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OSCILLATIONS OF NEUTRAL DELAY DIFFERENTIAL EQUATIONS

BY

G. LADAS AND Y. G. SFICAS

ABSTRACT. The oscillatory behavior of the solutions of the neutral delay differential equation

$$\frac{d}{dt}[x(t) - px(t-\tau)] + Q(t)x(t-\sigma) = 0, \; t \geq t_0,$$

where p, τ, and σ are positive constants and $Q \in C([t_0, \infty), \mathbb{R}^+)$, are studied.

1. **Introduction**. A neutral delay differential equation (NDDE for short) is a differential equation in which the highest order derivative of the unknown function appears with the argument t (present state) as well as one or more delayed arguments (past histories). See Driver [3] and [4], Bellman and Cooke [2], and Hale [5] for questions of existence, uniqueness, and continuous dependence. In general the theory of NDDEs presents extra complications and basic results which are true for delay differential equations are not true for neutral equations. For example Snow [16] has shown that even though the characteristic roots of a NDDE may all have negative real parts, it is possible for some solutions to be unbounded.

In this paper we deal with the oscillatory behavior of the solutions of linear NDDEs of the form

$$(1) \qquad \frac{d}{dt}[x(t) - px(t-\tau)] + Q(t)x(t-\sigma) = 0, \; t \geq t_0$$

where p, τ, and σ are positive constants and $Q \in C[[t_0, \infty), \mathbb{R}]$. Let $\phi \in C[[t_0 - m, t_0], \mathbb{R}]$ where $m = \max\{\tau, \sigma\}$. By a *solution of Eq. (1) with initial function* ϕ *at* t_0 we mean a function $x \in C[[t_0 - m, \infty), \mathbb{R}]$ such that $x(t) = \phi(t)$ for $t_0 - m \leq t \leq t_0, x(t) - px(t-\tau)$ is continuously differentiable for $t \geq t_0$, and x satisfies the equation

$$\frac{d}{dt}[x(t) - px(t-\tau)] + Q(t)x(t-\sigma) = 0 \text{ for } t \geq t_0.$$

Using the method of steps, it follows that for any continuous initial function ϕ there exists a unique solution of Eq. (1) valid for $t \geq t_0$.

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As is customary, a solution of Eq. (1) is called *oscillatory* if it has arbitrarily large zeros and *nonoscillatory* if it is eventually positive or negative. Although the oscillation theory of delay differential equations has been extensively developed during the past few years (see, for example, [1], [6] and [8]-[15] and the references cited therein), there is hardly any theory, at this time, dealing with the oscillatory behavior of NDDEs. [For some results about second order nonlinear neutral equations and for μ-like continuous functions see [17] and [7] respectively.]

2. **Constant Coefficients**. Consider the NDDE

$$\frac{d}{dt}[x(t) - px(t-\tau)] + qx(t-\sigma) = 0, \; t \geq t_0 \tag{2}$$

where p, q, τ, and σ are positive constants. The characteristic equation of Eq. (2) is

$$F(\lambda) \equiv \lambda - p\lambda e^{-\lambda\tau} + qe^{-\lambda\sigma} = 0. \tag{3}$$

Observe the following:

(α) For $p \leq 1$, $F(\lambda) = \lambda(1 - pe^{-\lambda\tau}) + qe^{-\lambda\sigma} > 0$ for $\lambda \geq 0$.

(β) For $p = 1$, $F(\lambda) = -\lambda(e^{-\lambda\tau} - 1) + qe^{-\lambda\sigma} > 0$ for $\lambda \leq 0$.

(γ) For $p > 0$ and $q\sigma e > 1$, $F(\lambda) = (\lambda + qe^{-\lambda\sigma}) - p\lambda e^{-\lambda\tau}$

$$\geq \frac{ln(q\sigma e)}{\sigma} - p\lambda e^{-\lambda\tau} > 0 \text{ for } \lambda \leq 0.$$

Motivated by the above observations we established the following results.

THEOREM 1. *Assume* $p < 1$, *Then every nonoscillatory solution of Eq.* (2) *tends to zero as* $t \to \infty$.

THEOREM 2. *Assume* $p = 1$. *Then every solution of Eq.* (2) *oscillates.*

THEOREM 3. *Assume* $p < 1$ *and* $q\sigma e > 1$. *Then every solution of Eq.* (2) *oscillates.*

Finally, without any motivation from the characteristic equation (3), we establish the following oscillation result.

THEOREM 4. *Assume* $p < 1$, $\sigma > \tau$, *and* $q(\sigma - \tau) > (1-p)/e$. *Then every solution of Eq.* (*) *oscillates.*

As a corollary, the above result gives sufficient conditions for Eq. (3) to have no real roots.

Extensions of these results to equations with the coefficient q variable are given together with their proofs in Section 3. The proofs of the above Theorems will therefore be omitted.

EXAMPLES. 1) The NDDE

$$\frac{d}{dt}\left[x(t) - \frac{1}{2e}x(t-1)\right] + \frac{1}{2e}x(t-1) = 0,\ t \geq 0 \tag{4}$$

satisfies the hypothesis of Theorem 1 and therefore every solution of Eq. (4) tends to zero as $t \to \infty$. For example, $x(t) = e^{-t}$ is such a solution. On the other hand, Eq. (4) does not satisfy the hypothesis $q\sigma e > 1$ of Theorem 3. Therefore it is not surprising that Eq. (4) has nonoscillatory solutions, for example, $x(t) = e^{-t}$.

2) The NDDE

$$\frac{d}{dt}[x(t) - (1+e)x(t-1)] + x(t-1) = 0,\ t \geq 0$$

does not satisfy the hypothesis of any of the Theorems 1–4. Therefore, it is not surprising that it has a solution namely, $x(t) = e^{t}$, which is nonoscillatory and does not approach zero as $t \to \infty$. Examples 1 and 2 demonstrate that if either of the hypotheses of Theorem 3 fails, the conclusion may fail too.

3) The NDDE

$$\frac{d}{dt}\left[x(t) - \frac{1}{2}x(t-\pi)\right] + \frac{3}{2}x\left(t - \frac{\pi}{2}\right) = 0 \tag{5}$$

satisfies the hypotheses of Theorem 3 and so all solutions of Eq. (5) oscillate. For example, sin t and cos t are oscillatory solutions of Eq. (5). Note, however that the hypotheses of Theorem 4 are not satisfied.

4) The NDDE

$$\frac{d}{dt}[x(t) - px(t-2\pi)] + qx\left(t - \frac{5\pi}{2}\right) = 0 \tag{6}$$

with $q = 1/5\pi e$ and $p = 1 - 1/5\pi e$ satisfies the hypotheses of Theorem 4 and so all solutions of Eq. (6) oscillate. For example, sin t and cos t are oscillatory solutions of Eq. (6). Note, however, that the hypotheses of Theorem 3 are not satisfied.

5) The NDDE

$$\frac{d}{dt}[x(t) - x(t-\pi)] + 2x\left(t - \frac{\pi}{2}\right) = 0$$

satisfies the hypotheses of Theorem 2 and so every solution of this equation oscillates. For example, sin t and cos t are oscillatory solutions.

3. **Variable Coefficients**. Consider the NDDE

$$\frac{d}{dt}[x(t) - px(t-\tau)] + Q(t)x(t-\sigma) = 0,\ t \geq t_0 \tag{7}$$

where p, τ, and σ are positive constants and $Q \in C[[t_0, \infty), \mathbb{R}^+]$. Theorems 5 through 8 below are extensions of Theorems 1 through 4 respectively.

THEOREM 5. *Assume*

$$p < 1 \tag{8}$$

and

$$\int_{t_0}^{\infty} Q(s)ds = \infty \tag{9}$$

Then every nonoscillatory solution of Eq. (7) tends to zero as $t \to \infty$.

THEOREM 6. *Assume condition* (9) *and that*

$$p = 1. \tag{10}$$

Then every solution of Eq. (7) oscillates.

THEOREM 7. *Assume condition* (8) *and that*

$$\lim_{t\to\infty} \int_{t-\sigma}^{t} Q(s)ds > \frac{1}{e}. \tag{11}$$

Then every solution of Eq. (7) oscillates.

THEOREM 8. *Assume condition* (8), $\sigma > \tau$, *Q is τ-periodic, and that*

$$\lim_{t\to\infty} \int_{t-(\sigma-\tau)}^{t} Q(s)ds > \frac{1-p}{e}. \tag{12}$$

Then every solution of Eq. (7) oscillates.

The following Lemma will be useful in the proofs of the Theorems.

LEMMA 1. *Let f, g: $[t_0, \infty) \to \mathbb{R}$ be such that*

$$f(t) = g(t) - pg(t-c),\ t \geq t_0. \tag{13}$$

Assume p and c are constants with $0 < p \leq 1$, g is bounded on $[t_0, \infty)$, and $\lim_{t\to\infty} f(t) \equiv l$ exists. Then the following statements hold:

(i) $p = 1$ implies $l = 0$;

(ii) $p < 1$ implies $\lim_{t\to\infty} g(t)$ exists.

PROOF: Let $\{t_n\}$ and $\{t_n'\}$ be sequences of points in $[t_0, \infty)$ which converge to ∞ as $n \to \infty$ and such that

$$\lim_{n\to\infty} g(t_n) = \overline{\lim_{t\to\infty}}\, g(t) \equiv s$$

and

$$\lim_{n\to\infty} g(t_n') = \underline{\lim_{t\to\infty}}\, g(t) \equiv i.$$

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From (13),

$$g(t_n - c) = \frac{g(t_n) - f(t_n)}{p}, \; n = 1, 2, \ldots$$

and taking limits as $n + \infty$ we obtain

$$s \geq \lim_{n \to \infty} g(t_n - c) = \frac{s - l}{p}$$

or $l \geq s(1 - p)$. In a similar way we find $l \leq i(1 - p)$. When $p = 1$, clearly $l = 0$ and (i) is proved. When $p < 1$,

$$s \leq \frac{l}{1 - p} \leq i$$

from which it follows that $s = i$ and (ii) is proved. The proof of the Lemma is complete.

Proof of Theorem 5. Since the negative of a solution of Eq. (7) is also a solution, it suffices to show that every eventually positive solution $x(t)$ of Eq. (7) tends to zero as $t \to \infty$. First, we will prove that every eventually positive solution $x(t)$ of Eq. (7) is bounded *Set* $z(t) = x(t) - px(t - \tau)$. Then, from Eq. (7), $z'(t) = -Q(t)x(t - \sigma) \leq 0$ which implies that $z(t)$ is eventually decreasing and so in particular $z(t) < x(t - \sigma)$. Using this in Eq. (7) we find $z'(t) + Q(t)z(t) \leq 0$ for t sufficiently large, say, $t \geq t_2 \geq t_1$. Then

$$\frac{d}{dt}\left[z(t)e^{\int_{t_2}^{t} Q(s)\,ds}\right] \leq 0, \; t \geq t_2$$

and so

$$z(t)e^{\int_{t_2}^{t} Q(s)\,ds} \leq z(t_2), \; t \geq t_2.$$

Let $B = \max\{z(t_2), 0\}$. Then for $t \geq t_2$

$$x(t) \leq px(t - \tau) + B\exp\left(-\int_{t_2}^{t} Q(s)\,ds\right). \tag{14}$$

Assume, for the sake of contradiction, that $x(t)$ is not bounded. Then, there exists a sequence $\{t_n\}$ such that

$$\lim_{n \to \infty} t_n = \infty, \; \lim_{n \to \infty} x(t_n) = \infty \text{ and } x(t_n) = \max_{t_2 \leq s \leq t_n} x(s).$$

From (14), for n sufficiently large, we then obtain

$$x(t_n) < px(t - \tau) + B\exp\left(-\int_{t_2}^{t_n} Q(s)\,ds\right) \leq px(t_n) + B\exp\left(-\int_{t_2}^{t_n} Q(s)\,ds\right).$$

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Hence

$$x(t_n) < \frac{B}{1-p} \exp\left(-\int_{t_2}^{t_n} Q(s)\,ds\right)$$

which as $n \to \infty$ leads to a contradiction. This establishes our claim that $x(t)$ is bounded. Applying Lemma 1(ii) with $f(t) = z(t)$, $g(t) = x(t)$, and $c = \tau$ we obtain the conclusion that $\lim_{t\to\infty} x(t)$ exists. Since $x(t) > 0$ this limit, call it L, is nonnegative. Assume, for sake of contradiction, that $L > 0$. Then, from Eq. (7)

$$\frac{d}{dt}[x(\mathrm{t}) - px(t-\tau)] + Q(t)\frac{L}{2} \leq 0$$

for t sufficiently large, say, $t \geq t_3 \geq t_2$. Integrating from t_3 to t we obtain

$$(15) \qquad [x(\mathrm{t}) - px(t-\tau)] + \frac{l}{2}\int_{t_2}^{t} Q(s)\,ds) \leq [x(t_3) - px(t_3 - \tau)].$$

Since $x(t)$ is bounded and since, in view of (9), the integral in (15) diverges to ∞ as $t \to \infty$, it follows that (15) leads to a contradiction as $t \to \infty$. The proof of Theorem 5 is complete.

Proof of Theorem 6. Assume, for the sake of contradiction, that Eq. (7) has an eventually positive solution $x(t)$. Set $z(t) = x(t) - x(t-\tau)$. It follows from Eq. (7) that $z(t)$ is eventually decreasing. Since $z(t)$ cannot be eventually identically zero, it follows that either $z(t)$ is eventually negative or $z(t)$ is eventually positive. If $z(t)$ were eventually negative then $x(t)$ would be bounded and from Lemma 1(i), $\lim_{t\to\infty} z(t) = 0$. This implies that $z(t)$ is eventually positive. Thus

$$(16) \qquad x(t) - x(t-\tau) > 0$$

for t sufficiently large, say, $t \geq t_1 \geq t_0$. Integrating Eq. (7) from t_1 to t we find

$$[x(t) - x(t-\tau)] - [x(t_1) - x(t_1 - \tau)] + \int_{t_1}^{t} Q(s)x(s-\sigma)\,ds = 0$$

which implies that

$$(17) \qquad \int_{t_1}^{t} Q(s)x(s-\sigma)\,ds \leq x(t_1), \; t \geq t_1.$$

Let $m = \min x(s-\sigma)$ for $t_1 \leq s \leq t_1 + \tau$. Then, in view of (16), $x(s-\sigma) \geq m$ for $s \geq t_1$ and (17) implies that

$$m\int_{t_1}^{t} Q(s)\,ds \leq x(t_1), \; t \geq t_1$$

which as $t \to \infty$ contradicts (9). The proof of Theorem 6 is complete.

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Proof of Theorem 7. Assume, for the sake of contradiction, that Eq. (7) has an eventually positive solution $x(t)$. By Theorem 5, it follows that $\lim_{t\to\infty} x(t) = 0$. Set $z(t) = x(t) - px(t - \tau)$. Then $z(t)$ decreases to zero and, in view of (11) $z(t)$ cannot be eventually identically zero which implies that eventually

$$z(t) > 0. \tag{18}$$

But $z(t) < x(t)$ and Eq. (7) implies that eventually

$$z'(t) + Q(t)z(t - \sigma) < 0. \tag{19}$$

From [11] and [8] we know that condition (11) implies that (19) cannot have an eventually positive solution. This contradicts (18) and completes the proof of the theorem.

Proof of Theorem 8. Let $x(t)$ and $z(t)$ as in the proof of Theorem 7 and set $w(t) = z(t) - pz(t - \tau)$. Since Q is τ-periodic, it is easily seen that z and w are continuously differentiable solutions of Eq. (7). We have $z'(t) = -Q(t)x(t - \sigma)$ and $w'(t) = -Q(t)z(t - \sigma)$. Now using the fact that $x(t)$ tends to zero as $t \to \infty$, it follows that each of z and w is nonincreasing and tends to zero as $t \to \infty$. It then follows that eventually

$$w(t) > 0. \tag{20}$$

Furthermore, $w'(t - \tau) = -Q(t - \tau)z(t - \tau - \sigma) = -Q(t)z(t - \tau - \sigma) \leq -Q(t)z(t - \sigma) = w'(t)$. Then from the equation

$$w'(t) - pw'(t - \tau) + Q(t)w(t - \sigma) = 0$$

we obtain the delay inequality

$$w'(t - \tau) - pw'(t - \tau) + Q(t)w(t - \sigma) \leq 0$$

or

$$w'(t) + \frac{1}{1 - p} Q(t)w[t - (\sigma - \tau)] \leq 0.$$

But in view of (12), see [11] and [8], the last inequality cannot have an eventually positive solution. This contradicts (20) and the proof is complete.

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Department of Mathematics
University of Rhode Island
Kingston, Rhode Island 02881

Department of Mathematics
University of Ioannina
Ioannina 45332, Greece

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