\sigma, x)dxd\sigma$$
$$= \int_{\Gamma\times X} F(\sigma, \sigma x)J'(\sigma, x)d(\sigma, x);$$

together with Theorem 4.1 (i) this yields that $J = J'$ locally almost everywhere in $\Gamma \times X$, and we have completed the construction of J'. Henceforth we shall write J instead of J'. We restate our final result:

4.13. THEOREM. *For all $\sigma \in \Gamma$ and all $k \in L_1(X)$,*

$$\int_X k(x)dx = \int_X k(\sigma x)J(\sigma, x)dx.$$

Indeed our new J has all the properties of the one defined earlier. In particular,

4.14. THEOREM. *For all $\sigma, \tau \in \Gamma$,*

$$J(\sigma\tau, x) = J(\sigma, \tau x)J(\tau, x) \qquad \textit{and} \qquad J(\sigma, \sigma^{-1}x)J(\sigma^{-1}, x) = 1$$

for locally almost all $x \in X$.

Proof. If $k \in L_1(X)$, then by Theorem 4.13,

$$\int_X k(\sigma\tau x)J(\sigma\tau, x)dx = \int_X k(x)dx = \int_X k(\sigma x)J(\sigma, x)dx$$
$$= \int_X k(\sigma\tau x)J(\sigma, \tau x)J(\tau, x)dx.$$

The second assertion follows from the first one.

We mention that we do not know if the statements in Theorem 4.14 are true for all $x \in X$. We conjecture that they are. For later use we prove the following

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4.15. LEMMA. *Let* $\tau \in \Gamma$. *Then for locally almost all* $(\sigma, x) \in \Gamma \times X$,

$$J(\sigma\tau, x) = J(\sigma, \tau x)J(\tau, x).$$

Proof. For any $F \in C_c(\Gamma \times X)$,

$$\begin{aligned}\int_{\Gamma\times X} F(\sigma, x)J(\sigma\tau, x)d(\sigma, x) &= \int_\Gamma [\int_X F(\sigma, x)J(\sigma\tau, x)dx]d\sigma \\ &= \int_\Gamma [\int_X F(\sigma, x)J(\sigma, \tau x)J(\tau, x)dx]d\sigma \\ &= \int_{\Gamma\times X} F(\sigma, x)J(\sigma, \tau x)J(\tau, x)d(\sigma, x).\end{aligned}$$

We conclude this section with the introduction of the shift on $L_p(X)$, $p \in [1, \infty]$. For $\sigma \in \Gamma$ and $k \in L_p(X)$, we define the shift k_σ of k by

$$k_\sigma(x) = k(\sigma x)J(\sigma, x)^{1/p}, \qquad x \in X.$$

Thus $((k_\sigma)_\tau) = k_{\sigma\tau}$ for all $\sigma, \tau \in \Gamma$. We remark that if m_X is invariant under Γ, then $J(\sigma, x) = 1$ and hence $k_\sigma(x) = k(\sigma x)$, $\sigma \in \Gamma$, $x \in X$. Also we mention that such a definition of the shift was impossible until J was defined for all $\sigma \in \Gamma$.

4.16. THEOREM. *If* $k \in L_p(X)$, *then for each* $\sigma \in \Gamma$, $k_\sigma \in L_p(X)$ *and* $||k||_p = ||k_\sigma||_p$.

Proof. Cf. Theorem 4.13.

Indeed, Theorem 4.16 tells us that for $f \in L_1(\Gamma)$ and $k \in L_p(X)$, the convolution $f * k$ given in Theorem 4.2 may be written as

$$(f * k)(x) = \int_\Gamma f(\sigma)k_{\sigma^{-1}}(x)d\sigma,$$

for locally almost all $x \in X$. This formula is similar to the classical definition of convolution. Note that if $k \in L_1(\Gamma)$, then k_σ can be written in terms of the convolution of k with elements of $M(\Gamma)$. In fact, let δ_σ be the point mass at $\sigma \in \Gamma$. Then

4.17. THEOREM. *Let* $k \in L_1(X)$ *and* $\sigma \in \Gamma$. *Then* $k_\sigma = \delta_{\sigma^{-1}} * k$.

Proof. Let $k \in L_1(X)$, $\sigma \in \Gamma$, and let $Y \subseteq X$ be any Borel set. By the definition of convolution, and by Theorem 4.13,

$$\begin{aligned}\int_X \xi_Y(x)k(\sigma x)J(\sigma, x)dx &= \int_X \xi_Y(\sigma^{-1} x)k(x)dx \\ &= \int_X\int_\Gamma \xi_Y(\tau x)k(x)d\delta_{\sigma^{-1}}(\tau)dx \\ &= \int_X \xi_Y(x)[\delta_{\sigma^{-1}} * k)(x)]dx,\end{aligned}$$

so that $k_\sigma(x) = (\delta_{\sigma^{-1}} * k)(x)$ for almost all $x \in X$.

We should also mention that if $k \in L_p(X)$ and $j \in L_q(X)$, then

$$j(k_\sigma) = (j_{\sigma^{-1}})(k)$$

for any $\sigma \in \Gamma$, since by Theorem 4.13,

$$\begin{aligned} j(k_\sigma) &= \int_X j(x)k(\sigma x)J(\sigma, x)^{1/q}dx \\ &= \int_X j(\sigma^{-1} x)k(x)J(\sigma, \sigma^{-1} x)^{1/q}J(o^{-1}, x)dx \\ &= \int_X j(\sigma^{-1} x)J(\sigma^{-1}, x)^{1/p}k(x)dx \\ &= (j_{\sigma^{-1}})(k). \end{aligned}$$

We finish this section by showing that the shift is a continuous function.

4.18. THEOREM. *Let $p \in [1, \infty)$. For $f \in L_1(\Gamma)$, $k \in L_p(X)$, and $\sigma \in \Gamma$, $(f * k)_\sigma = f_\sigma * k$. Hence k_σ depends continuously on $\sigma \in \Gamma$.*

Proof. Since $f \in L_1(\Gamma)$, there exists a sigma-compact set Φ in Γ such that $f(\sigma) = 0$, for all $\sigma \notin \Phi$ (**(3**, Theorem 11.40**)**. From Lemma 4.15, we know that for locally almost all $x \in X$, $J(\tau^{-1} \sigma, x) = J(\tau^{-1}, \sigma x)J(\sigma, x)$ for almost every $\tau \in \Phi$. For all these $x \in X$,

$$\begin{aligned} (f * k)_\sigma(x) &= \int_\Gamma f(\tau)k(\tau^{-1} \sigma x)J(\tau^{-1}, \sigma x)^{1/p}J(\sigma, x)^{1/p}d\tau \\ &= \int_\Phi f(\tau)k(\tau^{-1}, \sigma x)J(\tau^{-1}, x\sigma)^{1/p}d\tau \\ &= \int_\Gamma f(\sigma\tau)k(\tau^{-1} x)J(\tau^{-1}, x)^{1/p}d\tau \\ &= (f_\sigma * k)(x), \end{aligned}$$

and the first statement is proved. For the second, we use the fact that $L_p(X)$ is factorable, by Theorem 4.11. Thus $k = f * h$, where $f \in L_1(\Gamma)$ and $h \in L_p(X)$. Thus $k_\sigma = (f * h)_\sigma = f_\sigma * h$. The continuity of the shift on $L_1(\Gamma)$ and the continuity of the convolution (Theorem 4.6) complete the proof that k_σ is a continuous function of σ.

5. The $(L_1(\Gamma), L_p(X))$-homomorphisms. In this section we apply the previous results. The first two theorems generalize the major portions of **(2**, Theorems 3.10 and 3.11**)**. Let $L_p(X), p \in [1, \infty]$, and $L_1(\Gamma)$ be modules over $L_1(\Gamma)$.

5.1. DEFINITION. *Let $p \in [1, \infty]$. We call a linear continuous map*

$$R : L_1(\Gamma) \to L_p(X)$$

an $(L_1(\Gamma), L_p(X))$-homomorphism if

$$f * R(g) = R(f * g), \qquad f, g \in L_1(\Gamma).$$

These homomorphisms form a Banach space which we denote by $\mathfrak{R}(L_1(\Gamma), L_p(X))$. On account of Theorem 4.6, every $j \in L_p(X)$ gives rise to an element $R \in \mathfrak{R}(L_1(\Gamma), L_p(X))$ through $R(f) = f * j$, $f \in L_1(\Gamma)$; thus $L_p(X)$ can be embedded in $\mathfrak{R}(L_1(\Gamma), L_p(X))$. In fact, for $p \in (1, \infty]$, we have

5.2. THEOREM. *For $p \in (1, \infty]$, the formula*

$$R(f) = f * j, \qquad f \in L_1(\Gamma),$$

yields a linear isometry between $L_p(X)$ and $\mathfrak{R}(L_1(\Gamma), L_p(X))$.

Proof. To any $j \in L_p(X)$ there corresponds a unique element R of $\mathfrak{R}(L_1(\Gamma), L_p(X))$, by our previous comments. Since $L_p(X)$ is an $L_1(\Gamma)$-module, $||j||_p \geqslant ||R||$. On the other hand, let $R \in \mathfrak{R}(L_1(\Gamma), L_p(X))$. Also let $\{e_\iota: \iota \in I\}$ be an approximate identity in $L_1(\Gamma)$ with $||e_\iota|| = 1$ for each ι. In addition, let $f \in L_1(\Gamma)$ and $k \in L_q(X)$, where $1/p + 1/q = 1$ and $q = 1$ if $p = \infty$. Then Lemma 4.7 yields

$$(R(f))k = \lim_\iota (R(f * e_\iota))k = \lim_\iota (f * R(e_\iota))k = \lim_\iota (R(e_\iota))(f' * k).$$

By Theorem 4.11, $F(k)$ can be defined for every $k \in L_q(X)$, by the equation $F(k) = \lim_\iota (R(e_\iota))k$. Evidently F is linear. Furthermore, if $k \in L_q(X)$, then

$$|F(k)| \leqslant \sup_\iota ||(Re_\iota)||\, ||k||_q \leqslant ||R|| \sup_\iota ||e_\iota||_1\, ||k||_q \leqslant ||R||\, ||k||_q,$$

so F is continuous. Hence $F \in L_p(X)$ and there exists a unique $j \in L_p(X)$ for which $F(k) = j(k)$, $k \in L_q(X)$. In addition, $||j||_p \leqslant ||R||$. Furthermore, for $f \in L_1(\Gamma)$, $k \in L_q(X)$,

$$(R(f))k = F(f' * k) = j(f' * k) = (f * j)k,$$

so that $R(f) = f * j$, for all $f \in L_1(\Gamma)$. Thus the correspondence is an isometry, which is obviously linear.

The essential ingredients of Theorem 5.2 are that $L_p(X)$ is isometric to the dual space of $L_q(X)$, and that $L_1(\Gamma) * L_q(X)$ is dense in $L_q(X)$. Both these properties of $L_q(X)$ are lacking when $p = 1$ (i.e., $q = \infty$). Actually, $\mathfrak{R}(L_1(\Gamma), L_1(X))$ usually is not isometric to $L_1(X)$. For instance, it is well known that $\mathfrak{R}(L_1(\Gamma), L_1(\Gamma))$ is isometric to $M(\Gamma)$, which is equal to $L_1(\Gamma)$ only when Γ is discrete. Nevertheless, for $p = 1$, the situation is not hopeless. In the rest of this section we identify $k \in L_1(X)$ with the element of $M(X)$ absolute continuous with respect to m_X which k defines. Thus for a Borel set $Y \subseteq X$ we write

$$k(Y) = \int_Y k(x)dx.$$

5.3. THEOREM. *The formula $R(f) = f * \mu, f \in L_1(\Gamma)$, yields a linear isometry from $\mathfrak{R}(L_1)(\Gamma), L_1(X))$ into $M(X)$. The image is*

$$N = \{\mu \in M(X) : f * \mu \ll m_X,\ f \in L_1(\Gamma)\}.$$

Proof. By Theorem 3.1, each $\mu \in N$ defines a unique $R \in \mathfrak{R}(L_1(\Gamma), L_1(X))$ by the formula $R(f) = f * \mu, f \in L_1(\Gamma)$, and $||R|| \leqslant ||\mu||$. On the other hand, let $(e_\iota)_{\iota \in I}$ be an approximate identity in $L_1(\Gamma)$ with $||e_\iota|| \leqslant 1$ and let $R \in \mathfrak{R}(L_1(\Gamma), L_1(X))$. Just as in Theorem 5.2 define $\mu \in M(X)$ such that *lim*$_\iota\, R(e_\iota)k = \mu(k)$, where this time $k \in C_\infty(X)$. Then Lemma 4.10 tells us

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that μ is uniquely defined, and as in Theorem 5.2, we have $R(f)k = \mu(f' * k)$, $k \in C_\infty(X)$, $f \in L_1(\Gamma)$. An easy calculation yields $R(f) = f * \mu$, for all $f \in L_1(\Gamma)$. Furthermore, it is clear that $||R|| \geqslant ||\mu||$. Finally, since $R(f) \in L_1(X)$, it follows that $f * \mu$ is absolutely continuous with respect to m_X, for all $f \in L_1(\Gamma)$, so that $\mu \in N$. Again the isometry is clearly linear.

Since $L_1(X)$ is an $L_1(\Gamma)$-module, $L_1(X) \subseteq N \subseteq M(X)$. On the other hand, N may reside anywhere from $L_1(X)$ to $M(X)$. In order to describe more precisely the nature of N we give a tractable characterization of N.

5.4. THEOREM. *An element μ of $M(X)$ is in N if and only if for every compact $K \subseteq X$ with $m_X(K) = 0$, we have $\mu(\sigma K) = 0$ for locally almost all $\sigma \in \Gamma$.*

Proof. By (3.3),

$$(f * \mu)(K) = \int_\Gamma f(\sigma)\mu(\sigma^{-1}K)d\sigma, \qquad \textit{for } f \in L_1(\Gamma),\ K \subseteq X \text{ compact};$$

it is readily seen that $\mu \in N$ if and only if the condition holds.

Examples: (i) Let Γ be discrete. Then the condition on $\mu \in M(X)$ for μ to be in N is that if $m_X(K) = 0$, then $\mu(\sigma K) = 0$ for *all* $\sigma \in \Gamma$, and in particular, $\mu(K) = 0$, whence $\mu \ll m_X$. Thus in this case, $N = L_1(X)$. This also occurs when $\pi(\sigma, x) = x$, for all $\sigma \in \Gamma$, $x \in X$. On the other hand, if $X = \Gamma$, and $m_X = m_\Gamma$, then $N = M(X)$ because $L_1(\Gamma)$ is an ideal in $M(\Gamma)$.

(ii) Let R be the additive group of the reals, with the ordinary topology, and let m be the Lebesgue measure on R. Let $\Gamma = R$ and $X = R \cup \{\infty\}$, the one-point compactification of R, and let δ_∞ be the point mass at ∞. Finally let m_X be defined by $m_X(Y) = m(R \cap Y) + \delta_\infty(Y)$, Y Borel, and define the action of Γ on X by

$$\pi(\sigma, x) = x + \sigma, \qquad x \in R,\ \sigma \in \Gamma,$$

$$\pi(\sigma, \infty) = \infty, \qquad \sigma \in \Gamma.$$

Then it is easy to see that $N = M(X)$, and of course, $L_1(X) \neq M(X)$.

(iii) Let Γ be the circle group with its usual topology and its Haar measure m_Γ. Let $X = \Gamma \times \Gamma$ and $m_X = m_\Gamma \times m_\Gamma$. For $\sigma \in \Gamma$ and $(x, y) \in X$, let $\pi(\sigma, (x, y)) = (\sigma x, y)$. Then N contains no point masses, so $N \neq M(X)$. On the other hand, $N \neq L_1(X)$ because N contains $\mu \in M(X)$, where

$$\mu(Y) = \int_\Gamma \xi_Y(1, \sigma)d\sigma, \qquad Y \subseteq X \text{ Borel}.$$

We now continue our analysis of N.

5.5. THEOREM. *If $\mu \in N$, then the support of μ is contained in the support of m_X.*

Proof. Without loss of generality, let μ be a real-valued measure with positive and negative parts μ^+, μ^- respectively. Assume that the conclusion is false, i.e., that there is a compact $K \subseteq X$ such that K and the support of m_X have void intersection, and such that $\mu(K) \neq 0$. Such a K can be written as $K_1 \cup K_2$, where $K_1 \cap K_2 = \emptyset$ and $\mu|_{K_1} \geqslant 0$, $\mu|_{K_2} \leqslant 0$. We may further assume that $\mu|_{K_1} \neq 0$. Then there is a compact subset L of K_1 such that $\mu^+(L) > 0$ and $\mu^-(L) = 0$. Since μ^- is regular, there is an open neighbourhood U of L such that $\mu^-(U) < \mu^+(L)$. Let Φ be an open neighbourhood of $1 \in \Gamma$ such that $\bar{\Phi}^{-1}\,\bar{\Phi}L \subseteq U$ and that $\bar{\Phi}L$ intersects the support of m_X in the void set. From $m_X(\bar{\Phi}L) = 0$ and $\mu \in N$ we may conclude that $\mu(\sigma^{-1}\,\bar{\Phi}L) = 0$ for locally almost all $\sigma \in \Gamma$. In particular there must be a $\sigma_0 \in \Phi$ such that $\mu({\sigma_0}^{-1}\,\bar{\Phi}L) = 0$. Then

$$\mu^+(L) = \mu({\sigma_0}^{-1}\,\sigma_0\,L) = \mu({\sigma_0}^{-1}\,\bar{\Phi}L) - \mu({\sigma_0}^{-1}\,\bar{\Phi}L\backslash L)$$
$$\leqslant 0 + \mu^-({\sigma_0}^{-1}\,\bar{\Phi}L\backslash L) \leqslant \mu^-(U) < \mu^+(L),$$

which is a contradiction. This yields the theorem.

Henceforth let Γ_0 be an open sigma-compact subgroup of Γ.

5.6. THEOREM. *The following conditions on $y \in X$ are equivalent:*

(i) *The point mass δ_y is in N.*

(ii) *If $K \subseteq X$ is compact and $m_X(K) = 0$, then $\sigma y \notin K$ for locally almost all $\sigma \in \Gamma$.*

(iii) *If Φ is a Borel subset of Γ with positive measure, then the outer measure of Φy is positive.*

(iv) $m_X(\Gamma_0\,y) > 0$.

Proof. That (i) is equivalent to (ii) follows from Theorem 5.4. Next, if $m_\Gamma(\Phi) > 0$, then Φ contains a compact set Ψ of positive measure. Let $K = \Psi y$, and note that K is compact. Then (ii) implies that $m_X(\Psi y) > 0$; thus (ii) implies (iii). Next, (iii) implies (iv) because Γ_0 and $\Gamma_0\,y$ are sigma-compact and because $m_\Gamma(\Gamma_0) > 0$. To prove that (iv) implies (ii), let K be a compact subset of X such that $m_X(K) = 0$. If L is a compact subset of X, then

$$\int_L\int_{\Gamma_0} \xi_K(\sigma x)d\sigma dx = \int_{\Gamma_0}\int_L \xi_K(\sigma x)dxd\sigma = \int_{\Gamma_0} m_X(\sigma^{-1}\,K \cap L)d\sigma$$
$$\leqslant \int_{\Gamma_0} m_X(\sigma^{-1}\,K)d\sigma = 0;$$

hence

$$\int_{\Gamma_0} \xi_K(\sigma x)d\sigma = 0$$

for locally almost all $x \in X$. However, since $\Gamma_0\,y$ has positive measure, there exists a $\tau \in \Gamma_0$ such that

$$\int_{\Gamma_0} \xi_K(\sigma\tau y)d\sigma = 0.$$

If Δ is the modular function on Γ, then

$$\int_{\Gamma_0} \xi_K(\sigma y)d\sigma = \Delta(\tau)\int_{\Gamma_0} \xi_K(\sigma\tau y)d\sigma = 0,$$

so that $\sigma y \notin K$ for almost all $\sigma \in \Gamma_0$. Since $m_X(\phi\Gamma_0\,y) > 0$ for all $\phi \in \Gamma$, and since $\{\phi L_0 : \phi \in \Gamma\}$ forms an open cover of Γ, it follows that $\sigma y \notin K$ locally almost everywhere in Γ.

5.7. THEOREM. *The following conditions are equivalent:*

(i) $N = M(X)$.

(ii) *N contains all the point masses.*

(iii) *If* $\mu \in M(X)$ *and if* μ_σ *is continuous as a function of* $\sigma \in \Gamma$, *then* $\mu \in L_1(X)$.

(iv) *If* $\mu \in M(X)$ *and if for each compact* $K \subseteq X$, $\mu(\sigma K)$ *is continuous as a function of* $\sigma \in \Gamma$, *then* $\mu \in L_1(X)$.

Proof. That (i) implies (ii) is trivial. To show that (ii) implies (i), let $\mu \in M(X)$ and $f \in L_1(\Gamma)$. Assume that $K \subseteq X$ is compact and $m_X(K) = 0$. Then

$$(f * \mu)(K) = \int_X \int_\Gamma \xi_K(\sigma x) f(\sigma) d\sigma d\mu(x) = \int_X \int_\Gamma [\int_X \xi_K(\sigma y) d\delta_x(y)] f(\sigma) d\sigma d\mu(x)$$
$$= \int_X \int_X \int_\Gamma \xi_K(\sigma y) f(\sigma) d\sigma d\delta_x(y) d\mu(x) = \int_X (f * \delta_x)(K) d\mu(x).$$

By hypothesis, $f * \delta_x \ll m_X$ for each $x \in X$, so $(f * \delta_x)(K) = 0$; hence $(f * \mu)(K) = 0$. This implies that $f * \mu \in L_1(X)$. Thus $\mu \in N$.

To show that (i) implies (iii), we note that if μ_σ is continuous as a function of σ, then $\mu \in L_1(\Gamma) * M(X)$ by Theorem 3.2, so that

$$\mu \in L_1(\Gamma) * M(X) = L_1(\Gamma) * N \subseteq L_1(\Gamma).$$

On the other hand, if $\mu \in M(X)$ and $\mu \in L_1(\Gamma) * M(X)$, then by Theorem 3.2, μ_σ is continuous as a function of σ, so by (iii), $\mu \in L_1(X)$. Thus

$$L_1(\Gamma) * M(X) \subseteq L_1(X),$$

whence $N = M(X)$ by the definition of N.

We now prove that (i) implies (iv). Note that if $\mu(\sigma K)$ is continuous for every compact $K \subseteq X$, then by Theorem 3.3 there is a $\nu \in M(X)$ such that

$$\nu \in L_1(\Gamma) * M(X) = L_1(\Gamma) * N \subseteq L_1(X), \qquad \text{and } \mu \ll \nu,$$

which means that $\mu \in L_1(X)$ also. Conversely, (iv) implies (i) by Theorem 3.3.

W. Rudin **(6)** noted that (iv) holds if $X = \Gamma$ and if Γ is abelian. In our context this implication follows from the fact that $N = M(X)$ if $X = \Gamma$. We can extend his result by use of the preceding theorems. Let X be a factor space of Γ with the induced topology and action of Γ. With Γ_0 open in Γ, it follows that $\Gamma_0 y$ is open in X, for each $y \in X$. The support of the induced m_X is the whole space X, and since the projection map of Γ onto X is open, $m_X(\Gamma_0 y) > 0$. Thus N contains δ_y by Theorem 5.6. However, $y \in X$ is arbitrary. Thus $N = M(x)$, according to Theorem 5.7. Thus part (iv) of Theorem 5.7 is true, provided X is a factor space of Γ.

Our final theorems concern the cases in which $N = M(X)$.

5.8. THEOREM. *If* $N = M(X)$, *then the support of* m_X *is* X *and every positive quasi-invariant Radon measure on* X *is absolutely continuous with respect to* m_X.

Proof. Since N contains all the point masses, the support of m_X is X, by Theorem 5.5. Let ν be a positive quasi-invariant Radon measure on X. Then Theorem 3.5 (with ν in place of m_X) implies that if $\mu \in M(X)$ and if $\mu \ll \nu$, then μ is absolutely continuous with respect to some element of $L_1(\Gamma) * M(X)$. But $N = M(X)$ implies $L_1(\Gamma) * M(X) \subseteq L_1(X, m_X)$. Thus whenever $\mu \ll \nu$, then $\mu \ll m_X$. Hence $\nu \ll m_X$.

We mention that even if $N = M(X)$, m_X need not be absolutely continuous with respect to every positive quasi-invariant Radon measure on X. As in one of the examples preceding Theorem 5.5. let R denote the reals, and let $\Gamma = R$ and $X = R \cup \{\infty\}$, the one-point compactification of R. As we mentioned earlier, $N = M(X)$. Let $\nu(Y) = m_X(R \cap Y)$, $Y \subseteq X$ Borel, so that so that ν is an invariant measure whose support is X. However, m_X is not absolutely continuous with respect to ν.

Given a group Γ acting as a transformation group on the locally compact Hausdorff space X, we say that Γ acts transitively on X if for any $x, y \in X$ there is a $\sigma \in \Gamma$ such that $\sigma x = y$. The orbit of $x \in X$ is the set Γx. For $y \in X$, let $\pi_y : \Gamma \rightarrow \Gamma y$ be defined by $\pi_y(\sigma) = \sigma y$, all $\sigma \in \Gamma$.

5.9. LEMMA. *If $N = M(X)$, then there is a $y \in X$ for which $\Gamma_0 y$ and Γy are open sets. For such an element y, the map π_y is open and Γy is homeomorphic to the factor space Γ/Γ_y, where $\Gamma_y = \{\sigma \in \Gamma : \sigma y = y\}$.*

Proof. Let U be an open subset of X with finite measure. If $y \in U$, then the set $\{\sigma \in \Gamma_0 : \sigma y \in U\}$ is open and non-empty, and therefore has positive measure. However, $N = M(X)$ by hypothesis, so δ_y is in N, and hence, by Theorem 5.6 ((i) implies (iii)), $\Gamma_0 y \cap U$ has positive measure. Moreover, since $m_X(U) < \infty$, this means that U can intersect at most a countable collection of Γ_0-orbits, say $\Gamma_0 y_1, \Gamma_0 y_2, \ldots$. But each $\Gamma_0 y_i$ is sigma-compact, so that each $\Gamma_0 y_i \cap U$ is the union of countably many sets each of which is closed in U. By the Baire category theorem one of the sets $\Gamma_0 y_i$, say $\Gamma_0 y$, contains a non-empty set V which is open in U and thus open in X. Then $\Gamma_0 y = \bigcup_{\sigma \in \Gamma_0} \sigma V$, which is open. Similarly, $\Gamma y = \bigcup_{\sigma \in \Gamma} \sigma V$ is open. Remember that Γ_0 is a sigma-compact group and acts transitively on the locally compact space $\Gamma_0 y$. Therefore, according to **(5**, Theorem 20**)**, the restriction of π_y to Γ_0 is open as a map from Γ_0 onto $\Gamma_0 y$, and therefore as a map from Γ_0 into Γy. But then for any $\sigma \in \Gamma$, π_y defines an open map from $\sigma\Gamma_0$ into Γy. In addition, the various $\sigma\Gamma_0$ form an open covering of Γ. Thus π_y is open. The rest of the proof follows easily.

5.10. THEOREM. *Let $N = M(X)$. Then every orbit $\Gamma y, y \in X$, is the intersection of a closed and an open subset of X. For each $y \in X$, the map π_y is open. Also, Γy is homeomorphic to the factor space Γ/Γ_y, where $\Gamma_y = \{\sigma \in \Gamma : \sigma y = y\}$.*

Proof. We define a well-ordering $\{\Gamma y_\iota : \iota < \iota_\infty\}$ of the orbits in X as follows. Let ι_0 be an ordinal number and assume that Γy_ι has been defined for all

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$\iota < \iota_0$, so that $\bigcup\{\Gamma y_\iota : \iota < \iota_0\}$ is open. If $\bigcup\{\Gamma y_\iota : \iota < \iota_0\} = X$, we call $\iota_0 = \iota_\infty$ and stop the induction process. However, if $\bigcup\{\Gamma y_\iota : \iota < \iota_0\} \neq X$, then $X_{\iota_0} = X \backslash \bigcup\{\Gamma y_\iota : \iota < \iota_0\}$ is closed in X; hence it is locally compact. In addition, X_{ι_0} is invariant under Γ, and for $y \in X_{\iota_0}$, $\Gamma_0 y$ has positive measure. Therefore, by Lemma 5.9, there is a $y_{\iota_0} \in X_{\iota_0}$ such that $\Gamma_0 y_{\iota_0}$ and Γy_{ι_0} are open in X_{ι_0}. Then

$$X \backslash \bigcup\{\Gamma y_\iota : \iota \leqslant \iota_0\} = X_{\iota_0} \backslash \Gamma y_{\iota_0}$$

is closed, so $\bigcup\{\Gamma y_\iota : \iota \leqslant \iota_0\}$ is open and we have prepared the way for the next induction step. By this construction, for every $\iota < \iota_\infty$, Γy_ι is open in X_ι and X_ι itself is closed in X. Thus Γy_ι is the intersection of a closed and an open set. That the map π_y is open and that Γy is homeomorphic to Γ/Γ_y follows directly from the lemma.

5.11. THEOREM. *If $N = M(X)$, and if Γ acts transitively on X, then X is homeomorphic to a factor space of Γ.*

Proof. Apply Theorem 5.10 directly.

We mention that the map $\pi(\text{o}, y) : \Gamma \to X$ is *not* open, even though π_y is open. We also mention two unsolved questions. First, exactly when does $N = L_1(X)$ hold. Secondly, if $\mu \in N$ and if $\nu \ll \mu$, then is $\nu \in N$?

Added in proof. Recently, E. Hewitt and also A. Figà-Talamanca and P. Curtis have generalized Cohen's result.

REFERENCES

1. P. J. Cohen, *Factorization in group algebras*, Duke Math. J., *26* (1959), 199–205.
2. S. L. Gulick, T. S. Liu, and A. C. M. van Rooij, *Group algebra modules*, I, Can. J. Math. (this issue).
3. E. Hewitt and K. A. Ross, *Abstract harmonic analysis, Part* I (Berlin, 1963).
4. A. M. Macbeath and S. Świerczkowski, *Inherited measures*, Proc. Roy. Soc. Edinburgh, *65* (1961), 237–246.
5. L. S. Pontryagin, *Topologische Gruppen*, I (Leipzig, 1957).
6. W. Rudin, *Measure algebras on abelian groups*, Bull. Amer. Math. Soc., *65* (1959), 227–247.

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