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LETTERS TO THE EDITOR

RANDOM SET AND COVERAGE MEASURE

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Abstract

It is well known that a random set determines its random coverage measure. The paper gives a necessary and sufficient condition for the reverse implication. An equivalent formulation of the condition constitutes a first step in the search for a way to recognize a random measure as being the random coverage measure of a random set.

RANDOM MEASURE; RANDOM COVERAGE MEASURE

1. Introduction

The interest in random closed sets (Matheron (1975)) has increased during recent years along with the increasing development of their applications. These sets are a cornerstone in the model approach to stereology, as can be seen in Stoyan (1990). Given a random set Φ, its associated random coverage measure is a partial description of it. Recent results in stereology are focused on the estimation of random set characteristics related to this associated coverage measure, in particular its second-moment measure (Cruz-Orive (1989), Jensen et al. (1990)).

Obviously, a random set always determines its coverage measure. But under what conditions does the coverage measure determine the distribution of a random set? The answer to this question can be found as a corollary to another more general question: how can one recognize a random measure μ as being the random coverage measure of a random set with distribution determined by μ?

2. Results

In $\mathbb{R}^k$ with Borel σ-field β^k, a random (d, k)-set Φ is defined as a measurable mapping from a probability space into the measurable space of ν-rectifiable closed sets in $\mathbb{R}^k$, where ν stands for the corresponding d-dimensional Hausdorff measure in $\mathbb{R}^k$ (see Jensen et al. (1990) and Zähle (1982) for more details). This random set determines a unique random coverage measure defined as $\mu_\Phi(B) = \nu(\Phi \cap B)$, $B \in \beta^k$. The following theorem establishes which condition the random closed set must satisfy in order to recover its distribution from the associated coverage measure.

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Theorem 1. Let Φ be a random (d, k)-set and μ_Φ its associated coverage measure. The distribution of Φ is recoverable from μ_Φ if and only if

$$P\{\Phi \cap K \neq \varnothing,\ \mu_\Phi(K \oplus \varepsilon B) = 0 \quad \text{for some } \varepsilon\} = 0 \tag{2.1}$$

for all compact sets K, where $\oplus$ denotes Minkowski addition and B is the open unit ball in $\mathbb{R}^k$.

Proof. The distribution of Φ is determined by the probabilities $T(K) = P\{\Phi \cap K \neq \varnothing\}$ for every compact K (see Matheron (1975)).

Notice that

$$\mu_\Phi(K \oplus \varepsilon B) > 0, \qquad \forall \varepsilon > 0 \Rightarrow \Phi \cap K \neq \varnothing. \tag{2.2}$$

On the other hand, $\{\Phi \cap K \neq \varnothing\}$ can be written as the disjoint decomposition

$$\begin{aligned}\{\Phi \cap K \neq \varnothing\} = \{&\mu_\Phi(K \oplus \varepsilon B) > 0, \forall \varepsilon > 0\} \\ &\cup \{\Phi \cap K \neq \varnothing,\ \mu_\Phi(K \oplus \varepsilon B) = 0 \quad \text{for some } \varepsilon\},\end{aligned} \tag{2.3}$$

and $\{\mu_\Phi(K \oplus \varepsilon B) > 0, \forall \varepsilon > 0\}$ being an event with its probability determined by the distribution of μ_Φ, the sufficiency of (2.1) is established.

To prove necessity, suppose that for some compact K and $\varepsilon > 0$, $P(\Phi \cap K \neq \varnothing, \mu_\Phi(K \oplus \varepsilon B) = 0) > 0$. Set

$$\Phi' = \begin{cases} \Phi - (K \oplus \varepsilon B), & \text{if } \mu_\Phi(K \oplus \varepsilon B) = 0 \\ \Phi, & \text{otherwise.} \end{cases}$$

It follows immediately that $\mu_\Phi = \mu_{\Phi'}$, but the disjoint decomposition

$$\{\Phi' \cap K = \varnothing\} = \{\Phi \cap K = \varnothing\} \cup \{\Phi \cap K \neq \varnothing,\ \mu_\Phi(K \oplus \varepsilon B) = 0\}$$

implies $P\{\Phi \cap K \neq \varnothing) > P(\Phi' \cap K \neq \varnothing)$.

If follows from the theorem that $P(\Phi \cap K \neq \varnothing) = \lim_{i \to \infty} P\{\mu_\Phi(K \oplus \varepsilon_i B) > 0\}$ for any sequence of ε_i decreasing to 0. In fact, we can associate with each random measure μ a random closed set Φ_μ, satisfying the condition (2.1), as follows.

Definition. Given a random measure μ let Φ_μ be the *random closed support* of μ, defined for any sequence of ε_i decreasing to 0 and for any $\{x_1, x_2, \cdots\}$ dense in $\mathbb{R}^k$ by

$$\Phi_\mu = \textstyle\bigcap_{i=1}^{\infty} \text{closure } \{x_j,\ \mu(x_j \oplus \varepsilon_i B) > 0\}.$$

Note that this definition is independent of the choice of sequences $\{\varepsilon_i\}$ and $\{x_1, x_2, \cdots\}$ and it allows us the following alternative formulation of Theorem 1.

Theorem 2. The distribution of the random closed set Φ is recoverable from μ_Φ if and only if Φ is distributed as the random closed support of μ_Φ.

Theorem 2 gives a natural answer to the second question in the introduction: a random measure μ will be the random coverage of a random closed set when its closed support Φ_μ has μ as its random coverage measure. But the fundamental question remains: what natural and verifiable conditions must be imposed on μ in order for it to have this property?

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